1. Notes on the Notes

These are lecture notes for a series of four lectures at MIT on hypergeometric functions in the style of Gelfand, Kapranov and Zelevinsky. Much of the material here is taken directly from [2], [3] and [1], though all errors are of course mine.

They are still under construction, and probably include a number of typos and omissions.

2. Origins

Consider the integral

\[ I(z) = \int_0^1 t^{-1/2}(1 - t)^{-1/2}(1 - zt)^{-1/2} dt = \int_0^1 \frac{dt}{\sqrt{zt^3 - (z + 1)t^2 + t}} \]

defining a period of the elliptic curve \( y^2 = zt^3 - (z + 1)t^2 + t \). For any fixed value of \( z \), it is difficult to evaluate exactly. If we let \( z \) roam free, however, we find that \( I(z) \) satisfies an ODE in \( z \) called the Picard-Fuchs equation which almost characterizes \( I(z) \) uniquely. By letting \( z \) run
off to infinity, one recovers a power series expansion of $I(z)$; the monodromy of $I(z)$ around the singularities of the ODE encodes valuable information about the moduli of elliptic curves.

Similarly, consider the integral

$$I(z_1, z_2) = \int (t_1 - t_2)\alpha(t_1 - z_1)\beta_1(t_2 - z_1)\beta_1(t_1 - z_2)\beta_2(t_2 - z_2)\frac{dt_1 dt_2}{t_1 t_2}$$

which arises as a correlation function in conformal field theory, measuring the interaction between events located at $z_1$ and $z_2$. Again it is difficult to evaluate for fixed $z_1, z_2$, but we shall see that it satisfies a beautiful PDE in $z_1, z_2$ which opens up a treasure trove of information on $I(z_1, z_2)$.

These notes will be concerned with general integrals of complex powers of complex polynomials, viewed as functions of the coefficients of said polynomials. These integrals satisfy differential equations discovered by Gelfand, Kapranov and Zelevinsky called generalized hypergeometric equations, which allow us to tackle them very explicitly. Along the way these GKZ equations will provide a testing ground for notions from toric geometry, microlocal analysis and perverse sheaves.

We conclude with a famous application of these integrals to the construction of solutions to the Knizhnik-Zamolodchikov equations and the construction of quantum groups at root of unity, generalizing example (2).

3. Gauss Hypergeometric Function

3.1. Beta integral. The very simplest such integral can be evaluated explicitly: Consider the beta function

**Definition 1.**

$$B(z, w) = \int_0^1 t^{z-1}(1 - t)^{w-1}dt$$

**Theorem 2.**

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z + w)}.$$

**Proof.** By definition,

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt.$$ 

Thus

$$\Gamma(z)\Gamma(w) = \int_0^\infty \int_0^\infty e^{-(t+s)}t^{z-1}s^{w-1}dtds.$$
Let $t = xy, s = x(1 - y)$, so that $x = t + s, y = \frac{t}{t+s}$. Then

$$
\int_{0}^{\infty} \int_{0}^{\infty} e^{-(t+s)s^{z-1}s^{w-1}} dtds = \int_{0}^{1} \int_{0}^{\infty} e^{-x^{x}z^{z+w-2}y^{z-1}(1-y)^{w-1}} \left| \frac{\partial s}{\partial x} \frac{\partial t}{\partial y} - \frac{\partial s}{\partial y} \frac{\partial t}{\partial x} \right| dxdy
$$

$$
= \int_{0}^{1} \int_{0}^{\infty} e^{-x^{x}z^{z+w-1}y^{z-1}(1-y)^{w-1}} dxdy
$$

$$
= \Gamma(z+w) \int_{0}^{1} y^{z-1}(1-y)^{w-1} dy.
$$

\[ \square \]

3.2. **Gauss hypergeometric function.** Consider Euler’s integral, which specializes to the elliptic integral (1) for appropriate choice of $a, b, c$.

(3) 

$$
I(z) = \int_{0}^{1} t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt
$$

It converges for $\text{Re}(b) > 0, \text{Re}(c - b) > 0$. A priori, it is a multivalued function of $z \in \mathbb{C} \setminus [0, 1]$, which depends on a choice of branch for the integrand. It famously satisfies the differential equation in $z$ with singularities at $z = 0, 1, \infty$.

$$
z(1-z)F'' + (c - (a + b + 1)z)F' - abF = 0
$$

which defines an analytic continuation to a multivalued function on $\mathbb{C} \setminus 0, 1$. Let us derive this equation in a manner that will generalize to more complex integrals. First we add some variables: define

$$
F(w_1, z_1, w_2, z_2) = \int_{0}^{1} t^{b-1}(w_1 - z_1t)^{c-b-1}(w_2 - z_2t)^{-a} dt
$$

The usual Gauss hypergeometric function is obtained by $F(1, 1, 1, z)$. It is easily seen to satisfy

$$
\partial_{z_1} \partial_{w_2} F = \partial_{w_1} \partial_{z_2} F
$$

and

$$
F(\lambda w_1, \lambda z_1, w_2, z_2) = \lambda^{c-b-1} F(w_1, z_1, w_2, z_2)
$$

$$
F(w_1, z_1, \lambda w_2, \lambda z_2) = \lambda^{-a} F(w_1, z_1, w_2, z_2)
$$

$$
F(w_1, \lambda z_1, w_2, \lambda z_2) = \lambda^{-b} F(w_1, z_1, w_2, z_2)
$$

Differentiating the above equations gives

$$
(z_1 \partial_{z_1} + w_1 \partial_{w_1}) F = (c - b - 1) F
$$

$$
(z_2 \partial_{z_2} + w_2 \partial_{w_2}) F = -a F
$$

$$
(z_1 \partial_{z_1} + z_2 \partial_{z_2}) F = -b F
$$
Writing \( \theta_1 = z_1 \partial_{z_1} \) and \( \overline{\theta}_1 = w_1 \partial_{w_1} \), and likewise for \( \theta_2, \overline{\theta}_2 \), we can record the above relations concisely as

\[
\begin{aligned}
z_2 w_1 \theta_1 \theta_2 &= z_1 w_2 \theta_2 \overline{\theta}_1 \\
\theta_1 + \overline{\theta}_1 &= c - b - 1 \\
\theta_2 + \overline{\theta}_2 &= -a \\
\theta_1 + \theta_2 &= -b
\end{aligned}
\]

Substituting the last three equations in the first and setting \( w_1 = z_1 = w_2 = 1 \) and \( \theta_2 = \theta = \frac{\partial}{\partial z} \), we obtain

\[
(4) \quad \left( z(\theta + a)(\theta + b) - \theta(\theta + c - 1) \right) F = 0.
\]

The reader may check that this is the Gauss hypergeometric equation. By construction, \( I(z) \) is a solution to (4) on its domain. Hence we have a natural (multivalued) analytic continuation of \( I(z) \) away from the singularities of (4) at \( z = 0, 1, \infty \). Note that we can also obtain this analytic continuation by deforming the contour of integration to avoid \( z^{-1} \) as \( z^{-1} \) crosses the interval \([0, 1]\).

Suppose instead we want a series solution

\[
F(z) = z^r \sum_{n=0}^{\infty} f_n z^n
\]

Since \( \theta z^{n+r} = (n + r)z^{n+r} \), equation (4) determines the recurrence relation

\[
f_{n-1}(n + r + a)(n + r + b) = f_n(n + r)(n + r - 1 + c).
\]

The condition \( f_{n-1} = 0 \) forces \( r = 0 \) or \( r = 1 - c \). Thus we obtain the two solutions

\[
\begin{aligned}
F^{(0)}(z) &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_nn!} z^n \\
F^{(1-c)}(z) &= z^{1-c} \sum_{n=0}^{\infty} \frac{(1-c+a)_n(1-c+b)_n}{(2-c)_nn!} z^n
\end{aligned}
\]

Both series converge in a punctured neighborhood of \( z = 0 \); \( F^{(0)}(z) \) is single-valued, whereas \( F^{(1-c)}(z) \) picks up a factor of \( \exp(2\pi i(1-c)) \) as \( z \) loops around the origin.

Since (4) is a degree two equation, it has a two dimensional space of solutions. Hence \( I(z) = AF^{(0)}(z) + BF^{(1-c)}(z) \) for some \( A, B \) wherever both sides are defined. In fact it is clear that \( B = 0 \), since by inspection \( I(z) \) has trivial monodromy around \( z = 0 \).
By expanding the term \((1 - \frac{zt}{c})^{-a}\) in the Euler integral (6) according to the binomial formula, we obtain an infinite sum of Beta integrals; a quick computation yields

\[
I(z) = \frac{\Gamma(b)\Gamma(b - c)}{\Gamma(c)} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n = \frac{\Gamma(b)\Gamma(b - c)}{\Gamma(c)} F^{(0)}(z).
\]

If we wish to find a second integral solution, we can try integrating along a line from 0 to \(\infty\) or 1 to \(\infty\). One must then check that the integral converges for the given values of \(a, b, c\). Alternatively, we can avoid issues of convergence by choosing closed integration cycles, so long as the restriction of the integrand to the cycle is single valued. These are called pochhammer cycles; they are spanned by the kernel of the abelianization map \(\pi_1(C - 0, 1, z^{-1}) \rightarrow H_1(C - 0, 1, z^{-1}, \mathbb{Z})\).

4. REVIEW OF \(\mathcal{D}\)-MODULES

We establish some basic tools to study the PDEs satisfied by our integrals. For further details, see eg \(\text{??}\). Let \(V\) be a vector space, with coordinates \(z_1, \ldots, z_n\). We write \(\mathcal{D} = \mathbb{C}\langle \partial_i, z_i \rangle\) for the ring\(^1\) of differential operators on \(V\), with relations

\[
[\partial_i, z_i] = 1, \quad [\partial_i, \partial_j] = [z_i, z_j] = 0.
\]

As vector spaces, we have

\[
\mathcal{D} = \bigoplus_{\alpha} \mathcal{O}(V) \partial^{\alpha}
\]

where \(\alpha = (\alpha_1, \ldots, \alpha_n)\) runs over all multi-indices. \(\mathcal{D}\) admits an increasing filtration \(F^i \subset \mathcal{D}\) by order, where \(F^i \subset \mathcal{D}\) is the subspace

\[
F^i = \bigoplus_{|\alpha| \leq i} \mathcal{O}(V) \partial^{\alpha}.
\]

Then \(\text{Gr}_F \mathcal{D} = \mathbb{C}[z_i, p_i] = \mathcal{O}[T^*V]\). There is a notion of good filtration on a \(\mathcal{D}\)-module, \(F^i \subset \mathcal{M}\), such that \(\text{Gr}_F \mathcal{M}\) defines a coherent \(\mathcal{O}[T^*V]\)-module. We do not give the definition here; it will be enough to know that any such filtration arises from a surjection

\[
\phi: \mathcal{D}^\oplus N \rightarrow \mathcal{M}
\]

\[
F^i \mathcal{M} := \phi \left( \bigoplus_{k=1}^{N} F^{i-n_k} \mathcal{D} \right)
\]

where \(n_k \in \mathbb{Z}\). We will only consider such \(\mathcal{D}\)-modules from now on.

**Definition 3.** The characteristic variety \(Ch(\mathcal{M}) \subset T^*V\) is the support of \(\text{Gr}_F \mathcal{M}\).

\(^1\)Since we work exclusively on vector spaces, we will avoid sheafy constructions whenever possible.
One easily sees that $Ch(\mathcal{M})$ is a conical subvariety of $T^*V$, i.e. it is stable under dilations of the cotangent fibers. The commutation relations in $\mathcal{D}$ show that if $m \in \mathcal{M}$ and $z_i(m) = \partial_i(m) = 0$, then $m = 0$. This is the mathematical origin of the Heisenberg uncertainty principle. For us, it has the following important consequence:

**Theorem 4** (Sato-Kawai-Kashiwara). $Ch(\mathcal{M})$ is an involutive\(^2\) subvariety of $T^*V$.

In particular, $\dim Ch(\mathcal{M}) \geq \dim V$.

**Definition 5.** $\mathcal{M}$ is a holonomic $\mathcal{D}$ module if $\dim Ch(\mathcal{M}) = \dim V$.

Thus for $\mathcal{M}$ holonomic, $Ch(\mathcal{M})$ is a union of conical lagrangian subvarieties of $T^*V$. By a general theorem, any smooth conical lagrangian is the conormal bundle of some subvariety of $V$; the components of $Ch(\mathcal{M})$ will be closures of such objects.

Let $C \subset Ch(\mathcal{M})$ be an irreducible component, and let $Gr_F \mathcal{M}_C$ be the localization of $Gr_F \mathcal{M}$ at the generic point of $C$. In other words, it is the pullback to $Gr \mathcal{M}$ to $\mathcal{O}[T^*V]_{p_C}$ where $p_C$ is the defining ideal of $C$. We define the multiplicity $m_C$ of $Gr_F \mathcal{M}_C$ along $C$ to be the length of $Gr_F \mathcal{M}_C$ as a module over $\mathcal{O}[T^*V]_{p_C}$.

**Definition 6.** We define the characteristic cycle by $CC(\mathcal{M}) := \sum m_C[C]$.

**Proposition 7.** Let $\mathcal{M} = \mathcal{D}/\mathfrak{I}$ where $\mathfrak{I} \subset \mathcal{D}$ is a left ideal. Then $Gr_F \mathcal{M} = Gr_F \mathcal{D}/Gr_F \mathfrak{I}$.

Given $P \in F^i\mathcal{D}$, $P \notin F^{i-1}\mathcal{D}$. Then the image of $P$ in $F^i\mathcal{D}/F^{i-1}\mathcal{D}$ is called the symbol of $P$. The previous proposition tell us that $Gr_F \mathcal{M}$ is determined by the ideal of symbols.

**Caveat.** The symbols of a set of generators of an ideal may not generate the ideal of symbols. For instance, $\mathcal{D}/\mathcal{D}(x, \partial) = 0$, but $\mathbb{C}[z, p]/(z, p) \neq 0$.

**Example.** Let $H$ be a complex torus and let $\mathfrak{h}$ be its Lie algebra. Any $\beta \in \mathfrak{h}$ defines a vector field $Z_\beta$ on $H$. Pick $\gamma \in \mathfrak{h}^*$. Then we can define the $\mathcal{D}$-module $\mathcal{L}(\gamma)$ on $H$ by

$$\mathcal{L}(\gamma) := \mathcal{D}/\mathcal{D}(Z_\beta - \beta(\gamma)).$$

It corresponds to the local system on $H$ with monodromy $\exp(2\pi i \gamma)$. $Gr_F \mathcal{L}(\gamma)$ is the quotient of $\mathbb{C}[z_i, p_i]$ by the ideal generated by

$$\sum \beta(i)z_ip_i$$

for $\beta \in \mathfrak{h}$. This cuts out the zero section in $T^*H$.

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\(^2\)A subspace $W \subset V$ of a symplectic vector space is involutive if it contains its orthogonal space $W^\perp$, i.e. the space of vectors pairing to zero with $W$ under the symplectic form.
Example. Let $H$ and $\gamma$ be as above, and let $H$ act on a torus $H'$. Let $O \subset V$ be a closed orbit of $H$. Let $I_O \subset O(H')$ be the ideal of functions vanishing on $O$, and let

$$M := \mathcal{D} / \mathcal{D} \cdot I_O + \mathcal{D} (Z_\beta - \beta(\gamma))$$

Then $Ch(M)$ is the conormal bundle of $O$ in $T^*H'$.

**Definition 8.** Let $M$ be a $\mathcal{D}$-module on a vector space $V$. The Fourier transform $\hat{M}$ is a $\mathcal{D}$-module on $V^*$ isomorphic to $M$ as a complex vector space, with action $\partial_x(m) := xm, u(m) := \partial_u m$ for $x \in V^*, u \in V$. In other words, differentiation and multiplication are interchanged, just as in the Fourier transform familiar from undergraduate analysis.

4.1. $\mathcal{D}$-modules and flat connections. Let $E$ be a vector bundle on $X$, $\mathcal{E}$ its sheaf of sections, and $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^1$ a $\mathbb{C}$-linear map satisfying the Leibniz rule $\nabla(fs) = df \cdot s + \nabla(s)$. We call $\nabla$ a connection, and say it is flat if $\nabla^2 = 0$. This makes $\mathcal{E}$ into a $\mathcal{D}$-module via $Z(s) := \nabla(s)(Z)$. It is not hard to see that $Ch(\mathcal{E}) = X \subset T^*X$.

5. The general Euler integral

Let $U$ be a manifold, and let $\mathcal{L}$ be a local system on $U$. We can define the group $H_n(Y, \mathcal{L})$ of $\mathcal{L}$-twisted $n$-dimensional cycles in $Y$. A representative $\sigma \in H_n(U, \mathcal{L})$ is a closed $n$-cycle carrying a flat section of $\mathcal{L}$.

Now suppose we have a fiber bundle $\pi : \mathcal{U} \rightarrow \mathcal{G}$, and a local system $\mathcal{L}$ on $\mathcal{U}$. Then the cohomology sheaf $R^n\pi_1\mathcal{L}$ with fibers $H^n_c(\mathcal{U}_s, \mathcal{L}) = H_n(\mathcal{U}_s, \mathcal{L})$ forms a vector bundle endowed with a flat connection over $\mathcal{G}$ called the Gauss-Manin connection.

In our setting, we begin with a collection of Laurent polynomials $P_i(t) : (\mathbb{C}^*)^k \rightarrow \mathbb{C}$, each given by a sum of monomials with exponents lying in some finite set $A_i \subset \mathbb{Z}^k$:

$$P_i(t) = P_i(z, t) = \sum_{\omega \in A_i} z_\omega t^\omega.$$

The space of such polynomials is parametrized by $\prod C^{A_i}$, with coordinates $z_\omega$. Let $Z(P_i)$ be the locus of zeros of $P_i(t)$, and $U_P = (\mathbb{C}^*)^k \setminus \bigcup_i Z(P_i)$. Then for $\alpha_i, \beta_j \in \mathbb{C}$,

$$P_i(t)^{\alpha_1} \cdots P_m(t)^{\alpha_m} t_1^{\beta_1} \cdots t_n^{\beta_k}$$

is a multivalued section of a rank one local system $\mathcal{L}(\alpha, \beta)$ on $U_P$. For any $\sigma \in H_n(U_P, \mathcal{L}(\alpha, \beta^*))$, define

$$F_\sigma := \oint_\sigma P_1(t)^{\alpha_1} \cdots P_m(t)^{\alpha_m} t_1^{\beta_1} \cdots t_n^{\beta_k} dt_1 \cdots dt_k$$

We now wish to vary the coefficients $z_\omega$ of $P_i(t)$. Let $\pi : \mathcal{U}_P \rightarrow C^{A_i}$ be the family with fiber above $z \in \prod C^{A_i}$ given by $(\mathbb{C}^*)^k \setminus \bigcup Z(P_i(z, t))$. $\mathcal{L}(\alpha, \beta)$ extends naturally to $\mathcal{U}_P$. There is an open
\( \mathcal{S} \subset \mathbb{C}^{A_i} \) on which \( H_n(U_P, L(\alpha, \beta)^\ast) \) forms a vector bundle with flat connection; let \( V \subset \mathcal{S} \), and let \( \sigma \) be a flat section of \( R^n\pi_!L \) over \( V \). This defines a function \( F_\sigma(z) \) on \( V \).

We will define a system of differential equations on \( \mathbb{C}^A \), of which \( F_\sigma(z) \) is a solution where defined. We will then forget about \( F \) for a while and study this system on its own terms. We will eventually see that for generic choices, \( F_\sigma(z) \) for different \( \sigma \) give a full set of solutions.

6. Defining the General GKZ Hypergeometric System

Let \( A \subset \mathbb{Z}^m \times \mathbb{Z}^k = \mathbb{Z}^n \) be a generating set, satisfying

- \( A \) generates \( \mathbb{Z}^n \).
- There is a linear map \( h : \mathbb{Z}^n \rightarrow \mathbb{Z} \) we have \( h(\omega) = 1 \) for all \( \omega \in A \).

Let \( \gamma \in \mathbb{C}^n \). Let \( \mathbb{L} \subset \mathbb{Z}^A \) be the lattice of relations among the \( \omega \), so that we have the exact sequence

\[
0 \rightarrow \mathbb{L} \rightarrow \mathbb{Z}^A \rightarrow \mathbb{Z}^n \rightarrow 0.
\]

Given \( l \in \mathbb{L} \), let \( l^+ \subset A \) be support of the positive coefficients, and \( l^- \subset A \) be the support of the negative coefficients. For \( l \in \mathbb{L} \), define

\[
\Box_l := \prod_{l^+} \partial^l_{\omega} - \prod_{l^-} \partial^l_{\omega}.
\]

Let \( H = \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{C}^* \) and let \( \mathfrak{h} \) be its Lie algebra. For \( x \in \mathfrak{h} \), define

\[
Z_x := \sum_\omega x(\omega)z_\omega \partial_\omega.
\]

Let \( \gamma \in \mathfrak{h} \).

Definition 9.

\[
\mathcal{M}(\gamma) := D/\Box_l + D(Z_x - x(\gamma)).
\]

Remark. The second equations have a simple interpretation. The map \( \mathbb{Z}^A \rightarrow \mathbb{Z}^n \) defines an action of \( H \) on \( \mathbb{C}^A \), and the vector field \( Z_\beta \) is the image of \( \beta \in H \). Hence the equations involving \( Z_\beta \) are equivalent to the homogeneity of a solution with respect to \( H \).

Now suppose \( A_1, \ldots, A_m \subset \mathbb{Z}^k \) generate \( \mathbb{Z}^k \), and further suppose each \( A_i \) contains 0. Let \( A \subset \mathbb{Z}^m \times \mathbb{Z}^k \) be the union \( \bigcup_i e_i \times A_i \) where \( e_i \) are a basis of \( \mathbb{Z}^m \). Then \( A \) generates \( \mathbb{Z}^m \times \mathbb{Z}^k \), and we have

Proposition 10. The integral in (6) is a solution of the GKZ equations (9), with \( \alpha_j = \gamma_j \), \( \beta_i = -\gamma_{m+i} + 1 \).
Proof. We write \( \int \phi(\alpha_i, \beta_j) \, dt \) for the integral in (6), where \( \phi(\alpha_i, \beta_j) = P_1(t)^{\alpha_1} \cdots P_m(t)^{\alpha_m} t_1^{\beta_1} t_2^{\beta_2} \cdots t_k^{\beta_k} \).

The equations involving \( \Box_l \) are satisfied by the integrand \( \phi(\alpha_i, \beta_j) \) itself. To see this, let \( \omega = e_i \times \omega' \in A \). We have the equality

\[
\partial_\omega \phi(\alpha_i, \beta_j) = \alpha_i \phi(\alpha_i - 1, \beta + \omega').
\]

But since \( L \) is the kernel of the map \( Z^A \rightarrow Z^m \times Z^k \), by definition for any \( l \in L \) we have

\[
\sum_{l^+} l_\omega \omega + \sum_{l^-} l_\omega \omega = 0.
\]

Hence

\[
\prod_{l^+} \partial_{\omega}^{l^+} \phi(\alpha_i, \beta_j) = \prod_{l^-} \partial_{\omega}^{l^-} \phi(\alpha_i, \beta_j)
\]

or equivalently \( \Box_l \phi(\alpha_i, \beta_j) = 0 \). The relations involving \( Z_x \) are straightforward, and we leave them as an exercise.

How strong a condition is this? Pretty strong: near generic \( z \), there is a finite dimensional space of solutions, as we show in the next section.

Example. The Gauss hypergeometric function: Let \( A_1 = A_2 = \{0, 1\} \subset \mathbb{Z}^1 \). Then \( A \subset \mathbb{Z}^2 \times \mathbb{Z} \) is given by the four vertices of a unit square: \((1, 0, 0), (1, 0, 1), (0, 1, 0), (0, 1, 1)\). Write \( w_1, z_1, w_2, z_2 \) for the corresponding coordinates. We have

\[
0 \rightarrow \mathbb{L} \rightarrow \mathbb{Z}^4 \rightarrow \mathbb{Z}^3 \rightarrow 0
\]

where \( \mathbb{L} \) is spanned by \((1, -1, -1, 1)\). Hence we recover the Gauss hypergeometric equations.

7. Holonomicity, Fourier Transform and Singular Locus

Proposition 11. \( \mathcal{M} \) is a holonomic \( \mathcal{D} \)-module.

We prove this below. Consider the subvariety \( Y_A \subset \mathbb{P}(\mathcal{C}^A) \) cut out by the Fourier transforms of (7)

\[
\hat{\Box}_l = \prod_{l^+} \zeta_{\omega}^{l^+} - \prod_{l^-} \zeta_{\omega}^{l^-}.
\]

By assumption, we have a linear \( h : \mathbb{Z}^n \rightarrow \mathbb{Z} \) with \( h(A) = 1 \). It follows that \( \Box_l \) is homogenous for all \( l \) and \( Y_A \) is the cone of a projective variety \( X_A \subset \mathbb{P}(\mathcal{C}^A) \). Since we assume \( A \) generates the lattice of characters of \( H \), it also follows that \( \iota : H \rightarrow H \cdot 1, t \mapsto (t^\omega)_{\omega \in A} \) is an isomorphism.

Proposition 12. \( Y_A \subset \mathbb{C}^A \) is the closure of the orbit \( H \cdot 1 \subset \mathbb{C}^A \).

Proof. This We have an embedding \( R[\mathcal{C}^A] \hookrightarrow R[(\mathbb{C}^*)^A] \). It is easy to see that the ideal of \( H \cdot 1 \subset (\mathbb{C}^*)^A \) is spanned by \( \{\chi - 1 : \chi \in R[(\mathbb{C}^*)^A]^H\} \). Such \( \chi \) are multiplicatively generated by

\[
\chi_t := \prod_{l^+} \zeta_{\omega}^{l^+} / \prod_{l^-} \zeta_{\omega}^{l^-}.
\]
Thus the relations $\widehat{\square}_i$ generate $R[\mathbb{C}^A]^H = R[(\mathbb{C}^*)^A]^H \cap R[\mathbb{C}^A]$.  

Clearly, $\widehat{\mathcal{M}}$ is supported on $Y_A$. We can say more:

**Proposition 13.** Let $\Box, Z$ be the symbols of $\Box, Z$.

$$Gr_{\mathcal{F}}\mathcal{M} = \mathbb{C}[z_i, p_i]/(\Box, Z)$$

**Proof.** First we must show that the $Z_i$ form a regular sequence in $\mathcal{D}/\Box$. Then the Koszul complex

$$(10) \quad \ldots \rightarrow \mathcal{D}/\Box \otimes \bigwedge^i \mathfrak{h} \rightarrow \mathcal{D}/\Box \otimes \bigwedge^{i-1} \mathfrak{h} \rightarrow \ldots \rightarrow \mathcal{D}/\Box \rightarrow \mathcal{D}/\Box, Z \rightarrow 0$$

is exact. We define a non-commutative analog

$$(11) \quad \ldots \rightarrow \mathcal{D}/\Box \otimes \bigwedge^i \mathfrak{h} \rightarrow \mathcal{D}/\Box \otimes \bigwedge^{i-1} \mathfrak{h} \rightarrow \ldots \rightarrow \mathcal{D}/\Box \rightarrow \mathcal{D}/\Box, Z \rightarrow 0$$

where $d : \mathcal{D}/\Box \otimes \bigwedge^i \mathfrak{h} \rightarrow \mathcal{D}/\Box \otimes \bigwedge^{i-1} \mathfrak{h}$ is defined by sending $P \wedge e_i$ to $PZ_i$. This is well-defined since $[Z_i, Z_j] = 0, [Z_i, \Box] = c_i \Box$ with $c_{ij}$ constant. One shows that (11) is exact by comparing with (10). Taking the associated graded complex, and noting that $Gr \mathcal{D}/\Box = \mathcal{D}/\Box$, we obtain the result.  

**Proof of proposition 11.** We have $Ch(\mathcal{M}) \subset T^*V$. We claim that under the natural identification of $T^*V$ with $T^*V^*$, $Ch(\mathcal{M})$ is contained in the union of conormal bundles to $H$-orbits in $Y_A$. Indeed, given any $(v, w) \in Ch(\mathcal{M})$, let $X(w)$ be the $H$-orbit of $w$. Then $T_wX(w)$ is spanned by the image of $\mathfrak{h}$. But given $\zeta \in \mathfrak{h}$, we have $\langle v, \zeta \rangle = \langle Z_\zeta(v, w) \rangle = 0$.

Since $Y_A$ contains finitely many $H$-orbits, $Ch(\mathcal{M})$ is contained in finitely many lagrangians, hence $\mathcal{M}$ is holonomic.

We investigate the structure of $Y_A$ and $X_A$.

**Proposition 14.** The orbits of $H$ in $X_A$ are indexed by the faces $\Gamma \subset Q$, where $Q$ is the convex hull of $A \subset \mathbb{R}^n$.

**Proof.** An orbit $H \cdot v$ is contained in $X_A = \overline{H \cdot 1}$ iff there is a map from the punctured disk $\sigma : \mathbb{D}^* \rightarrow H \cdot 1 = (\mathbb{C}^*)^n$ whose completion maps 0 into $H \cdot v$. Factor $\sigma$ as $\mathbb{D}^* \overset{\tilde{\sigma}}{\twoheadrightarrow} H \overset{\iota}{\rightarrow} H \cdot 1$, and write

$$\tilde{\sigma}(z) = (c_1z^{a_1} + \ldots, c_nz^{a_n} + \ldots)$$

where the . . . indicate higher order terms. The $a_i$ determine a cocharacter $a : \mathbb{C}^* \rightarrow H$ whereas $c_i$ determines an element $c \in H$. Let $\Gamma \subset Q$ be the facet of $Q$ along which $\langle a, \omega \rangle$ is maximal. Then $\lim_{z \rightarrow 0} \sigma(z)$ is the point $c \cdot e_\Gamma \in \mathbb{C}^A$ with coordinates $z_\omega = c^\omega$ for $\omega \in \Gamma, z_\omega = 0$ otherwise.
Write \( X_\Gamma \subset X_A \) for the orbit indexed by \( \Gamma \). We describe the singular locus of \( M \), i.e. the image of \( Ch(M) \setminus C^A \) under the projection to \( C^A \). It should be related to the locus \( z_\omega \) where the topology of \( U_P = (C^*)^k \setminus Z(P_i(t)) \) changes.

**Definition 15.** Let \( S \subset \mathbb{P}^{n-1} \) be a variety. The projective dual \( S^\vee \subset \mathbb{P}^{n-1} \) parametrizes hyperplanes tangent to \( S \).

**Example.** The projective dual of a nondegenerate quadric \( a_{ij}x_ix_j = 0 \) in \( \mathbb{P}^n \) is given by the inverse quadric \( a_{ij}^{-1}x_ix_j \) in \( \mathbb{P}^n \), identified with its dual via the bilinear form. The dual of a degenerate quadric is defined the same way, on a linear subspace of the dual \( \mathbb{P}^n \) orthogonal to the kernel of the form. The dual of the space of \( m \times n \) matrices of rank \( r \) is the space of \( m \times n \) matrices of rank \( m - r \).

**Proposition 16.** The projectivization of the singular locus \( \mathbb{P}(\nabla A) \) of \( M \) is the union of the projective duals of \( X_\Gamma \).

**Proof.** \( Ch(M) \) is the union of conormal bundles to the \( X_\Gamma \). But \( Ch(M) \) is identified with \( Ch(M) \) under the natural isomorphism \( T^*V \cong T^*V^* \). Under this identification, a point \( v \in V \) lies in the image of the conormal to \( X_\Gamma \subset V^* \) iff there exists \( x \in X_\Gamma \) such that \( v(u) = 0 \) for all \( u \in T_xX_\Gamma \). In other words, the hyperplane in \( V^* \) defined by \( u \) is tangent to \( X_\Gamma \).

Let \( \mathcal{S} = \mathbb{C}^{|A|} \setminus \nabla_A \). Recalling the map \( \pi : \mathcal{U}_P \rightarrow \mathcal{S} \) and the local system \( \mathcal{L}(\alpha, \beta) \), we can define a map of constructible sheaves

\[
\Xi : R^n\pi_*\mathcal{L}(\alpha, \beta) \rightarrow \text{Hom}(\mathcal{M}(\gamma), \mathcal{O})
\]

over \( \mathcal{S} \), taking a family of cycles to the corresponding family of integrals.

### 8. Constructing solutions at infinity

We want to know the number of local solutions over the regular locus. We begin by producing a family of very explicit solutions; we will show later that they span all solutions. Recall that \( \mathcal{M} \) depends on a choice of \( \gamma \in \mathfrak{h} \). Write \( \mu : \mathbb{C}^A \rightarrow \mathfrak{h} \) for the map induced by \( \mathbb{Z}^A \rightarrow \mathbb{Z}^n \).

**Definition 17.** Let \( \tilde{\gamma} \in \mu^{-1}(\gamma) \). Define

\[
\Psi(\tilde{\gamma}) = \sum_{l \in L} \frac{z^{\tilde{\gamma}_\omega + l\omega - 1}}{\prod_{\omega} \Gamma(\tilde{\gamma}_\omega + l\omega - 1)}.
\]

**Proposition 18.** \( \Psi(\tilde{\gamma}) \) is a formal solution of (9).

**Proof.** For \( s \in \mathbb{C}^n \), let \( \Gamma(s) := \prod_{\omega} \Gamma(s_\omega) \). We can rewrite

\[
\Psi(\tilde{\gamma}) = \sum_{s \in L + \tilde{\gamma}} \frac{z^s}{\Gamma(s)}.
\]
Note that
\[ \partial \omega \frac{z^s}{\Gamma(s)} = \frac{z^{s-1}}{\Gamma(s-1)}. \]
Thus we have
\[ \square \Psi(\tilde{\gamma}) = \prod_{\omega \in l^+} \partial \omega \Psi(\tilde{\gamma}) - \prod_{\omega \in l^-} \partial \omega \Psi(\tilde{\gamma}) = \sum_{s \in \mathbb{L} + \tilde{\gamma} - l^+} \frac{z^s}{\Gamma(s)} - \sum_{s \in \mathbb{L} + \tilde{\gamma} - l^-} \frac{z^s}{\Gamma(s)}. \]
But \( l^+ - l^- = l \in \mathbb{L} \), hence the two sums cancel. The equations involving \( Z_x \) are simply conditions on the exponent of \( z \), and are satisfied by each term individually.

For generic \( \tilde{\gamma} \), this series involves unbounded powers of \( z \) in all directions, hence it is typically non-convergent. However, by picking \( \tilde{\gamma} \) appropriately, we may obtain vanishing of coefficients outside of a cone.

**Definition 19.** A base \( I \subset A \) is a subset forming a basis of \( \mathbb{R}^n \).

Let \( I \subset A \) be a base, and let \( \Pi(I, \tilde{\gamma}) = \{ \tilde{\gamma} \in \mu^{-1}(\gamma) : \tilde{\gamma}_i \in \mathbb{Z} \text{ for } i \notin I \} \). \( \Pi(I, \tilde{\gamma}) \) is clearly stable under translations by \( \mathbb{L} \subset \mathbb{Z}^n \).

**Proposition 20.** Let \( \tilde{\gamma} \in \Pi(I, \tilde{\gamma}) \). Then \( \psi(\tilde{\gamma}) \) is a product of a monomial and a power series. For \( z \in \mathbb{C}^A \), let \( \log |z| := (\log |z_1|, \ldots, \log |z_n|) \in \mathbb{R}^A \). Let \( C(I) \subset \mathbb{L} \subset \mathbb{Z}^A \) be the cone \( l \in \mathbb{L} : l_\omega \geq 0 \text{ for } \omega \notin I \). Let \( C(I) \subset \mathbb{R}^A \) be the dual cone. This series converges for \( \log |z| \in U \) where \( U \subset C(I) \) is a translate of \( C(I) \).

*Proof.* Straightforward; apply the ratio test to the series solution.

**Definition 21.** Let \( K \) be a real vector space and \( \mathbb{K} \subset K \) a lattice. Then we define translation invariant measure on \( K \) by setting the volume of an elementary simplex to be one.

**Proposition 22.**
\[ |\Pi(I, \tilde{\gamma})| = \text{vol } I \]
where \( \text{vol } I \) is the volume of the convex hull of \( I \).

**Definition 23.** A triangulation \( T \) of \( Q, A \) is a decomposition of \( Q \) into simplices with vertices in \( A \). A triangulation is regular (coherent) if it arises from a convex function.

**Proposition 24.** The solutions attached to the simplices of a regular triangulation have a common domain of convergence.

**Proposition 25.** Regular triangulations always exist.

*Proof.* Let \( \Phi : A \to \mathbb{R} \) be a function. Consider the half-lines \( \{(\omega, x) \in \mathbb{R}^n \times \mathbb{R} : x \leq \Phi(\omega)\} \). Their convex hull \( G_\Phi \) is an unbounded polyhedron in \( \mathbb{R}^n \times \mathbb{R} \). Its projection to \( \mathbb{R}^n \) may or may not define a triangulation; we claim that it does for generic \( \Phi \). Indeed, suppose some face \( \Gamma \subset G_\Phi \) is not a
simplex, i.e. contains more than $n$ points. The locus of $\Phi$ for which this holds is a codimension one subvariety of $\mathbb{R}^A$; since there are finitely many $\Gamma \subset G_{\Phi}$, we are done.

**Definition 26.** $\gamma \in \mathfrak{h}$ is non-resonant if for all proper faces $\Gamma \subset Q$, we have $\gamma \notin \text{Lin}_{\mathbb{R}}(\Gamma) + \mathbb{Z}^n$.

**Proposition 27.** For $\gamma$ non-resonant, the solutions $\phi(I, \widehat{\gamma})$ attached to a coherent triangulation of $Q$ are linearly independent.

It follows that the rank of $\mathcal{M}$ along its regular locus is at least the number of simplices in a regular triangulation of $Q$. In fact we have:

**Proposition 28.** Suppose $A$ admits a unimodular triangulation. The restriction of $R\text{Hom}(\mathcal{M}, \mathcal{O})$ to $(\mathbb{C}^*)^A \setminus \nabla_A$ is a local system of rank $\text{vol}(Q)$.

More generally, we compute the number of ‘microsolutions’ over various components of $\nabla_A$. Let $H(\Gamma)$ be the torus attached to the lattice $\mathbb{Z}^n / \Gamma \cap A$. Let $P(\Gamma)$, $Q(\Gamma)$ be the projections of the convex hulls of $A$, $A \setminus \Gamma$ respectively. Then

**Proposition 29.** The multiplicity of $CC(\mathcal{M})$ along $X_\Gamma^r$ is $\text{vol}(P(\Gamma)) - \text{vol}(Q(\Gamma))$.

**Proof.** Any component $L \subset Ch(\mathcal{M})$ is the closure of the conormal bundle $N_S V \subset T^*V$ to some $S \subset V$. The multiplicity of $L$ in $CC(\mathcal{M})$ can be computed by picking a half-dimensional line $K \subset T^*V$ intersecting $L$ in a single point of its open locus, and taking the length of the scheme-theoretic intersection $K \cap CC(\mathcal{M})$.

This is the local degree of the scheme $X_\Gamma \subset X$. By a theorem of [...] it is given by $\text{vol}(P(\Gamma)) - \text{vol}(Q(\Gamma))$. □

**Proposition 30.** The generic normal locus to $X_\Gamma$ is given by $i(\Gamma)$ branches, toric wrt $H(\Gamma)$.

For each regular triangulation $T$ of $Q$, $A$, we have a basis of solutions analytic for all $\log(z) \in U(T) \subset \mathbb{R}^A$, corresponding to points $z$ sufficiently deep inside a cone $C(T)$. Such cones are obviously stable under $H$; we consider their projections $N(T) \subset \mathbb{R}^d = \mathbb{L} \otimes \mathbb{R}$.

**Proposition 31.** The cones $N(T)$ tile $\mathbb{R}^d$.

**Proof.** A generic $\Phi \in \mathbb{R}^n$ is regular. □

The dual polytope to the fan composed of the $C(T)$ is called the secondary polytope $\Sigma(A)$ of $A$.

We pause to recall our original function $P(t)$. $\Sigma(A)$ has a prime factorization whose factors are the discriminants of functions attached to faces; in particular, one can recover (the newton polytope of) the discriminant of $P(t)$. 
9. Irreducibility of \( \mathcal{M} \)

Recall:

**Definition 32.** \( \gamma \) is said to be *non-resonant* if for all proper faces \( \Gamma \subset Q \), we have \( \gamma \notin \mathbb{Z}^n + \mathbb{C}(\Gamma) \).

As we shall see, this is equivalent to the local systems on \( X_A \) appearing in the Fourier transform of \( \mathcal{M} \) having non-trivial monodromy around all codimension-one strata.

**Proposition 33.** Let \( \gamma \) be a non-resonant parameter. Then the monodromy representation of \( \mathcal{M}(\gamma) \) along \( \mathcal{S} \) is irreducible.

**Proof.** In short, one shows that the Fourier transform is an irreducible perverse sheaf; since the Fourier transform is an equivalence of abelian categories, \( \mathcal{M} \) must also be irreducible as a perverse sheaf. But the structure theory of perverse sheaves then tells us that \( \mathcal{M} \) must restrict to an irreducible local system along its open stratum. \( \square \)

**Theorem 34.** For non-resonant parameters, there is a full collection of cycles, i.e.

\[
\mathcal{M}(\gamma) = R^n \pi_! \mathcal{L}(\gamma)
\]

**Theorem 35.** For non-resonant \( \gamma \) the natural map

\[
R\pi_! \mathcal{L}(\gamma) \rightarrow R\pi_* \mathcal{L}(\gamma)
\]

is an isomorphism, and \( R\phi_* \mathcal{L}(\gamma) \) is an irreducible perverse sheaf.

**Remark.** For an introduction to perverse sheaves, see for instance [4].

From now on, we assume \( \gamma \) is non-resonant. We begin by showing the following:

**Proposition 36.** Let \( X^0_A \) be the open orbit, and let \( \mathcal{K}(-\gamma) \) be the rank one local system on \( X^0_A \cong H \) with monodromy \( \exp(-\gamma) \). Let \( j : X^0_A \rightarrow X \) be the inclusion. Then \( \gamma \) is non-resonant iff

\[
c : j_! \mathcal{K}(-\gamma) \rightarrow Rj_* \mathcal{K}(-\gamma)
\]

is an isomorphism.

**Proof.** Follows from a local description of \( X_\Gamma \subset X \). \( \square \)

**Proposition 37.** \( j_! \mathcal{K}(-\gamma) \) is an irreducible perverse sheaf.

**Proof.** \( \mathcal{K}(-\gamma) \) is of course irreducible. Thus so is \( j_* j_! \mathcal{K}(-\gamma) \), which is the image of \( j_! \mathcal{K}(-\gamma) \) in \( Rj_* \mathcal{K}(-\gamma) \). But since the former is irreducible, and the map is nonzero, it must be isomorphic to its image. \( \square \)
Let $\mathcal{E}$ be a conical complex of constructible sheaves on a vector space $V$. Here conical means it is smooth along orbits of the dilating $\mathbb{C}^*$-action. Let $V \xrightarrow{p_1} T^*V \xrightarrow{p_2} V^*$ be the two projections, and let $G \subset T^*V, G = \{(x, y) : \langle x, y \rangle \geq 0\}$. Then we define the Fourier transform

**Definition 38.**

$$\mathcal{F}(\mathcal{E}) := Rp_2_* R\Gamma_G(p_1^*\mathcal{E}).$$

$\mathcal{F}(\mathcal{E})$ is also conical, and $\mathcal{F}$ defines an equivalence of abelian categories between conical perverse sheaves on $V$ and $V^*$.

**Proposition 39.** We have an isomorphism $R\pi_* \mathcal{L}(\gamma) \cong \mathcal{F}(j_! K(-\gamma))$.

The key point is that for $z \in V, (y, t) \in X_A^0 \subset V^*$, we have $\langle z, t \rangle = \sum_i y_i \sum \omega z \omega t^\omega = \sum_i y_i P_i(t)$ where $(y, t) : H \to H \cdot 1 \subset V^*$ is the natural parametrization. Thus $G \cap V \times X^0$ is the locus $\{z, y, t : \sum_i y_i P_i(t) \geq 0\}$.

[to be completed]

10. **The Case of a Product of Linear Factors**

Now suppose each $P_i(t)$ is the defining equation of a hyperplane $H_i \subset \mathbb{C}^n$, i.e. $P_i(t) = (\sum_{j=1}^n z_{ij} t_j)$. Write $\mathcal{C} = \{H\}$ for the collection of such hyperplanes; we call it a ‘hyperplane arrangement’.

Write $U = \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{C}} H$. We will assume for simplicity that each hyperplane $H$ is defined by an equation with real coefficients.

In this case we have an explicit description of the cohomologies $H^\bullet(\mathbb{C}^n \setminus H_i, \mathbb{Z})$ and more generally $H^\bullet(\mathbb{C}^n \setminus H_i, \mathcal{L}(\alpha))$. For $H \in \mathcal{C}$ and $P_H(t)$ a defining polynomial, we have the element $\omega_H = \frac{dP_H(t)}{P_H(t)} \in \Omega_1(U)$. We will see that these generate the cohomology, subject to the ‘Orlik-Solomon’ relations.

**Definition 40.** We say a subset $S \subset \mathcal{C}$ is in general position if $\bigcap_{H \in S} H$ has codimension $|S|$.

**Definition 41.** The Orlik-Solomon algebra $\mathscr{A}^\bullet$ has (anticommuting) generators $\omega_H, H \in \mathcal{C}$ and relations as follows. Pick an ordering $H_1, \ldots, H_n$ of $\mathcal{C}$.

$$\prod_{H \in S} \omega_H = 0$$

and

$$\sum_{i \in S} (-1)^i \prod_{H \in S, H \neq H_i} \omega_H = 0$$

for all $S \subset \mathcal{C}$ not in general position.

**Theorem 42 (OS).** The assignment $\omega_H = \frac{dP_H(t)}{P_H(t)}$ defines an isomorphism

$$\mathscr{A}^\bullet \cong H^\bullet(U, \mathbb{Z}).$$
Suppose instead we want the cohomology with coefficients in a local system $\mathcal{L}(\alpha)$. Define $d(\alpha): \mathcal{A}^\bullet \to \mathcal{A}^{\bullet+1}$ as multiplication by $\omega(\alpha) = \sum_{H \in \mathcal{C}} \alpha(H)H$.

**Proposition 43.** For generic $\alpha$, we have
\[
H^\bullet(\mathcal{A}^\bullet, d(\alpha)) \cong H^\bullet(U, \mathcal{L}(\alpha)).
\]

**REFERENCES**


