EISENSTEIN ORIENTATION: CORE DUMP

MARK BEHRENS

This informal note is meant as a followup to my previous status report [Beh] to my co-conspirators Mike Hopkins and Niko Naumann, with the goal of constructing a generalization of the $\hat{A}$ and Witten orientations to TAF. The associated Hirzebruch series is supposed to be expressed in terms of Eisenstein series on Shimura varieties. The present note describes an *unsuccessful* attempt to produce an orientation of $\mathrm{TAF}_{E(1)}$.

The material in [Beh] is somewhat complementary to the material here, but the basic strategy (pulling back Eisenstein series a la Shimura) is the same. The optimistic tone of [Beh] must be tempered by the fact that the very last equation there is wrong - I misunderstood a coset result of Shimura's, that brings in a constant of proportionality expressible as a product of L-functions. The whole business becomes more complicated, and I can't get it to come together. Also, when I wrote the notes [Beh], I thought this would solve the problem, as Harris-Li-Skinner [HLS06] constructed Eisenstein measure for quasi-split unitary groups. Now that I actually have *read* Harris-Li-Skinner, I realize that their Eisenstein measure is "perpendicular" to the one we want: they give a p-adic interpolation based on p-adic interpolation of the associated Hecke characters, where they insist that the Hecke characters vary in p-adic families with the factor at ∞ fixed! We want to *only* vary the factors at ∞, so their work is largely inapplicable to our problem. More on this at the end of the introduction.

Fix

$$\begin{aligned} F &= \text{quadratic imaginary extension of } \mathbb{Q} \\ p &= \text{prime which splits as } u\bar{u} \text{ in } F \\ V &= F \text{ vector space of dimension } n \\ \langle -, - \rangle &= \text{alternating hermitian form on } V \text{ of signature } (1, n-1) \end{aligned}$$

Let U_V and GU_V be the associated unitary and unitary similitude groups. Let $K^{p,\infty}$ be a maximal compact open in $GU(\mathbb{A}^{p,\infty})$, and let $\mathrm{TAF}_{U_V}(K^{p,\infty})$ be the associated p-complete spectrum of topological automorphic forms.

We seek an E_∞ orientation

$$MO\langle 4n \rangle \to \mathrm{TAF}_{E(1)}.$$

By Ando-Hopkins-Rezk, we know this is provided by a sequence

$$b_{2k} \in \pi_{4k}\mathrm{TAF}_{\mathbb{Q}}, \quad k \geq n$$

Date: October 3, 2009.

satisfying

(1) We have
$$b_{2k} \equiv b_{2k}^{Miller} \mod \mathbb{Z}$$
where $b_{2k}^{Miller} \in \pi_{2k}\mathrm{TAF}_{\mathbb{Q}/\mathbb{Z}}$ are Miller's universal Bernoulli numbers.

(2) For c a topological generator of $\mathbb{Z}_p^\times$, there exists a $\mathbb{V}$-valued measure $\mu^{(c)}$ on $\mathbb{Z}_p^\times$ whose moments are given by
$$\int_{\mathbb{Z}_p^\times} x^{k-1} d\mu^{(c)} = (c^k - 1)b_k^*, \qquad k \geq 2n.$$
Here, $b_k = 0$ for k odd, $\mathbb{V} = K_0^\wedge(\mathrm{TAF})$ is the Katz-Hida space of p-adic automorphic forms, and $b_k^* = \log_1(b_k)$, where $\log_1$ is Rezk's $K(1)$-local logarithm.

Since $\pi_{2k}\mathrm{TAF}_{\mathbb{Q}}$ is the space of rational holomorphic automorphic forms of scaler weight k for GU_V, the b_k may be regarded as automorphic forms. Our candidate is
$$b_k = \mathbb{E}_k$$
where $\mathbb{E}_k$ are suitably normalized holomorphic Eisenstein series on GU.

Throughout, we write
$$V = V_0 \oplus V_1 \oplus \bar{V}_0.$$
Here, V_1 is anisotropic, V_0 is totally isotropic, and $\bar{V}_0$ is dual to V_0:
$$\dim_F V_0 = \dim_F \bar{V}_0 = 1$$
$$\dim_F V_1 = n - 2$$

Our strategy is as follows:

(1) reduce the problem to finding a measure $\mu_1^{(\ell)}$ on $\mathbb{Z}_p$ with moments $(\ell^k - 1)\mathbb{E}_k$. The desired measure $\mu^{(c)}$ is obtained by restricting $\mu_1^{(\ell)}$ to $\mathbb{Z}_p^\times \subset \mathbb{Z}_p$.

(2) Define a Hermitian F-vector space
$$W = V_0 \oplus V_1 \oplus \bar{V}_0 \oplus -V_1.$$
Then W has signature $(n-1, n-1)$, and a corresponding decomposition
$$W = W_0 \oplus \bar{W}_0.$$
Use Shimura's computation of the Fourier expansion of certain Eisenstein series $E_k'(z;\ell)$ on U_W to argue that a suitable normalization $\mathbb{E}_k'(z;\ell)$ p-adically interpolates to a measure $\nu^{(\ell)}$ on $\mathbb{Z}_p$.

(3) Under the canonical inclusion $U_V \hookrightarrow U_W$, the measure $\nu^{(\ell)}$ pulls back to a measure $\mu_2^{(\ell)}$ with moments
$$\int_{\mathbb{Z}_p} x^{k-1} d\mu_2^{(\ell)} = c_k(\ell^k - 1)\mathbb{E}_k.$$
Then, analyzing the p-adic properties of c_k, deduce the existence of the desired measure $\mu_1^{(\ell)}$

(4) Determine the agreement with the Miller invariants by analyzing the values of the $\mathbb{E}_k$ on certain CM points. This essentially involves pullback under the composite

$$U_{V_1} \hookrightarrow U_V \hookrightarrow U_W.$$

WARNING: some of the definitions/statements which follow may be slightly incorrect, missing minor normalization factors, etc to not get me bogged down. More seriously I will utterly ignore the possibility of *any* class numbers being larger than 1, because they will add distracting details. These "moral" statements are just meant to illustrate where it all falls apart for me. In particular, adelic language is suppressed, and thus some statements are (deliberately) inaccurate. They are accurate enough to sketch the main points.

Other constructions of Eisenstein measures in the literature. I know of several different (recent) papers by number theorists related to the subject matter of this note:

Harris-Li-Skinner [HLS06]**:** Harris-Li-Skinner construct an Eisenstein measure on $U(m, m)$. Originally, I envisioned we could pull-back their measure to the desired measure on $U(1, n-1)$. However, the moments of their measure are all automorphic forms of a fixed weight k, but "change level". We want a fixed level, and varying weights. The paper is still a great resource for p-adic automorphic forms, automorphic induction, integrality of Eisenstein series, etc... In fact, the paper uses Shimura's computation of the Fourier coefficients for unitary Eisenstein series [Shi97] in an essential way, as well as Hida's generalized "q-expansion principle".

Panchishkin [Pan00]**:** Panchishkin constructs an Eisenstein measure for $Sp(m)$. This is closely related, and again uses Shimura's Fourier expansion computations [Shi83]. The moments are of varying weights, which make this work closely related to what we want. The work pre-dates Hida's work on p-adic automorphic forms, an therefore the "measure" that is constructed seems to live in a more nebulous space.

Hsieh [Hsib]**,** [Hsia]**:** Ming-Luh Hsieh (student of Eric Urban) seems to have the most relevant work, as is represented in some VERY recent preprints. The author constructs Eisenstein measures on $U(1, 2)$ and $U(1, 3)$. In fact, the method this author uses is *exactly* the Shimura pullback method that I've been advocating and describe more fully in this note. Clearly, these papers deserve close scrutiny by anyone wishing to do something for $U(1, n-1)$. But I grow tired and impatient, and have not been able to muster the effort of reading the combined 100 pages of preprints on the subject Hsieh has produced.

1. Hermitian symmetric domains, factors of automorphy, and some Eisenstein series

1.1. **The case of U_V.** Under the decomposition

$$V = V_0 \oplus V_1 \oplus \bar{V}_0$$

we write matrices $g \in U_V$ as

$$g = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Then we have a symmetric hermitian domain given by

$$\mathcal{H}_V = \{(z, w) \in \mathbb{C} \times \mathbb{C}^{n-2} \, : \, i(z^* - z) > \langle y, y \rangle_{V_1}\}$$

with action

$$g \cdot (z, w) = \left(\frac{a_1 z + b_1 w + c_1}{a_3 z + b_3 w + c_3}, \frac{a_2 z + b_2 w + c_2}{a_3 z + b_3 w + c_3} \right).$$

For a weight k we define a factor of automorphy

$$J_k(g, z, w) = (a_3 z + b_3 w + c_3)^{-k}.$$

We fix lattices L_i of V_i, and take $\bar{L}_0$ to be the lattice dual to L_0. Thus we have a lattice

$$L = L_0 \oplus L_1 \oplus \bar{L}_0$$

in V. We define a "level 1" congruence subgroup

$$\Gamma = \{g \in U_V(\mathbb{Q}) \, : \, gL = L\}.$$

More generally, for a "level N", we define

$$\Gamma_0(N) = \{g \in \Gamma \, : \, a_3 L_0 \subseteq N \bar{L}_0, \; b_3 L_1 \subseteq N \bar{L}_0, \; b_2 \equiv 1 \mod N\}.$$

This congruence subgroup is the one associated to the compact group D^ψ of [Shi97, 20.6.8].

The relevant "weight k holomorphic automorphic forms of level N" are the holomorphic functions on $\mathcal{H}_V$ that satisfy

$$f \|_k \gamma = f$$

for every $\gamma \in \Gamma_0(N)$ where

$$(f \|_k \gamma)(z, w) = J_k(\gamma, z, w) f(\gamma \cdot (z, w)).$$

It is not necessary to specify a "growth condition at the cusp" [Shi00, Sec. 5.2] unless $n = 2$.

We define a parabolic subgroup $P \subseteq U_V$ to be the stabilizer of the totally isotropic subspace $\bar{V}_0$. It consists of matrices of the form

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

We define Eisenstein series

$$E_k(z, w) = \sum_{g \in (\Gamma \cap P) \backslash \Gamma} J_k(g, z, w),$$

$$E_k(z, w; N) = \sum_{g \in (\Gamma_0(N) \cap P) \backslash \Gamma_0(N)} J_k(g, z, w).$$

These series converge for $k > 2n - 2$

1.2. **The case of U_W.** Under the decomposition

$$W = W_0 \oplus \bar{W}_0$$

we write matrices in $g \in U_W$ as

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then we have a symmetric hermitian domain given by

$$\begin{aligned}\mathcal{H}_W &= \{z \in M_{n-1}(\mathbb{C}) \,:\, i(z^* - z) > 0\} \\ &= \{x + iy \in M_{n-1}(\mathbb{C}) \,:\, x, y \in \mathrm{Her}_{n-1}(F),\ y > 0\}\end{aligned}$$

with action

$$g \cdot z = \frac{az + b}{cz + d}.$$

For a weight k we define a factor of automorphy

$$J_k(g, z) = (cz + d)^{-k}.$$

The lattices L_0 and L_1 give rise to a natural choice of lattice L_2 of W_0. Letting $\bar{L}_2$ be the dual lattice, we get a lattice

$$L' = L_2 \oplus \bar{L}_2$$

in W. We define a "level 1" congruence subgroup

$$\Gamma' = \{g \in U_W(\mathbb{Q}) \,:\, gL' = L'\}.$$

More generally, for a "level N", we define

$$\Gamma'_0(N) = \{g \in \Gamma' \,:\, cW_0 \subseteq N\bar{W}_0\}.$$

The relevant "weight k holomorphic automorphic forms of level N" are the holomorphic functions on $\mathcal{H}_W$ that satisfy

$$f\|_k\gamma = f$$

for every $\gamma \in \Gamma'_0(N)$ where

$$(f\|_k\gamma)(z) = J_k(\gamma, z) f(\gamma \cdot z).$$

It is not necessary to specify a "growth condition at the cusp" [Shi00, Sec. 5.2] unless $n = 2$.

We define a parabolic subgroup $P' \subseteq U_W$ to be the stabilizer of the totally isotropic subspace $\bar{W}_0$. It consists of matrices of the form

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

We define Eisenstein series

$$\begin{aligned}E'_k(z) &= \sum_{g \in (\Gamma' \cap P')\backslash \Gamma'} J_k(g, z), \\ E'_k(z; N) &= \sum_{g \in (\Gamma'_0(N) \cap P')\backslash \Gamma'_0(N)} J_k(g, z).\end{aligned}$$

These series converge for $k > 2n - 2$.

2. Fourier expansions

Let $m = n - 1$, and fix $N > 1$. A holomorphic automorphic form on U_W has a Fourier expansion:

$$f(z) = \sum_{\beta \in \mathrm{Her}_m(F)} c_\beta q^\beta$$

where

$$q^\beta(z) = \exp(2\pi i \mathrm{Tr}(\beta z)).$$

The method of computation uses adelic methods to compute the Fourier coefficients as an Euler product of "Whittaker integrals" - this is discussed in [Shi83]. Shimura found that, if $N > 1$, the Fourier expansion of

$$E_k^*(z; N) := E_k'(z; N)\|_k \begin{bmatrix} 0 & -1_m \\ 1_m & 0 \end{bmatrix}$$

had better properties than that for $E_k'(z; N)$: the main advantage is $c_\beta(E_k^*(-; N)) = 0$ for β which are not of full rank.

Theorem 2.1 ([Shi97, Thm. 19.2])**.** Let $\mathcal{L}$ be the dual lattice to $\mathrm{Her}_m(\mathcal{O}_F)$ in $\mathrm{Her}_m(F)$ under the pairing

$$(\alpha, \beta) \mapsto \mathrm{Tr}(\alpha\beta).$$

We have $c_\beta(E^*(-, N)) = 0$ unless $\beta \in N^{-1}\mathcal{L}$ and β is positive definite. In this case, we have

$$c_\beta(E'(z; N)) = C \cdot \xi_k \cdot N^{-m^2} \cdot \alpha_N(\beta, k)$$

where:

$$C = 2^{m(m-1)/2} |\Delta_{F/\mathbb{Q}}|^{-m(m-1)/4}$$

$$\xi_k = \frac{2^{mk-m(m-1)}(-i)^{mk}\pi^{mk-m(m-1)/2} \det \beta^{k-m}}{\prod_{j=1}^m (k-j)!}$$

$$\alpha_N(\beta, k) = \Lambda_N(k)^{-1} \prod_{\ell \in S_\beta} f_{\beta,\ell}(\ell^{-k})$$

and:

$$\Delta_{F/\mathbb{Q}} = \text{the discriminant of } F/\mathbb{Q}$$

$$\Lambda_N(k) = \prod_{j=0}^{m-1} L^N(k-j, \tau^j)$$

$$\tau = \text{Dirichlet character associated to } F/\mathbb{Q}$$

$$L^N(s, \chi) = \text{Dirichlet series of } \chi \text{ with Euler factors at primes dividing } N \text{ removed}$$

$$f_{\beta,\ell}(t) = \text{integral polynomial with constant term 1}$$

$$S_\beta = \{\ell \not| N \,:\, \ell \text{ ramified in } F \text{ or } \det(\beta) \notin \mathbb{Z}_\ell^\times\}$$

Note that the set S_β above is a finite set of primes which depends on β. For such $\ell \in S_\beta$, the polynomials $f_{\beta,\ell}$ and $f_{N^{-1}\beta,\ell}$ agree.

In summary, we have, for $\beta \in \mathcal{L}$ positive definite:

$$c_{N^{-1}\beta} = (-i/N)^{mk} \left[\prod_{j=0}^{m-1} \frac{(2\pi)^{k-j}}{|\Delta_{F/\mathbb{Q}}|^{j/2}(k-j-1)! \cdot L^N(k-j,\tau^j)} \right] \det(\beta)^{k-m} \prod_{\ell \in S_\beta} f_{\beta,\ell}(\ell^{-k}).$$

We will normalize these Eisenstein series and define

$$\mathbb{E}^*_k(z;N) := (iN)^{mk} \left[\prod_{j=0}^{m-1} \frac{|\Delta_{F/\mathbb{Q}}|^{j/2}(k-j-1)! \cdot L^N(k-j,\tau^j)}{(2\pi)^{k-j}} \right] E^*_k(z;N).$$

We similarly normalize the $E'_k(z;N)$:

$$\mathbb{E}'_k(z;N) := (iN)^{mk} \left[\prod_{j=0}^{m-1} \frac{|\Delta_{F/\mathbb{Q}}|^{j/2}(k-j-1)! \cdot L^N(k-j,\tau^j)}{(2\pi)^{k-j}} \right] E'_k(z;N).$$

Observe that we have

$$\mathbb{E}'_k(z;N) = \mathbb{E}^*_k(z;N)\|_k \begin{bmatrix} 0 & -1_m \\ 1_m & 0 \end{bmatrix}.$$

3. p-integral and p-adic automorphic forms

3.1. **The case of U_W.** Recall $m = n-1$. The Shimura stack $Sh(U_W)$ is a moduli stack of tuples whose R-points (for p-complete R) are given by

$$(A', i', \lambda')/R$$

where

$$\begin{aligned} A' &= \text{abelian scheme over } R \text{ of dimension } 2m, \\ i' &: \mathcal{O}_F \hookrightarrow \mathrm{End}(A), \\ \lambda' &= \text{polarization of } A' \text{ compatible with } i \end{aligned}$$

such that:

(1) the formal group $A'(u)_{inf}$ is m-dimensional,
(2) the Tate module (away from p, at a geometric fiber) with Weil pairing $(\widehat{T}^p(A'_s), \langle -,- \rangle_{\lambda'})$ is isomorphic to $(L'^p, \mathrm{Tr}_{F/\mathbb{Q}}\langle -,- \rangle'_W)$.

For us, an "automorphic form for U_W over $\mathbb{Z}_p$ of weight k" is a rule f which, for p-complete rings R gives

$$(A', i', \lambda', (v_i))/R \mapsto f(A', i', \lambda', (v_i)) \in R$$

where (v_i) is a framing of $\mathrm{Lie}\, A(u)$, satisfying

$$f(A', i', \lambda', (\alpha(v_i)) = \det(\alpha)^k f(A', i', \lambda', (v_i))$$

for all $\alpha \in GL_m(R)$. We will denote the space of these things $(M'_k)_{\mathbb{Z}_p}$.

Hida has proven a q-expansion principle for these things, which implies that if f is a modular form for U_W over $\mathbb{C}$ of weight k whose q-expansion has coefficients which lie in $\mathbb{Z}_p$ (i.e., the coefficients of the q-expansion of f lies in $\mathcal{O}_{F,(p)}$, and we regard these as being in $\mathbb{Z}_p \cong \mathcal{O}_{F,u}$) then there is a corresponding $f \in (M'_k)_{\mathbb{Z}_p}$.

A "p-adic automorphic form for U_W" is a rule f which, for p-complete rings R gives

$$(A', i', \lambda', \phi')/R \mapsto f(A', i', \lambda', \phi') \in R$$

where ϕ' is an isomorphism of formal groups

$$\phi' : (\widehat{\mathbb{G}}_m)^{\times m} \xrightarrow{\cong} A(u)_{inf}.$$

We will denote the space of these things $\mathbb{V}'$. Note that using the canonical isomorphism

$$\mathrm{End}(\widehat{\mathbb{G}}_m) = \mathbb{Z}_p$$

we have

$$\mathrm{Aut}((\widehat{\mathbb{G}}_m)^m) = GL_m(\mathbb{Z}_p).$$

We therefore get an action of $\alpha \in GL_m(\mathbb{Z}_p)$ on $\mathbb{V}'$ by

$$([\alpha] \cdot f)(A', i', \lambda', \phi') = f(A', i', \lambda', \phi' \circ \alpha).$$

We let $\mathbb{V}'[k]$ denote the subspace consisting of those f for which

$$[\alpha] \cdot f = \det(\alpha)^k f$$

for all α.

Let $\frac{d}{dt}$ be the standard vector in $\mathrm{Lie}\, \mathbb{G}_m$. There is an embedding

$$(M'_k)_{\mathbb{Z}_p} \to \mathbb{V}'[k]$$
$$f \mapsto \widehat{f}$$

given by

$$\widehat{f}(A', i', \lambda', \phi') = f(A', i', \lambda', ((\phi')_* \frac{\partial}{\partial t_i})).$$

3.2. **The case of U_V.** The Shimura stack $Sh(U_V)$ is a moduli stack of tuples whose R-points (for p-complete R) are given by

$$(A, i, \lambda)/R$$

where

$$\begin{aligned} A &= \text{abelian scheme over } R \text{ of dimension } n, \\ i &: \mathcal{O}_F \hookrightarrow \mathrm{End}(A), \\ \lambda &= \text{polarization of } A \text{ compatible with } i \end{aligned}$$

such that:

(1) the formal group $A(u)_{inf}$ is 1-dimensional,
(2) the Tate module (away from p, at a geometric fiber) with Weil pairing $(\widehat{T}^p(A_s), \langle -, - \rangle_\lambda)$ is isomorphic to $(L^p, \mathrm{Tr}_{F/\mathbb{Q}}\langle -, - \rangle_V)$.

For us, an "automorphic form for U_V over $\mathbb{Z}_p$ of weight k" is a rule f which, for p-complete rings R gives

$$(A, i, \lambda, v)/R \mapsto f(A, i, \lambda, v) \in R$$

where $v \in \mathrm{Lie}\, A(u)$ is non-zero, satisfying

$$f(A, i, \lambda, \alpha v) = \alpha^k f(A, i, \lambda, v)$$

for all $\alpha \in R^\times$. We will denote the space of these things $(M_k)_{\mathbb{Z}_p}$.

A "p-adic automorphic form for U_V" is a rule f which, for p-complete rings R gives

$$(A, i, \lambda, \phi)/R \mapsto f(A, i, \lambda, \phi) \in R$$

where ϕ is an isomorphism of formal groups

$$\phi : \widehat{\mathbb{G}}_m \xrightarrow{\cong} A(u)_{inf}.$$

We will denote the space of these things $\mathbb{V}$. Note that using the canonical isomorphism

$$\mathrm{End}(\widehat{\mathbb{G}}_m) = \mathbb{Z}_p$$

we have

$$\mathrm{Aut}(\widehat{\mathbb{G}}_m) = \mathbb{Z}_p^\times.$$

We therefore get an action of $\alpha \in \mathbb{Z}_p^\times$ on $\mathbb{V}'$ by

$$([\alpha] \cdot f)(A, i, \lambda, \phi) = f(A, i, \lambda, \phi \circ \alpha).$$

We let $\mathbb{V}[k]$ denote the subspace consisting of those f for which

$$[\alpha] \cdot f = \alpha^k f$$

for all α.

There is an embedding

$$\begin{aligned} (M_k)_{\mathbb{Z}_p} &\to \mathbb{V}[k] \\ f &\mapsto \widehat{f} \end{aligned}$$

given by

$$\widehat{f}(A, i, \lambda, \phi) = f(A, i, \lambda, ((\phi)_* \frac{d}{dt})).$$

4. Eisenstein measure for U_W

Using the q-expansion principle, and the explicit form of the Fourier expansions, we deduce that there is a $\mathbb{V}'$-valued measure $\nu^{(N)}$ on $\mathbb{Z}_p$ with moments

$$\int_{\mathbb{Z}_p} x^{k-1} d\nu^{(N)} = \widehat{\mathbb{E}}_k(N).$$

Do I really know this? I think if p divides N, the argument of [Pan00] works. If p does not divide N, we know the argument works in the case of $n = 2$ (classical Eisenstein series) by the explicit form of the polynomials $f_{\beta,\ell}$. Here we need the factor of $\det(\beta)^{k-m}$ to correct for some p's in the denominators: see Section 9.

5. Pullback of Eisenstein series

The Eisenstein series pull back in many different senses.

5.1. **Analytic pullback.** The embedding

$$V = V_0 \oplus V_1 \oplus \bar{V}_0 \hookrightarrow V_0 \oplus V_1 \oplus \bar{V}_0 \oplus -V_1 = W$$

gives an embedding

$$\iota_V^W : U_V \hookrightarrow U_W$$

and hence an embedding of symmetric hermitian domains

$$\iota_V^W : \mathcal{H}_V \hookrightarrow \mathcal{H}_W$$

which has a conceptual interpretation in terms of "complex structures" (see [Beh]). In terms of our explicit presentations of the symmetric hermitian domains, ι_V^W is given by some elaborate formulas. We are interested in the pullback of the Eisenstein series $\mathbb{E}'_k(N)$ along ι_V^W.

This is done in [Shi97, 22.6.6] - see also [Shi00, Proof of Thm. 26.13] for a "synopsis". I have had a terribly difficult time understanding these formulas.

Basically, in our case, it seems to roughly say (modulo a "mass term" which I'm deliberately omitting, although for Shimura it is a principle object of study - this goes with my "class number 1" deception):

$$\left[\prod_{j=0}^{n-3} L^N(k-j;\tau^j)\right](i_V^W)^* E'_k(N) = \left[\prod_{j=0}^{n-3} L_F^N(-j;\chi_{-k})\right] E_k(N).$$

Here, χ_{-k} is the Hecke character whose associated ideal character satisfies

$$\chi_{-k}((\alpha)) = \alpha^{-k}.$$

(Because I am pretending all class numbers are 1, it suffices to define on priciple ideals). Note that some Euler factors in the above L-functions for F may need to be tweaked to account for places of $\mathbb{Q}$ which do not split in F and which do not divide N, for which the corresponding localization of $\langle -, - \rangle_W$ is not hyperbolic.

We deduce that

$$(i_V^W)^*(\mathbb{E}'_k) = (iN)^{(n-1)k} \left[\prod_{j=0}^{n-3} \frac{|\Delta_{F/\mathbb{Q}}|^{j/2}(k-j-1)! \cdot L_F^N(-j,\chi_{-k})}{(2\pi)^{k-j}}\right] \cdot \frac{|\Delta_{F/\mathbb{Q}}|^{(n-2)/2}(k-n+1)! \cdot L^N(k-n+2,\tau^{n-2})}{(2\pi)^{k-n+2}} E_k(N).$$

5.2. **algebraic interpretation.** The map of Shimura varieties

$$Sh_V \to Sh_W$$

has a moduli theoretic interpretation.

Namely, (ignoring class number issues), there is a "unique" polarized $\mathcal{O}_F$-linear abelian variety $(A_1, i_1, \lambda_1)/\mathbb{Z}_p$ of dimension $n-2$ whose p-divisible summand $A_1(u)$ is of dimension $n-2$, and for which there is an isomorphism

$$(\widehat{T}^p(A_1), \langle -, - \rangle_{\lambda_1}) \cong (\widehat{L}_1^p, -\mathrm{Tr}_{F/\mathbb{Q}}\langle -, - \rangle_{V_1}).$$

(In reality, class numbers make there be a finite scheme's worth of such (A_1, i_1, λ_1)'s - think elliptic curves with complex multiplication by F.)

The p-divisible group $A(u)$ is necessarily formal, and there exists an isomorphism

$$\phi_1 : \widehat{\mathbb{G}}_m^{n-2} \xrightarrow{\cong} A(u).$$

I want to say that there is some preferred choice that is somehow dictated by the form $-\langle -, - \rangle_{V_1}$, but for the life of me I can't seem to come up with how.

The map on Shimura stacks is given on R-points by

$$(A, i, \lambda) \mapsto (A \times A_1, i \times i_1, \lambda \times \lambda_1).$$

For any automorphic form $f \in (M'_k)_{\mathbb{Z}_p}$, the pullback $(i_V^W)^* f \in (M_k)_{\mathbb{Z}_p}$ is given by

$$((i_V^W)^* f)(A, i, \lambda, v) = f(A \times A_1, i \times i_1, \lambda \times \lambda_1, (v, (\phi_1)_* \frac{\partial}{\partial t_1}, \ldots, (\phi_1)_* \frac{\partial}{\partial t_{n-2}})).$$

We also get an induced map on p-adic automorphic forms

$$(i_V^W)^* : \mathbb{V}' \to \mathbb{V}$$

by

$$((i_V^W)^* f)(A, i, \lambda, \phi) = f(A \times A_1, i \times i_1, \lambda \times \lambda_1, \phi \times \phi_1).$$

6. Eisenstein measure for U_V

Define normalized Eisenstein series on U_V by

$$\mathbb{E}_k(N) = \left[\frac{N^k (k-1)! \cdot L_F^N(0, \chi_{-k})}{(-2\pi i)^k} \right] \cdot \left[\frac{L^N(k-n+2, \tau^{n-2})}{L_F^N(-n+2, \chi_{-k})} \right] E_k(N)$$

We deduce that

$$(i_V^W)^* \mathbb{E}'_k(N) = (iN)^{(n-2)k} \left[\prod_{j=1}^{n-2} \frac{|\Delta_{F/\mathbb{Q}}|^{j/2} (k-j-1)! \cdot L_F^N(-j, \chi_{-k})}{(2\pi)^{k-j}} \right] \mathbb{E}_k(N)$$

Define

$$L_F^{\infty,N}(-j, \chi_{-k}) = \frac{|\Delta_{F/\mathbb{Q}}|^{j/2} (k-j-1)! \cdot L_F^N(-j, \chi_{-k})}{(2\pi)^{k-j}}$$

Define a measure μ_2^N on $\mathbb{Z}_p$ valued in $\mathbb{V}$ by pulling back the measure $\nu^{(N)}$ along i_V^W. Thus we have

$$\int_{\mathbb{Z}_p} x^{k-1} \mu_2^{(N)} = (iN)^{(n-2)k} \prod_{j=1}^{n-2} L_F^{\infty,N}(-j, \chi_{-k}) \widehat{\mathbb{E}}_k(N).$$

Now we have [Shi97, Lemma 12.7(2)]

$$E_k = \sum_{\gamma \in \Gamma_0(N) \backslash \Gamma} E_k(N) \|_k \gamma.$$

Define

$$\mathbb{E}_k = \left[\frac{(k-1)! \cdot L_F(0, \chi_{-k})}{(-2\pi i)^k} \right] \cdot \left[\frac{L(k-n+2, \tau^{n-2})}{L_F(-n+2, \chi_{-k})} \right] E_k$$

We have

$$N^k L_{F,N}^{-1}(0,\chi_{-k}) \left[\frac{L_{F,N}(-n+2,\chi_{-k})}{L_N(k-n+2,\tau^{n-2})} \right] \mathbb{E}_k = \sum_{\gamma \in \Gamma_0(N)\backslash\Gamma} \mathbb{E}_k(N)\|_k \gamma \tag{6.1}$$

where L_N, $L_{F,N}$ denotes the products of Euler factors dividing N.

Alternatively, I would be perfectly satisfied with a measure $\widetilde{\mu}^{(N)}$ with moments

$$\int_{\mathbb{Z}_p} x^{k-1} d\widetilde{\mu}^{(N)} = (N^k - 1)\mathbb{E}_k$$

6.1. **What I would like, but don't have.** I would like a measure $\mu^{(N)}$ on $\mathbb{Z}_p$ valued in $\mathbb{V}$ with moments

$$\int_{\mathbb{Z}_p} x^{k-1} d\mu^{(N)} = N^k L_{F,N}^{-1}(0,\chi_{-k}) \left[\frac{L_{F,N}(-n+2,\chi_{-k})}{L_N(k-n+2,\tau^{n-2})} \right] \widehat{\mathbb{E}}_k.$$

Such a measure could be constructed from a measure $\mu_1^{(N)}$ with moments

$$\int_{\mathbb{Z}_p} x^{k-1} d\mu_1^{(N)} = \widehat{\mathbb{E}}_k(N).$$

But to get $\mu_1^{(N)}$ from $\mu_2^{(N)}$, we need to "divide" the moments of $\mu_2^{(N)}$ by

$$\prod_{j=1}^{n-2} L_F^{\infty,N}(-j,\chi_{-k}).$$

I don't think I can do this for $N > 1$.

Ironically, I think one can do this for $N = 1$. This is because of the form of the Katz 2-variable p-adic L-function (See [Yag82, Thm 9], where his $(k-1)!L_\infty(k,\bar{\psi}^{k+j})$ is $L_F^\infty(-j,\chi_{-(k+j)})$ in our notation.)

Alternatively, I would be perfectly satisfied with a measure $\widetilde{\mu}^{(N)}$ with measures

$$\int_{\mathbb{Z}_p} x^{k-1} d\widetilde{\mu}^{(N)} = (N^k - 1)\mathbb{E}_k$$

Presumably such a thing could be obtained using the remark in the paragraph above from a measure $\widetilde{\nu}^{(N)}$ valued in $\mathbb{V}'$ with moments

$$\int_{\mathbb{Z}_p} x^{k-1} d\widetilde{\nu}^{(N)} = (N^k - 1)\mathbb{E}'_k$$

but I don't have a handle on how to understand the Fourier expansion of $E'_k(N)$ when $N = 1$.

7. Eisenstein measure on $\mathbb{Z}_p^\times$

Up to this point we have only talked about constructing measures on $\mathbb{Z}_p$, and have ignored the effect of the logarithm $\log_1$. We will now explain how these two things probably happen "automatically".

Katz, in [Kat75], [Kat77], studies measures arising from power series $F(X)$ lying in (a suitable base change) of the coordinate ring of a formal group $\mathbb{G}$. In Katz's set-up, the association is given by

$$\int_{\mathbb{Z}_p} x^k d\mu_F = D^k F(0)$$

where D is an invariant differential operator "dual" to the coordinate x.

In [Kat75, Sec. IV] it is shown that

$$\int_{\mathbb{Z}_p^\times} x^k d\mu_F = \int_{\mathbb{Z}_p} x^k d\mu_{F^*}$$

where

$$F^*(X) = F(X) - \frac{1}{p} \sum_{\zeta \in \mathbb{G}[p]} F(X +_{\mathbb{G}} \zeta).$$

I think that this implies that

$$\int_{\mathbb{Z}_p^\times} x^k d\mu_F = \log_1 \int_{\mathbb{Z}_p} x^k d\mu.$$

for the F's under consideration.

8. Compatibility with Miller invariants

If everything else in this note compiled (which it doesn't), we would still be left with showing that $b_{2k} \equiv b_{2k}^{Miller} \mod \mathbb{Z}$.

How do we compute the Miller invariants? Let E be an elliptic curve with complex multiplication by F, chosen so that $\dim E(u) = 1$. Let $\bar{E}$ be the curve with conjugate multiplication, so that $\dim \bar{E}(u) = 0$. Let $(\bar{A}_1, \bar{i}_1, \bar{\lambda}_1)$ denote conjugate of the polarized F-linear abelian variety obtained from (A_1, i_1, λ_1) of Section 5.2 (i.e. conjugate F-linear structure) so that $\dim \bar{A}_1(u) = 0$. Then

$$(E \times \bar{E} \times \bar{A}_1, i_E \times \bar{i}_E \times \bar{i}_1, \lambda_E \times \bar{\lambda}_E \times \bar{\lambda}_1)$$

is a CM point of Sh_V.

Consider the following diagram of E_∞-ring spectra obtained by using the unit maps:

$$\begin{array}{ccc} S & \longrightarrow & \mathrm{TAF}_{E(1)} \\ \downarrow & & \downarrow \\ K & \longrightarrow & K \wedge \mathrm{TAF}_{E(1)} \end{array}$$

Since Miller invariants are functorial in E_∞ rings, we deduce that

$$b_{2k}^{Miller}(\mathrm{TAF}) \equiv b_{2k}^{Miller}(K) \mod \mathbb{Z}.$$

But

$$\pi_* K \wedge \mathrm{TAF} = \mathbb{V}[u^{\pm 1}].$$

where the right hand side is p-adic automorphic forms. The inclusion

$$\pi_0 K \to \mathbb{V}$$

is the inclusion of "constant" p-adic automorphic forms.

We therefore have two steps:

Step 1: Show that $\mathbb{E}_k$ is constant mod $\mathbb{Z}$.
Step 2: Show that $\mathbb{E}_k(x) \equiv b_{2k}^{Miller}(x) \mod \mathbb{Z}$ for one point $x \in Sh_V$.

I have no idea how to attack step 1. If an automorphic form is constant mod $\mathbb{Z}$ on CM-points, is it constant mod $\mathbb{Z}$? If this is true, then the strategy I outline for step 2 below could perhaps be adapted to handle step 1... (Actually, it looks like Hsieh and Hida may have also studied this in the context of Hilbert moduli spaces: "On the non-vanishing of Hecke L-values modulo p."?)

Step 2: Take the point x to be the CM point given by

$$x = (E \times \bar{E} \times \bar{A}_1, i_E \times \bar{i}_E \times \bar{i}_1, \lambda_E \times \bar{\lambda}_E \times \bar{\lambda}_1).$$

However, under the inclusion of Shimura stacks $Sh_V \hookrightarrow Sh_W$, the CM point x gets mapped to a CM point x' of Sh_W given by

$$x' = (E \times \bar{E} \times \bar{A}_1 \times A_1, i_E \times \bar{i}_E \times \bar{i}_1 \times i_1, \lambda_E \times \bar{\lambda}_E \times \bar{\lambda}_1 \times \lambda_1).$$

Of course, we then have

$$\mathbb{E}'_k(x') = i^{(n-2)k} \left[\prod_{j=1}^{n-2} \frac{|\Delta_{F/\mathbb{Q}}|^{j/2}(k-j-1)! \cdot L_F(-j, \chi_{-k})}{(2\pi)^{k-j}} \right] \mathbb{E}_k(x).$$

However, the point x' is essentially the Shimura stack Sh_{W_0}, for the maximal totally isotropic subspace $\bar{W}_0 \subset W$. Then, *again* using Shimura's pullback formula, this time for the inclusion of the group

$$U_{\bar{W}_0} \hookrightarrow U_W$$

[Shi97, 22.6.6], we get something "like" the following:

$$\mathbb{E}_k(x) = \frac{(k-1)! \cdot L_F(0, \chi_{-k})}{(-2\pi i)^k} = -\frac{BH_k}{k}$$

where BH_k are the "Bernoulli Hurewitz numbers". The inclusion of the CM point x induces an E_∞ map

$$\mathrm{TAF} \to K_E$$

where K_E is the "form of K-theory" associated to the elliptic curve E. Since (say by the restriction of the Witten orientation, and Dammerell's formula) we have

$$b_{2k}^{Miller}(K_E) = -\frac{BH_k}{k},$$

and step 2 would follow.

9. Example: the case of $n = 2$

Same notation, specialized to $n = 2$, we show that everything works. We have

$$\begin{aligned}
V &= V_0 \oplus \bar{V}_0 \\
SU_V(\mathbb{Q}) &= SL_2(\mathbb{Q}) \\
\mathcal{H}_V &= \text{upper half plane} \\
J_k(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, z) &= (cz+d)^{-k} \\
E_k(z) &= \frac{1}{2} \sum_{(m,n)=1} \frac{1}{(mz+n)^k} \\
E_k(z;N) &= \frac{1}{2} \sum_{\substack{(m,n)=1 \\ c \equiv 0 \pmod{N}}} \frac{1}{(mz+n)^k} \\
E_k^*(z;N) &= \sum_{\beta=1}^{\infty} c_{\beta/N} q^{\beta/N} \\
&= \frac{(-2\pi i)^k}{N^k \zeta^N(k)(k-1)!} \sum_{\beta=1}^{\infty} \beta^{k-1} \prod_{\substack{\ell \mid \beta \\ \ell \nmid N}} f_{\beta,\ell}(\ell^{-k}) q^{\beta/N} \\
f_{\beta,\ell}(t) &= 1 + \ell t + \ell^2 t^2 + \cdots + \ell^{\nu_\ell(\beta)} t^{\nu_\ell(\beta)}
\end{aligned}$$

We therefore get the usual q-expansion of the Eisenstein series $E_k^*(z;N) = E_k(z;N)\|_k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$:

$$E_k^*(z;N) = \frac{(-2\pi i)^k}{N^k \zeta^N(k)(k-1)!} \sum_{\beta=1}^{\infty} \sum_{\substack{d \mid \beta \\ (d,N)=1}} (\beta/d)^{k-1} q^{\beta/N}$$

The normalization conventions thus specialize to give:

$$\mathbb{E}_k^* = \sum_{\beta=1}^{\infty} \sum_{\substack{d \mid \beta \\ (d,N)=1}} (\beta/d)^{k-1} q^{\beta/N}.$$

In the $n = 2$ case, we have $V = W$, $E_k(z;N) = E'_k(z,N)$, and $\mathbb{E}_k(z;N) = \mathbb{E}'_k(z;N)$. The coefficients of this q-expansion are easily seen to p-adically interpolate to the moments of a measure (see [Kat75]). Therefore, the Eisenstein series

$$\mathbb{E}_k(z;N) = \mathbb{E}_k^*(z;N)\|_k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

p-adically interpolate to our desired $\mathbb{V}$-valued measure $\mu_2^{(N)} = \nu^{(N)}$ on $\mathbb{Z}_p$ with:

$$\int_{\mathbb{Z}_p} x^{k-1} \mu_2^{(N)} = \widehat{\mathbb{E}}_k(N).$$

Now, we have

$$E_k(z) = \sum_{[\gamma] \in \Gamma_0(N) \backslash \Gamma(1)} E_k(z;N)\|_k \gamma$$

which gives

$$N^k \zeta_N(k) \mathbb{E}_k(z) = \sum_{[\gamma] \in \Gamma_0(N) \backslash \Gamma(1)} \mathbb{E}_k(z; N) \|_k \gamma$$

We therefore get a measure $\mu_1^{(N)}$ with moments

$$\int_{\mathbb{Z}_p} x^{k-1} \mu_1^{(N)} = N^k \zeta_N(k) \widehat{\mathbb{E}}_k.$$

If $N = \ell$, a prime different from p, this specializes to give

$$\int_{\mathbb{Z}_p} x^{k-1} \mu_1^{(\ell)} = (\ell^k - 1) \widehat{\mathbb{E}}_k$$

which is the Katz Eisenstein measure.

Let me pause to point something out: the above construction differs from the construction of Katz in [Kat75] in that we make no use of his "Key Lemma". We *only* used the q-expansion of $E_k^*(z; \ell)$. From the point of view of adelic Fourier analysis (Whittaker integral technique) the computation of the coefficients of this Fourier series is actually much simpler than that of $E_k(z)$: this is because the corresponding function in the parabolically induced representation for $GL_2(\mathbb{A})$ is supported on the "big cell".

Note that we have

$$\mathbb{E}_k(z; N) = \frac{N^k \zeta^N(k)(k-1)!}{(-2\pi i)^k} \left(1 + \sum_{n=1}^{\infty} \sum_{\substack{d \mid n \\ (d,N)=1}} d^{k-1} q^n \right)$$

and in particular have

$$\mathbb{E}_k(z) = \frac{\zeta(k)(k-1)!}{(-2\pi i)^k} \left(1 + \sum_{n=1}^{\infty} \sum_{d \mid n} d^{k-1} q^n \right).$$

Thus, specializing the above to the constant terms gives a p-adic interpolation of the values

$$-(\ell^k - 1) B_k / k$$

giving the Kubota-Leopoldt ζ-measure.

As in [Kat75], the measure $\mu_1^{(\ell)}$ constructed above, when restricted to $\mathbb{Z}_p^\times$, has moments

$$\int_{\mathbb{Z}_p^\times} x^{k-1} \mu_1^{(\ell)} = (\ell^k - 1) \log_1 \widehat{\mathbb{E}}_k.$$

The verification of the equality of Miller invariants specializes to an analysis of Bernoulli-Hurwitz numbers mod $\mathbb{Z}$.

References

[Beh] Mark Behrens, *Orientations and Eisenstein seris*, handwritten digital notes, available at www-math.mit.edu/∼mbehrens/preprints.

[HLS06] Michael Harris, Jian-Shu Li, and Christopher M. Skinner, *p-adic L-functions for unitary Shimura varieties. I. Construction of the Eisenstein measure*, Doc. Math. (2006), no. Extra Vol., 393–464 (electronic). MR MR2290594 (2008d:11042)

[Hsia] Ming-Lun Hsieh, *Eisenstein congruence on unitary groups and Iwasawa main conjecture for CM fields*, available at http://www.math.ntu.edu.tw/∼mlhsieh/.

[Hsib] ———, *p-adic ordinary Eisenstein series and p-adic L-functions for unitary groups*, available at http://www.math.ntu.edu.tw/∼mlhsieh/.

[Kat75] Nicholas M. Katz, *p-adic L-functions via moduli of elliptic curves*, Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), Amer. Math. Soc., Providence, R. I., 1975, pp. 479–506. MR MR0432649 (55 #5635)

[Kat77] ———, *Formal groups and p-adic interpolation*, Journées Arithmétiques de Caen (Univ. Caen, Caen, 1976), Soc. Math. France, Paris, 1977, pp. pp 55–65. Astérisque No. 41–42. MR MR0441928 (56 #319)

[Pan00] A. A. Panchishkin, *On the Siegel-Eisenstein measure and its applications*, Proceedings of the Conference on p-adic Aspects of the Theory of Automorphic Representations (Jerusalem, 1998), vol. 120, 2000, pp. 467–509. MR MR1809631 (2002a:11054)

[Shi83] Goro Shimura, *On Eisenstein series*, Duke Math. J. **50** (1983), no. 2, 417–476. MR MR705034 (84k:10019)

[Shi97] ———, *Euler products and Eisenstein series*, CBMS Regional Conference Series in Mathematics, vol. 93, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1997. MR MR1450866 (98h:11057)

[Shi00] ———, *Arithmeticity in the theory of automorphic forms*, Mathematical Surveys and Monographs, vol. 82, American Mathematical Society, Providence, RI, 2000. MR MR1780262 (2001k:11086)

[Yag82] Rodney I. Yager, *A Kummer criterion for imaginary quadratic fields*, Compositio Math. **47** (1982), no. 1, 31–42. MR MR668779 (83k:12008)