LECTURE 7: COFIBERS

1. Mapping cone

For $X \in \text{Top}$, let $\text{Cone}(X)$ be the space $X \times I / X \times \{0\}$. The cone on X is contractible. For $f : X \to Y$, we define the *mapping cone* to be the pushout

$$\text{Cone}(f) = Y \cup_X \text{Cone}(X).$$

The mapping cone satisfies:

(1) If $i : A \hookrightarrow X$ is an inclusion, then there is an isomorphism $H^*(X, A) \cong \widetilde{H}^*(\text{Cone}(i))$.
(2) If $i : A \to X$ is a cofibration, then there is a homotopy equivalence $X/A \simeq \text{Cone}(i)$.

For $X \in \text{Top}_*$ there is a pointed analog. Let $C(X)$ be the space $X \wedge I$, also contractible. For $f : X \to Y$, we define the *reduced mapping cone* or *cofiber* to be the pushout

$$C(f) = Y \cup_X C(X).$$

2. Relative CW complexes

If X is obtained from A by iteratively adding cells, then we say $A \hookrightarrow X$ is a *relative CW complex*. In particular, if A is a subcomplex of X, then it is a relative CW complex.

We have seen that cofibrations have good cofibers. The following proposition states that cofibrations are not uncommon.

Proposition 2.1. If $A \hookrightarrow X$ is a relative CW complex, then it is a cofibration.

The proposition is proven by a sequence of lemmas.

Lemma 2.2. The inclusion $S^{n-1} \hookrightarrow D^n$ is a cofibration.

Lemma 2.3. Suppose that $i : A \to X$ is a cofibration, and that $f : A \to Y$ is a map. Then the inclusion $Y \to X \cup_A Y$ is a cofibration.

Lemma 2.4. Suppose that

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$$

is a sequence of cofibrations. Then the map $X_0 \to \varinjlim_i X_i$ is a cofibration.

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3. Exact sequence of a cofiber

We have the following lemma dual to the lemma for the homotopy fiber.

Lemma 3.1. Let $X \to Y$ be a map of unpointed spaces and let Z be a pointed space. Consider factorizations:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r] \arrow[d, "g"'] & \mathrm{Cone}(f) \arrow[dl, dotted, "\widetilde{g}"] \\ & Z & \end{tikzcd}$$

There is a bijective correspondence

$$\{\text{pointed factorizations } \widetilde{g}\}$$
$$\updownarrow$$
$$\{\text{null homotopies } gf \simeq *\}$$

Corollary 3.2. Let $X \to Y$ be a map of spaces, and let Z be a pointed space. Then the sequence

$$X \xrightarrow{f} Y \to \mathrm{Cone}(f)$$

induces an exact sequence of sets

$$[\mathrm{Cone}(f), Z]_* \to [Y, Z] \xrightarrow{f^*} [X, Z].$$

Corollary 3.3. Let $X \to Y$ be a map of pointed spaces, and let Z be a pointed space. Then the sequence

$$X \xrightarrow{f} Y \to C(f)$$

induces an exact sequence of sets

$$[C(f), Z]_* \to [Y, Z]_* \xrightarrow{f^*} [X, Z]_*.$$

Remark 3.4. We will prove later that there are isomorphisms

$$H^n(X, \pi) \cong [X, K(\pi, n)]$$
$$\widetilde{H}^n(X, \pi) \cong [X, K(\pi, n)]_*$$

The exact sequences of Corollaries 3.2 and 3.3 then recover part of the cohomology LES of a pair.