1. Compute the gradient of the following functions

   (a)
$$f(x,y) = (5y^3 + 2x^2y)^8$$

   (b)
$$F(\alpha,\beta) = \alpha^2 \ln \alpha^2 + \beta^2$$

   (c)
$$G(x,y,z) = e^{xz} \sin y/z$$

   (d)
$$S(u,v,w) = u \arctan(u\sqrt{w})$$

2. Compute the all the partial second derivatives of the following functions

   (a)
$$f(x,y) = 4x^3 - xy^2$$

   (b)
$$f(x,y,z) = x^k y^l z^m$$

   (c)
$$z = x^{-2y}$$

   (d)
$$v = r\cos(s+2t)$$

3. Find the linear approximation of the function
$$f(x,y,z) = x^3\sqrt{y^2+z^2}$$
   at the point $(2,3,4)$, and use it to estimate the number $(1.98)^3\sqrt{(3.01)^2 + (3.97).^2}$

4. If
$$z = y + f(x^2 - y^2), \tag{1}$$
   where f is differentiable, show that
$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x. \tag{2}$$

5. If $g(s,t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies
$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

6. If $u = x^2y^3 + z^4$, where $x = p + 3p^2$, $y = pe^p$ and $z = p\sin p$, use the chain rule to compute du/dp.

7. Find the maximum rate of change of $f(x,y) = x^2y + \sqrt{y}$ at the point $(2,1)$. In which direction does it occur?

8. If $z = xy + xe^{y/x}$, show that
$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy + z. \tag{3}$$

9. If $z = \sin(x + \sin t)$, show that

$$\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x \partial t} = \frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial x^2}. \tag{4}$$

10. If z depends on x and y implicitly following

$$\cos(xyz) = 1 + x^2y^2 + z^2, \tag{5}$$

using implicit differentiation find, $\partial_x z$ and $\partial_y z$.

11. Determine whether the following statements are true or false. If it is true explain why, otherwise find a counterexample.
    (a) $f_y(a, b) = \lim_{y\to b} \frac{f(a,y)-f(a,b)}{y-b}$.
    (b) $D_{\hat{\mathbf{k}}} f(x, y, z) = \partial_z f(x, y, z)$.
    (c) If $f(x, y) \to L$ as $(x, y) \to (a, b)$ along every straight line through (a, b), then $\lim_{(x,y)\to(a,b)} f(x, y) = L$.
    (d) If $f(x, y) = \ln y$ then $\nabla f = 1/y$.
    (e) If $f(x, y) = \sin x + \sin y$, then $-2 \le D_{\hat{\mathbf{u}}} f(x, y) \le 2$ for any unitary vector $\hat{\mathbf{u}}$.

12. Find the local minimum and maximum values and the saddle points of the functions
    (a) $f(x, y) = x^2 - xy + y^2 + 9x - 6y + 10$
    (b) $f(x, y) = x^3 - 6xy + 8y^3$
    (c) $f(x, y) = 3xy - x^2y - xy^2$
    (d) $f(x, y) = (x^2 + y)e^{y/2}$

13. We want to show that the maximum value of the function

$$f(x, y) = \frac{(ax + by + c)^2}{x^2 + y^2 + 1} \tag{6}$$

is $a^2 + b^2 + c^2$, where we suppose that $c \neq 0$.
    (a) Compute the gradient of $f(x, y)$.
    (b) (2 points) Show that the gradient is zero if

$$ax + by + c = 0 \tag{7}$$

or if

$$ay^2 + a - bxy - cx = 0,$$
$$bx^2 + b - axy - cy = 0.$$

Hint: you will need to factorize the expression that you obtained for the partial derivatives.
    (c) Check that if

$$ax + by + c = 0 \tag{8}$$

then $f(x, y) = 0$

(d) Check that $x = a/c$ and $y = b/c$ are solution of

$$\begin{aligned} ay^2 + a - bxy - cx &= 0, \\ bx^2 + b - axy - cy &= 0. \end{aligned}$$

(e) Check that $f(a/c, b/c) = a^2 + b^2 + c^2$

(f) Define the following two vectors:

$$\mathbf{u} = \langle a, b, c\rangle \text{ and } \mathbf{v} = \langle x, y, 1\rangle \tag{9}$$

using that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{u}| \cos\theta$; where θ is the angle between $\mathbf{u} \cdot \mathbf{v}$, show that

$$ax + by + c \leq (a^2 + b^2 + c^2)^{1/2}(x^2 + y^2 + 1^2)^{1/2}. \tag{10}$$

(g) Using the result from the question above show that

$$\frac{(ax + by + c)^2}{x^2 + y^2 + 1} \leq a^2 + b^2 + c^2 \tag{11}$$

(h) Argue, using the properties of the dot product, that the maximum is atteint at $x = a/c$ and $y = b/c$