Derived Algebraic Geometry I: Stable ∞-Categories

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1 Introduction

There is very useful analogy between topological spaces and chain complexes with values in an abelian category. For example, it is customary to speak of *homotopies* between chain maps, *contractible* complexes, and so forth. The analogue of the homotopy category of topological spaces is the *derived category* of an abelian category $\mathcal{A}$, a triangulated category which provides a good setting for many constructions in homological algebra. However, it has long been recognized that for many purposes the derived category is too crude: it identifies homotopic morphisms of chain complexes without remembering *why* they are homotopic. It is possible to correct this defect by viewing the derived category as the homotopy category of an underlying ∞-category $\mathcal{D}(\mathcal{A})$. The ∞-categories which arise in this way have special features which reflect their "additive" origins: they are *stable*.

The goal of this paper is to provide an introduction to the theory of stable ∞-categories. We will begin in §2 by introducing the definition of stability and some other basic terminology. In many ways, an arbitrary stable ∞-category $\mathcal{C}$ behaves like the derived category of an abelian category: in particular, we will see in §3 that for every stable ∞-category $\mathcal{C}$, the homotopy category $h\mathcal{C}$ is triangulated (Theorem 3.11). In §4 we will establish some other simple consequences of stability; for example, stable ∞-categories admit finite limits and colimits (Proposition 4.4).

The appropriate notion of *functor* between stable ∞-categories is an *exact* functor: that is, a functor which preserves finite colimits (or equivalently, finite limits: see Proposition 5.1). The collection of stable ∞-categories and exact functors between them can be organized into an ∞-category, which we will denote by $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Ex}}$. In §5, we will study the ∞-category $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Ex}}$; in particular, we will show that it is stable under limits and filtered colimits in $\mathcal{C}\mathrm{at}_{\infty}$. The formation of limits in $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Ex}}$ provides a tool for addressing the classical problem of "gluing in the derived category".

In §6, we will review the theory of t-structures on triangulated categories. We will see that, if $\mathcal{C}$ is a stable ∞-category, there is a close relationship between t-structures on the homotopy category $h\mathcal{C}$ and localizations of $\mathcal{C}$. We will revisit this subject in §16, where we show that, under suitable set-theoretic hypotheses (to be described in §15), we can construct a t-structure "generated" by an arbitrary collection of objects of $\mathcal{C}$.

The most important example of a stable ∞-category is the ∞-category $\mathcal{S}_{\infty}$ of *spectra*. The homotopy category of $\mathcal{S}_{\infty}$ can be identified with the classical *stable homotopy category*. There are many approaches to the construction of $\mathcal{S}_{\infty}$. In §9 we will adopt the most classical perspective: we begin by constructing an ∞-category $\mathcal{S}_{\infty}^{\mathrm{fin}}$ of *finite spectra*, obtained from the ∞-category of finite pointed spaces by formally inverting the suspension functor. The stability of $\mathcal{S}_{\infty}^{\mathrm{fin}}$ follows from the classical *homotopy excision theorem*. We can then define the ∞-category $\mathcal{S}_{\infty}$ as the ∞-category of Ind-objects of $\mathcal{S}_{\infty}^{\mathrm{fin}}$. The stability of $\mathcal{S}_{\infty}$ follows from a general result on Ind-objects (Proposition 8.3) which we will prove in §8.

There is another description of the ∞-category $\mathcal{S}_{\infty}$ which is perhaps more familiar: it can be viewed as the ∞-category of *infinite loop spaces*, obtained from the ∞-category $\mathcal{S}_{*}$ of pointed spaces by formally inverting the loop functor. More generally, one can begin with an arbitrary ∞-category $\mathcal{C}$, and construct a new ∞-category $\mathrm{Stab}(\mathcal{C})$ of *infinite loop objects of* $\mathcal{C}$. The ∞-category $\mathrm{Stab}(\mathcal{C})$ can be regarded as universal among stable ∞-categories which admits a left exact functor to $\mathcal{C}$ (Proposition 10.10). This leads to a characterization of $\mathcal{S}_{\infty}$ by a mapping property: namely, $\mathcal{S}_{\infty}$ is freely generated under colimits (as a stable ∞-category) by a single object, the *sphere spectrum* (Corollary 15.6).

In §11, we will return to the subject of homological algebra. We will explain how to pass from a suitable abelian category $\mathcal{A}$ to a stable ∞-category $\mathcal{D}^{+}(\mathcal{A})$, which we will call the *derived ∞-category of $\mathcal{A}$*. The homotopy category of $\mathcal{D}^{+}(\mathcal{A})$ can be identified with the classical derived category of $\mathcal{A}$.

Our final goal in this paper is to characterize $\mathcal{D}^{+}(\mathcal{A})$ by a universal mapping property. In §14, we will show that $\mathcal{D}^{+}(\mathcal{A})$ is universal among stable ∞-categories equipped with a suitable embedding of the ordinary category $\mathcal{A}$ (Corollary 14.13). To give the proof, we will need some ideas from the theory of nonabelian homological algebra (or "homotopical" algebra). We will give a brief review of this theory in §12, from the ∞-categorical point of view. In §13, we will present the same ideas in a more classical form, following Quillen's manuscript [18]. The comparison of these two perspectives is based on a rectification result (Proposition 13.2) which is of some independent interest.

The proof of Corollary 14.13 requires a well-known calculation for homotopy colimits of simplicial objects in model categories of simplicial objects. We have included the details of this calculation as an appendix.

The theory of stable ∞-categories is not really new: most of the results presented here are well-known to experts. There exists a sizable literature on the subject in the setting of *stable model categories* (see, for example, [8]). The theory of stable model categories is essentially equivalent to the notion of a *presentable* stable ∞-category, which we discuss in §15. For a brief account in the more flexible setting of Segal categories, we refer the reader to [22].

In this paper, we will use the language of ∞*-categories* (also called quasicategories or weak Kan complexes), as described in [11]. We will use the letter T to indicate references to [11]. For example, Theorem T.6.1.0.6 refers to Theorem 6.1.0.6 of [11].

2 Stable ∞-Categories

In this section, we will introduce our main object of study: stable ∞-categories. We begin with a brief review of some ideas from §T.7.2.2.

Definition 2.1. Let $\mathcal{C}$ be an ∞-category. A *zero object* of $\mathcal{C}$ is an object which is both initial and final. We will say that $\mathcal{C}$ is *pointed* if it contains a zero object.

In other words, an object $0 \in \mathcal{C}$ is zero if the spaces $\mathrm{Map}_{\mathcal{C}}(X, 0)$ and $\mathrm{Map}_{\mathcal{C}}(0, X)$ are both contractible for every object $X \in \mathcal{C}$. Note that if $\mathcal{C}$ contains a zero object, then that object is determined up to equivalence. More precisely, the full subcategory of $\mathcal{C}$ spanned by the zero objects is a contractible Kan complex (Proposition T.1.2.12.9).

Remark 2.2. Let $\mathcal{C}$ be an ∞-category with a zero object 0. For any $X, Y \in \mathcal{C}$, the natural map

$$\mathrm{Map}_{\mathcal{C}}(X, 0) \times \mathrm{Map}_{\mathcal{C}}(0, Y) \to \mathrm{Map}_{\mathcal{C}}(X, Y)$$

has contractible source. We therefore obtain a well defined morphism $X \to Y$ in the homotopy category $\mathrm{h}\mathcal{C}$, which we will refer to as the *zero morphism* and also denote by 0.

Definition 2.3. Let $\mathcal{C}$ be a pointed ∞-category. A *triangle* in $\mathcal{C}$ is a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$, depicted as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & Z \end{array}$$

where 0 is a zero object of $\mathcal{C}$. We will say that a triangle in $\mathcal{C}$ is *exact* if it is a pullback square, and *coexact* if it is a pushout square.

Remark 2.4. Let $\mathcal{C}$ be a pointed ∞-category. A triangle in $\mathcal{C}$ consists of the following data:

(1) A pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$.

(2) A 2-simplex in $\mathcal{C}$ corresponding to a diagram

$$\begin{array}{ccccc} & & Y & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} & \\ X & & \xrightarrow{\quad h \quad} & & Z \end{array}$$

in $\mathcal{C}$, which identifies h with the composition $g \circ f$.

(3) A 2-simplex

$$\begin{array}{ccccc} & & 0 & & \\ & \nearrow & & \searrow & \\ X & & \xrightarrow{h} & & Z \end{array}$$

in $\mathcal{C}$, which we may view as a *nullhomotopy* of h.

We will sometimes indicate a triangle by specifying only the pair of maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

with the data of (2) and (3) being implicitly assumed.

Definition 2.5. Let $\mathcal{C}$ be a pointed ∞-category containing a morphism $g : X \to Y$. A *kernel* of g is an exact triangle

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & Y. \end{array}$$

Dually, a *cokernel* for g is a coexact triangle

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z. \end{array}$$

We will sometimes abuse terminology by simply referring to W and Z as the kernel and cokernel of g.

Remark 2.6. Let $\mathcal{C}$ be a pointed ∞-category containing a morphism $f : X \to Y$. A cokernel of f, if it exists, is uniquely determined up to equivalence. More precisely, consider the full subcategory $\mathcal{E} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by the coexact triangles. Let $\theta : \mathcal{E} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ be the forgetful functor, which associates to a diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

the morphism $g : X \to Y$. Applying Proposition T.4.3.2.15 twice, we deduce that θ is a Kan fibration, whose fibers are either empty or contractible (depending on whether or not a morphism $g : X \to Y$ in $\mathcal{C}$ admits a cokernel). In particular, if every morphism in $\mathcal{C}$ admits a cokernel, then θ is a trivial Kan fibration, and therefore admits a section $\mathrm{coker} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$, which is well defined up to a contractible space of choices. We will often abuse notation by also letting $\mathrm{coker} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ denote the composition

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}) \to \mathcal{C},$$

where the second map is given by evaluation at the final object of $\Delta^1 \times \Delta^1$.

Remark 2.7. The functor $\mathrm{coker} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ can be identified with a left adjoint to the left Kan extension functor $\mathcal{C} \simeq \mathrm{Fun}(\{1\}, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$, which associates to each object $X \in \mathcal{C}$ a zero morphism $0 \to X$. It follows that coker preserves all colimits which exist in $\mathrm{Fun}(\Delta^1, \mathcal{C})$ (Proposition T.5.2.3.5).

Definition 2.8. An ∞-category $\mathcal{C}$ is *stable* if it satisfies the following conditions:

(1) There exists a zero object $0 \in \mathcal{C}$.

(2) Every morphism in $\mathcal{C}$ admits a kernel and a cokernel.

(3) A triangle in $\mathcal{C}$ is exact if and only if it is coexact.

Remark 2.9. Condition (3) of Definition 2.8 is analogous to the axiom for abelian categories which requires that the image of a morphism be isomorphic to its coimage.

Example 2.10. Recall that a *spectrum* consists of an infinite sequence of pointed topological spaces $\{X_i\}_{i\geq 0}$, together with homeomorphisms $X_i \simeq \Omega X_{i+1}$, where Ω denotes the loop space functor. The collection of spectra can be organized into a stable ∞-category $\mathcal{S}_\infty$. Moreover, $\mathcal{S}_\infty$ is in some sense the universal example of a stable ∞-category. This motivates the terminology of Definition 2.8: an ∞-category $\mathcal{C}$ is stable if it resembles the ∞-category $\mathcal{S}_\infty$, whose homotopy category $\mathrm{h}\mathcal{S}_\infty$ can be identified with the classical *stable homotopy category*. We will return to the theory of spectra (using a slightly different definition) in §9.

Example 2.11. Let $\mathcal{A}$ be an abelian category. Under mild hypotheses, we can construct a stable ∞-category $\mathcal{D}(\mathcal{A})$ whose homotopy category $\mathrm{h}\mathcal{D}(\mathcal{A})$ can be identified with the *derived category of* $\mathcal{A}$, in the sense of classical homological algebra. We will outline the construction of $\mathcal{D}(\mathcal{A})$ in §11.

Remark 2.12. If $\mathcal{C}$ is a stable ∞-category, then the opposite ∞-category $\mathcal{C}^{op}$ is also stable.

Remark 2.13. One attractive feature of the theory of stable ∞-categories is that stability is a property of ∞-categories, rather than additional data. The situation for additive categories is similar. Although additive categories are often presented as categories equipped with additional structure (an abelian group structure on all Hom-sets), this additional structure is in fact determined by the underlying category. If a category $\mathcal{C}$ has a zero object, finite sums, and finite products, then there always exists a unique map $A \oplus B \to A \times B$ which can be described by the matrix

$$\begin{bmatrix} \mathrm{id}_A & 0 \\ 0 & \mathrm{id}_B \end{bmatrix}.$$

If this morphism has an inverse $\phi_{A,B}$, then we may define a sum of two morphisms $f, g : X \to Y$ by defining $f+g$ to be the composition $X \to X \times X \stackrel{f,g}{\to} Y \times Y \stackrel{\phi_{Y,Y}}{\to} Y \oplus Y \to Y$. This definition endows each morphism set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ with the structure of a commutative monoid. If each $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is actually a group (in other words, if every morphism $f : X \to Y$ has an additive inverse), then $\mathcal{C}$ is an additive category. This statement has an analogue in the setting of stable ∞-categories: any stable ∞-category $\mathcal{C}$ is automatically enriched over the ∞-category of spectra. Since we do not wish to develop the language of enriched ∞-categories, we will not pursue this point further.

3 The Homotopy Category of a Stable ∞-Category

Our goal in this section is to show that if $\mathcal{C}$ is a stable ∞-category, then the homotopy category $\mathrm{h}\mathcal{C}$ is triangulated (Theorem 3.11). We begin by reviewing the definition of a triangulated category.

Definition 3.1 (Verdier). A *triangulated category* consists of the following data:

(1) An additive category $\mathcal{D}$.

(2) A translation functor

$$\mathcal{D} \to \mathcal{D}$$
$$X \mapsto X[1],$$

which is an equivalence of categories.

(3) A collection of *distinguished triangles*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

These data are required to satisfy the following axioms:

$(TR1)$ (a) Every morphism $f : X \to Y$ in $\mathcal{D}$ can be extended to distinguished triangle in $\mathcal{D}$.

(b) The collection of distinguished triangles is stable under isomorphism.

(c) Given an object $X \in \mathcal{D}$, the diagram

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$$

is a distinguished triangle.

$(TR2)$ A diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle if and only if the rotated diagram

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is a distinguished triangle.

$(TR3)$ Given a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle f} & & \downarrow & & \vdots & & \downarrow{\scriptstyle f[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

in which both horizontal rows are distinguished triangles, there exists a dotted arrow rendering the entire diagram commutative.

$(TR4)$ Suppose given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{u} Y/X \xrightarrow{d} X[1]$$

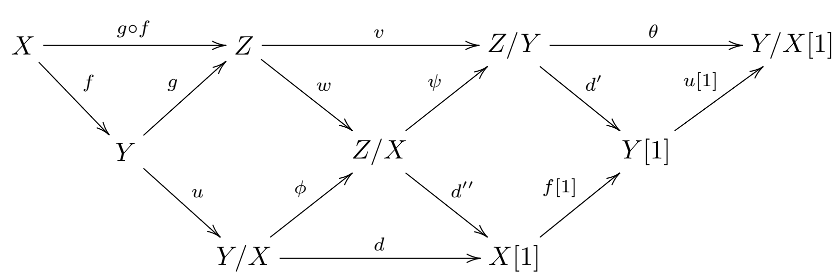
$$Y \xrightarrow{g} Z \xrightarrow{v} Z/Y \xrightarrow{d'} Y[1]$$

$$X \xrightarrow{g\circ f} Z \xrightarrow{w} Z/X \xrightarrow{d''} Z[1]$$

in $\mathcal{D}$. There exists a fourth distinguished triangle

$$Y/X \xrightarrow{\phi} Z/X \xrightarrow{\psi} Z/Y \xrightarrow{\theta} Y/X[1]$$

such that the diagram



commutes.

Remark 3.2. The theory of triangulated categories is an attempt to capture those features of stable ∞-categories which are visible at the level of homotopy categories. Triangulated categories which appear naturally in mathematics are usually equivalent to the homotopy categories of suitable stable ∞-categories.

We now consider the problem of constructing a triangulated structure on the homotopy category of an ∞-category $\mathcal{C}$. To begin the discussion, let us assume that $\mathcal{C}$ is an arbitrary pointed ∞-category. We $\mathcal{M}^{\Sigma}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by those diagrams

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & Y \end{array}$$

which are pushout squares, and such that 0 and $0'$ are zero objects of $\mathcal{C}$. If $\mathcal{C}$ admits cokernels, then we can use Proposition T.4.3.2.15 (twice) to conclude that evaluation at the initial vertex induces a trivial fibration $\mathcal{M}^{\Sigma} \to \mathcal{C}$. Let $s : \mathcal{C} \to \mathcal{M}^{\Sigma}$ be a section of this trivial fibration, and let $e : \mathcal{M}^{\Sigma} \to \mathcal{C}$ be the functor given by evaluation at the final vertex. The composition $e \circ s$ is a functor from $\mathcal{C}$ to itself, which we will denote by $\Sigma : \mathcal{C} \to \mathcal{C}$ and refer to as the *suspension functor* on $\mathcal{C}$. Dually, we define $\mathcal{M}^{\Omega}$ to be the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by diagrams as above which are pullback squares with 0 and $0'$ zero objects of $\mathcal{C}$. If $\mathcal{C}$ admits kernels, then the same argument shows that evaluation at the final vertex induces a trivial fibration $\mathcal{M}^{\Omega} \to \mathcal{C}$. If we let s' denote a section to this trivial fibration, then the composition of s' with evaluation at the initial vertex induces a functor from $\mathcal{C}$ to itself, which we will refer to as the *loop functor* and denote by $\Omega : \mathcal{C} \to \mathcal{C}$. If $\mathcal{C}$ is stable, then $\mathcal{M}^{\Omega} = \mathcal{M}^{\Sigma}$. It follows that Σ and Ω are mutually inverse equivalences from $\mathcal{C}$ to itself.

Remark 3.3. If the ∞-category $\mathcal{C}$ is not clear from context, then we will denote the suspension and loop functors $\Sigma, \Omega : \mathcal{C} \to \mathcal{C}$ by $\Sigma_{\mathcal{C}}$ and $\Omega_{\mathcal{C}}$, respectively.

Notation 3.4. If $\mathcal{C}$ is a stable ∞-category and $n \geq 0$, we let

$$X \mapsto X[n]$$

denote the nth power of the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ constructed above (this functor is well-defined up to canonical equivalence). If $n \leq 0$, we let $X \mapsto X[n]$ denote the $(-n)$th power of the loop functor Ω. We will use the same notation to indicate the induced functors on the homotopy category $\mathrm{h}\mathcal{C}$.

Remark 3.5. If the ∞-category $\mathcal{C}$ is pointed but not necessarily stable, the suspension and loop space functors need not be homotopy inverses but are nevertheless *adjoint* to one another.

If $\mathcal{C}$ is a pointed ∞-category containing a pair of objects X and Y, then the space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ has a natural base point, given by the zero map. Moreover, if $\mathcal{C}$ admits cokernels, then the suspension functor $\Sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is essentially characterized by the existence of natural homotopy equivalences

$$\mathrm{Map}_{\mathcal{C}}(\Sigma(X), Y) \to \Omega \, \mathrm{Map}_{\mathcal{C}}(X, Y).$$

In particular, we conclude that $\pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma(X), Y) \simeq \pi_1 \mathrm{Map}_{\mathcal{C}}(X, Y)$, so that $\pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma(X), Y)$ has the structure of a group (here the fundamental group of $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is taken with base point given by the zero map). Similarly, $\pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma^2(X), Y) \simeq \pi_2 \mathrm{Map}_{\mathcal{C}}(X, Y)$ has the structure of an *abelian group*. If the suspension functor $X \mapsto \Sigma(X)$ is an equivalence of ∞-categories, then for every $Z \in \mathcal{C}$ we can choose X such that $\Sigma^2(X) \simeq Z$ to deduce the existence of an abelian group structure on $\mathrm{Map}_{\mathcal{C}}(Z, Y)$. It is easy to see that this group structure depends functorially on $Z, Y \in \mathrm{h}\mathcal{C}$. We are therefore most of the way to proving the following result:

Lemma 3.6. *Let $\mathcal{C}$ be a pointed ∞-category which admits cokernels, and suppose that the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence. Then $\mathrm{h}\mathcal{C}$ is an additive category.*

Proof. The argument sketched above shows that $\mathrm{h}\mathcal{C}$ is (canonically) enriched over the category of abelian groups. It will therefore suffice to prove that $\mathrm{h}\mathcal{C}$ admits finite coproducts. We will prove a slightly stronger statement: the ∞-category $\mathcal{C}$ itself admits finite coproducts. Since $\mathcal{C}$ has an initial object, it will suffice to treat the case of pairwise coproducts. Let $X, Y \in \mathcal{C}$, and let $\mathrm{coker} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ be a cokernel functor, so that we have equivalences $X \simeq \mathrm{coker}(X[-1] \xrightarrow{u} 0)$ and $Y \simeq \mathrm{coker}\, 0 \xrightarrow{v} Y$. Proposition T.5.1.2.2 implies that u and v admit a coproduct in $\mathrm{Fun}(\Delta^1, \mathcal{C})$ (namely, the zero map $X[-1] \xrightarrow{0} Y$). Since the functor coker preserves coproducts (Remark 2.7), we conclude that X and Y admit a coproduct (which can be constructed as the cokernel of the zero map from $X[-1]$ to Y). □

Let $\mathcal{C}$ be a pointed ∞-category which admits cokernels. By construction, any diagram

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & Y \end{array}$$

which belongs to $\mathcal{M}$ determines a canonical isomorphism $X[1] \to Y$ in the homotopy category $\mathrm{h}\mathcal{C}$. We will need the following observation:

Lemma 3.7. *Let $\mathcal{C}$ be a pointed ∞-category which admits cokernels, and let*

$$\begin{array}{ccc} X & \xrightarrow{f} & 0 \\ \downarrow{\scriptstyle f'} & & \downarrow \\ 0' & \longrightarrow & Y \end{array}$$

be a diagram in $\mathcal{C}$, classifying a morphism $\theta \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X[1], Y)$. (Here 0 and $0'$ are zero objects of $\mathcal{C}$.) Then the transposed diagram

$$\begin{array}{ccc} X & \xrightarrow{f'} & 0' \\ \downarrow{\scriptstyle f} & & \downarrow \\ 0 & \longrightarrow & Y \end{array}$$

classifies the morphism $-\theta \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X[1], Y)$. Here $-\theta$ denotes the inverse of θ with respect to the group structure on $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X[1], Y) \simeq \pi_1 \mathrm{Map}_{\mathcal{C}}(X, Y)$.

Proof. Without loss of generality, we may suppose that $0 = 0'$ and $f = f'$. Let $\sigma : \Lambda^2_0 \to \mathcal{C}$ be the diagram

$$0 \xleftarrow{f} X \xrightarrow{f} 0.$$

For every diagram $p : K \to \mathcal{C}$, let $\mathcal{D}(p)$ denote the Kan complex $\mathcal{C}_{p/} \times_{\mathcal{C}} \{Y\}$. Then $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X[1], Y) \simeq \pi_0 \mathcal{D}(\sigma)$. We note that

$$\mathcal{D}(\sigma) \simeq \mathcal{D}(f) \times_{\mathcal{D}(X)} \mathcal{D}(f).$$

Since 0 is an initial object of $\mathcal{C}$, $\mathcal{D}(f)$ is contractible. In particular, there exists a point $q \in \mathcal{D}(f)$. Let

$$\mathcal{D}' = \mathcal{D}(f) \times_{\mathrm{Fun}(\{0\}, \mathcal{D}(X))} \mathrm{Fun}(\Delta^1, \mathcal{D}(X)) \times_{\mathrm{Fun}(\{1\}, \mathcal{D}(X))} \mathcal{D}(f)$$

$$\mathcal{D}'' = \{q\} \times_{\mathrm{Fun}(\{0\}, \mathcal{D}(X))} \mathrm{Fun}(\Delta^1, \mathcal{D}(X)) \times_{\mathrm{Fun}(\{1\}, \mathcal{D}(X))} \{q\}$$

so that we have canonical inclusions

$$\mathcal{D}'' \hookrightarrow \mathcal{D}' \hookleftarrow \mathcal{D}(\sigma).$$

The left map is a homotopy equivalence because $\mathcal{D}(f)$ is contractible, and the right map is a homotopy equivalence because the projection $\mathcal{D}(f) \to \mathcal{D}(X)$ is a Kan fibration. We observe that $\mathcal{D}''$ can be identified with the simplicial loop space of $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y)$ (taken with the base point determined by q, which we can identify with the zero map from X to Y). Each of the Kan complexes $\mathcal{D}(\sigma)$, $\mathcal{D}'$, $\mathcal{D}''$ is equipped with a canonical involution. On $\mathcal{D}(\sigma)$, this involution corresponds to the transposition of diagrams as in the statement of the lemma. On $\mathcal{D}''$, this involution corresponds to reversal of loops. The desired conclusion now follows from the observation that these involutions are compatible with the inclusions $\mathcal{D}'', \mathcal{D}(\sigma) \subseteq \mathcal{D}'$. $\square$

Definition 3.8. Let $\mathcal{C}$ be a pointed ∞-category which admits cokernels. Suppose given a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in the homotopy category $\mathrm{h}\mathcal{C}$. We will say that this diagram is a *distinguished triangle* if there exists a diagram $\Delta^1 \times \Delta^2 \to \mathcal{C}$ as shown

$$\begin{array}{ccccc} X & \xrightarrow{\widetilde{f}} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \widetilde{g}} & & \downarrow \\ 0' & \longrightarrow & Z & \xrightarrow{\widetilde{h}} & W, \end{array}$$

satisfying the following conditions:

- (i) The objects $0, 0' \in \mathcal{C}$ are zero.
- (ii) Both squares are pushout diagrams in $\mathcal{C}$.
- (iii) The morphisms $\widetilde{f}$ and $\widetilde{g}$ represent f and g, respectively.
- (iv) The map $h : Z \to X[1]$ is the composition of (the homotopy class of) $\widetilde{h}$ with the isomorphism $W \simeq X[1]$ determined by the outer rectangle.

Remark 3.9. We will generally only use Definition 3.8 in the case where $\mathcal{C}$ is a stable ∞-category. However, it will be convenient to have the terminology available in the case where $\mathcal{C}$ is not yet known to be stable.

The following result is an immediate consequence of Lemma 3.7:

Lemma 3.10. *Let $\mathcal{C}$ be a stable ∞-category. Suppose given a diagram $\Delta^2 \times \Delta^1 \to \mathcal{C}$, depicted as*

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow{\scriptstyle f} & & \downarrow \\ Y & \xrightarrow{g} & Z \\ \downarrow & & \downarrow{\scriptstyle h} \\ 0' & \longrightarrow & W, \end{array}$$

where both squares are pushouts and the objects $0, 0' \in \mathcal{C}$ are zero. Then the diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{-h'} X[1]$$

is a distinguished triangle in $\mathrm{h}\mathcal{C}$, where h' denotes the composition of h with the isomorphism $W \simeq X[1]$ determined by the outer square, and $-h'$ denotes the composition of h' with the map $-\mathrm{id} \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X[1], X[1]) \simeq \pi_1 \mathrm{Map}_{\mathcal{C}}(X, X[1])$.

We can now state the main result of this section:

Theorem 3.11. *Let $\mathcal{C}$ be a pointed ∞-category which admits cokernels, and suppose that the suspension functor Σ is an equivalence. Then the translation functor of Notation 3.4 and the class of distinguished triangles of Definition 3.8 endow* $\mathrm{h}\mathcal{C}$ *with the structure of a triangulated category.*

Remark 3.12. The hypotheses of Theorem 3.11 hold whenever $\mathcal{C}$ is stable. In fact, the hypotheses of Theorem 3.11 are *equivalent* to the stability of $\mathcal{C}$: see Corollary 10.12.

Proof. We must verify that Verdier's axioms $(TR1)$ through $(TR4)$ are satisfied.

$(TR1)$ Let $\mathcal{E} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^2, \mathcal{C})$ be the full subcategory spanned by those diagrams

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0' & \longrightarrow & Z & \longrightarrow & W \end{array}$$

of the form considered in Definition 3.8, and let $e : \mathcal{E} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ be the restriction to the upper left horizontal arrow. Repeated use of Proposition T.4.3.2.15 implies e is a trivial fibration. In particular, every morphism $f : X \to Y$ can be completed to a diagram belonging to $\mathcal{E}$. This proves (a). Part (b) is obvious, and (c) follows from the observation that if $f = \mathrm{id}_X$, then the object Z in the above diagram is a zero object of $\mathcal{C}$.

$(TR2)$ Suppose that

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle in $\mathrm{h}\mathcal{C}$, corresponding to a diagram $\sigma \in \mathcal{E}$ as depicted above. Extend σ to a diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0' & \longrightarrow & Z & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle u} \\ 0'' & \longrightarrow & 0''' & \longrightarrow & V \end{array}$$

where the lower right square is a pushout, and the objects $0'', 0''' \in \mathcal{C}$ are zero. We have a map between the squares

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & W \end{array} \qquad \begin{array}{ccc} Y & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0''' & \longrightarrow & V \end{array}$$

which induces a commutative diagram in the homotopy category $\mathrm{h}\mathcal{C}$

$$\begin{array}{ccc} W & \longrightarrow & X[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle f[1]} \\ V & \longrightarrow & Y[1] \end{array}$$

where the horizontal arrows are isomorphisms. Applying Lemma 3.10 to the rectangle on the right of the large diagram, we conclude that

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is a distinguished triangle in $\mathrm{h}\mathcal{C}$.

Conversely, suppose that

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is a distinguished triangle in $\mathrm{h}\mathcal{C}$. Since the functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence, we conclude that the triangle

$$Y[-1] \xrightarrow{g[-1]} Z[-1] \xrightarrow{h[-1]} X \xrightarrow{-f} Y$$

is distinguished. Applying the preceding argument twice, we conclude that the triangle

$$X \xrightarrow{-f} Y \xrightarrow{-g} Z \xrightarrow{-h} X[1]$$

is distinguished. We now conclude by applying $(TR1b)$.

$(TR3)$ Suppose distinguished triangles

$$X \xrightarrow{f} Y \to Z \to X[1]$$

$$X' \xrightarrow{f'} Y' \to Z' \to X'[1]$$

in $\mathrm{h}\mathcal{C}$. Without loss of generality, we may suppose that these triangles are induced by diagrams $\sigma, \sigma' \in \mathcal{E}$. Any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f'} & Y' \end{array}$$

in the homotopy category $\mathrm{h}\mathcal{C}$ can be lifted (nonuniquely) to a square in $\mathcal{C}$, which we may identify with a morphism $\phi : e(\sigma) \to e(\sigma')$ in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$. Since e is a trivial fibration of simplicial sets, ϕ can be lifted to a morphism $\sigma \to \sigma'$ in $\mathcal{E}$, which determines a natural transformation of distinguished triangles

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]. \end{array}$$

$(TR4)$ Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{C}$. In view of the fact that $e : \mathcal{E} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ is a trivial fibration, any distinguished triangle in $\mathrm{h}\mathcal{C}$ beginning with f, g, or $g \circ f$ is uniquely determined up to (nonunique) isomorphism. Consequently, it will suffice to prove that there exist *some* triple of distinguished triangles which satisfies the conclusions of $(TR4)$. To prove this, we construct a diagram in $\mathcal{C}$

$$\begin{array}{ccccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y/X & \longrightarrow & Z/X & \longrightarrow & X' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & Z/Y & \longrightarrow & Y' & \longrightarrow & (Y/X)' \end{array}$$

where 0 is a zero object of $\mathcal{C}$, and each square in the diagram is a pushout (more precisely, we apply Proposition T.4.3.2.15 repeatedly to construct a map from the nerve of the appropriate partially ordered set into $\mathcal{C}$). Restricting to appropriate rectangles contained in the diagram, we obtain isomorphisms $X' \simeq X[1]$, $Y' \simeq Y[1]$, $(Y/X)' \simeq Y/X[1]$, and four distinguished triangles

$$X \xrightarrow{f} Y \to Y/X \to X[1]$$

$$Y \xrightarrow{g} Z \to Z/Y \to Y[1]$$

$$X \xrightarrow{g \circ f} Z \to Z/X \to X[1]$$

$$Y/X \to Z/X \to Z/Y \to Y/X[1].$$

The commutativity in the homotopy category $\mathrm{h}\mathcal{C}$ required by $(TR4)$ follows from the (stronger) commutativity of the above diagram in $\mathcal{C}$ itself.

□

Remark 3.13. The definition of a stable ∞-category is quite a bit simpler than that of a triangulated category. In particular, the octahedral axiom $(TR4)$ is a consequence of ∞-categorical principles which are basic and easily motivated.

Notation 3.14. Let $\mathcal{C}$ be a stable ∞-category containing a pair of objects X and Y. We let $\mathrm{Ext}^n_{\mathcal{C}}(X, Y)$ denote the abelian group $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X[n], Y)$. If n is negative, this can be identified with the homotopy group $\pi_{-n} \mathrm{Map}_{\mathcal{C}}(X, Y)$. More generally, $\mathrm{Ext}^n_{\mathcal{C}}(X, Y)$ can be identified with the $(-n)$th homotopy group of an appropriate *spectrum* of maps from X to Y.

4 Properties of Stable ∞-Categories

According to Definition 2.8, a pointed ∞-category $\mathcal{C}$ is stable if it admits certain pushout squares and certain pullback squares, which are required to coincide with one another. Our goal in this section is to prove that a stable ∞-category $\mathcal{C}$ admits *all* finite limits and colimits, and that the pushout squares in $\mathcal{C}$ coincide with the pullback squares in general (Proposition 4.4). To prove this, we will need the following easy observation (which is quite useful in its own right):

Proposition 4.1. *Let $\mathcal{C}$ be a stable ∞-category, and let K be a simplicial set. Then the ∞-category $\mathrm{Fun}(K, \mathcal{C})$ is stable.*

Proof. This follows immediately from the fact that kernels and cokernels in $\mathrm{Fun}(K, \mathcal{C})$ can be computed pointwise (Proposition T.5.1.2.2). □

Definition 4.2. If $\mathcal{C}$ is stable ∞-category, and $\mathcal{C}_0$ is a full subcategory containing a zero object and stable under the formation of kernels and cokernels, then $\mathcal{C}_0$ is itself stable. In this case, we will say that $\mathcal{C}_0$ is a *stable subcategory* of $\mathcal{C}$.

Lemma 4.3. *Let $\mathcal{C}$ be a stable ∞-category, and let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory which is stable under cokernels and under translation. Then $\mathcal{C}'$ is a stable subcategory of $\mathcal{C}$.*

Proof. It will suffice to show that $\mathcal{C}'$ is stable under kernels. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Theorem 3.11 shows that there is a canonical equivalence $\ker(f) \simeq \mathrm{coker}(f)[-1]$. □

Proposition 4.4. *Let $\mathcal{C}$ be a pointed ∞-category. Then $\mathcal{C}$ is stable if and only if the following conditions are satisfied:*

(1) *The ∞-category $\mathcal{C}$ admits finite limits and colimits.*

(2) *A square*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

in $\mathcal{C}$ is a pushout if and only if it is a pullback.

Proof. Condition (1) implies the existence of kernels and cokernels in $\mathcal{C}$, and condition (2) implies that the exact triangles coincide with the coexact triangles. This proves the "if" direction.

Suppose now that $\mathcal{C}$ is stable. We begin by proving (1). It will suffice to show that $\mathcal{C}$ admits finite colimits; the dual argument will show that $\mathcal{C}$ admits finite limits as well. According to Proposition T.4.4.3.2, it will suffice to show that $\mathcal{C}$ admits coequalizers and finite coproducts. The existence of finite coproducts was established in Lemma 3.6. We now conclude by observing that a coequalizer for a diagram

$$X \underset{f'}{\overset{f}{\rightrightarrows}} Y$$

can be identified with $\mathrm{coker}(f - f')$.

We now show that every pushout square in $\mathcal{C}$ is a pullback; the converse will follow by a dual argument. Let $\mathcal{D} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ be the full subcategory spanned by the pullback squares. Then $\mathcal{D}$ is stable under finite limits and under translations. It follows from Lemma 4.3 that $\mathcal{D}$ is a stable subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$.

Let $i : \Lambda^2_0 \hookrightarrow \Delta^1 \times \Delta^1$ be the inclusion, and let $i_! : \mathrm{Fun}(\Lambda^2_0, \mathcal{C}) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ be a functor of left Kan extension. Then $i_!$ preserves finite colimits, and is therefore exact (Proposition 5.1). Let $\mathcal{D}' = i_!^{-1}\mathcal{D}$. Then $\mathcal{D}'$ is a stable subcategory of $\mathrm{Fun}(\Lambda^2_0, \mathcal{C})$; we wish to show that $\mathcal{D}' = \mathrm{Fun}(\Lambda^2_0, \mathcal{C})$. To prove this, we observe that any diagram

$$X' \leftarrow X \rightarrow X''$$

can be obtained as a (finite) colimit

$$e'_{X'} \coprod_{e'_X} e_X \coprod_{e''_X} e''_{X''}$$

where $e_X \in \mathrm{Fun}(\Lambda^2_0, \mathcal{C})$ denotes the diagram $X \leftarrow X \rightarrow X$, $e'_Z \in \mathrm{Fun}(\Lambda^2_0, \mathcal{C})$ denotes the diagram $Z \leftarrow 0 \rightarrow 0$, and $e''_Z \in \mathrm{Fun}(\Lambda^2_0, \mathcal{C})$ denotes the diagram $0 \leftarrow 0 \rightarrow Z$. It will therefore suffice to prove that pushout of any of these five diagrams is also a pullback. This follows immediately from the following more general observation: any pushout square

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow{\scriptstyle f} & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

in an (arbitrary) ∞-category $\mathcal{C}$ is also pullback square, provided that f is an equivalence. □

5 Exact Functors

Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between stable ∞-categories. Suppose that F carries zero objects into zero objects. It follows immediately that F carries triangles into triangles. If, in addition, F carries exact triangles into exact triangles, then we will say that F is *exact*. The exactness of a functor F admits the following alternative characterizations:

Proposition 5.1. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between stable ∞-categories. The following conditions are equivalent:*

(1) *The functor F is left exact. That is, F commutes with finite limits.*

(2) *The functor F is right exact. That is, F commutes with finite colimits.*

(3) *The functor F is exact.*

Proof. We will prove that (2) $\Leftrightarrow$ (3); the equivalence (1) $\Leftrightarrow$ (3) will follow by a dual argument. The implication (2) $\Rightarrow$ (3) is obvious. Conversely, suppose that F is exact. The proof of Proposition 4.4 shows that F preserves coequalizers, and the proof of Lemma 3.6 shows that F preserves finite coproducts. It follows that F preserves all finite colimits (see the proof of Proposition T.4.4.3.2). □

The identity functor from any stable ∞-category to itself is exact, and a composition of exact functors is exact. Consequently, there exists a subcategory $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}} \subseteq \mathcal{C}\mathrm{at}_\infty$ in which the objects are stable ∞-categories and the morphisms are the exact functors. Our next few results concern the stability properties of this subcategory.

Proposition 5.2. *Suppose given a homotopy Cartesian diagram of ∞-categories*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{G'} & \mathcal{C} \\ \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F} \\ \mathcal{D}' & \xrightarrow{G} & \mathcal{D}. \end{array}$$

Suppose further that $\mathcal{C}$, $\mathcal{D}'$, and $\mathcal{D}$ are stable, and that the functors F and G are exact. Then:

(1) *The ∞-category $\mathcal{C}'$ is stable.*

(2) *The functors F' and G' are exact.*

(3) *If $\mathcal{E}$ is a stable ∞-category, then a functor $H : \mathcal{E} \to \mathcal{C}'$ is exact if and only if the functors $F' \circ H$ and $G' \circ H$ are exact.*

Proof. Combine Proposition 4.4 with Lemma T.5.4.5.5. □

Proposition 5.3. *Let $\{\mathcal{C}_\alpha\}_{\alpha \in A}$ be a collection of stable ∞-categories. Then the product*

$$\mathcal{C} = \prod_{\alpha \in A} \mathcal{C}_\alpha$$

is stable. Moreover, for any stable ∞-category $\mathcal{D}$, a functor $F : \mathcal{D} \to \mathcal{C}$ is exact if and only if each of the compositions

$$\mathcal{D} \xrightarrow{F} \mathcal{C} \to \mathcal{C}_\alpha$$

is an exact functor.

Proof. This follows immediately from the fact that limits and colimits in $\mathcal{C}$ are computed pointwise. □

Theorem 5.4. *The ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$ admits small limits, and the inclusion*

$$\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}} \subseteq \mathcal{C}\mathrm{at}_\infty$$

preserves small limits.

Proof. Using Propositions 5.2 and 5.3, one can repeat the argument used to prove Proposition T.5.4.7.3. □

Proposition 5.5. *The ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$ admits small filtered colimits, and the inclusion $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}} \subseteq \mathcal{C}\mathrm{at}_\infty$ preserves filtered colimits.*

Proof. Let $\mathcal{I}$ be a filtered ∞-category, $p : \mathcal{I} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$ a diagram, which we will indicate by $\{\mathcal{C}_I\}_{I \in \mathcal{I}}$, and $\mathcal{C}$ a colimit of the induced diagram $\mathcal{I} \to \mathcal{C}\mathrm{at}_\infty$. We must prove:

- (i) The ∞-category $\mathcal{C}$ is stable.
- (ii) Each of the canonical functors $\theta_I : \mathcal{C}_I \to \mathcal{C}$ is exact.
- (iii) Given an arbitrary stable ∞-category $\mathcal{D}$, a functor $f : \mathcal{C} \to \mathcal{D}$ is exact if and only if each of the composite functors $\mathcal{C}_I \xrightarrow{\theta_I} \mathcal{C} \to \mathcal{D}$ is exact.

In view of Proposition 5.1, (ii) and (iii) follow immediately from Proposition T.5.5.6.9. The same result implies that $\mathcal{C}$ admits finite limits and colimits, and that each of the functors θ_I preserves finite limits and colimits.

To prove that $\mathcal{C}$ has a zero object, we select an object $I \in \mathcal{I}$. The functor $\mathcal{I} \to \mathcal{C}$ preserves initial and final objects. Since $\mathcal{C}_I$ has a zero object, so does $\mathcal{C}$.

We will complete the proof by showing that every exact triangle in $\mathcal{C}$ is coexact (the converse follows by the same argument). Fix a morphism $f : X \to Y$ in $\mathcal{C}$. Without loss of generality, we may suppose that there exists $I \in \mathcal{I}$ and a morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ in $\mathcal{C}_I$ such that $f = \theta_I(\widetilde{f})$ (Proposition T.5.4.1.2). Form a pullback diagram $\widetilde{\sigma}$

$$\begin{array}{ccc} \widetilde{W} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \widetilde{Y} \end{array}$$

in $\mathcal{C}_I$. Since $\mathcal{C}_I$ is stable, this diagram is also a pushout. It follows that $\theta_I(\sigma)$ is triangle $W \to X \xrightarrow{f} Y$ which is both exact and coexact in $\mathcal{C}$. □

6 t-Structures and Localizations

Let $\mathcal{C}$ be an ∞-category. Recall that we say a full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is a *localization* of $\mathcal{C}$ if the inclusion functor $\mathcal{C}' \subseteq \mathcal{C}$ has a left adjoint (§T.5.2.6). In this section, we will introduce a special class of localizations, called *t-localizations*, in the case where $\mathcal{C}$ is stable. We will further show that there is a bijective correspondence between t-localizations of $\mathcal{C}$ and *t-structures* on the triangulated category $\mathrm{h}\mathcal{C}$. We begin with a review of the classical theory of t-structures; for a more thorough introduction we refer the reader to [3].

Definition 6.1. Let $\mathcal{D}$ be a triangulated category. A *t-structure* on $\mathcal{D}$ is defined to be a pair of full subcategories $\mathcal{D}_{\geq 0}$, $\mathcal{D}_{\leq 0}$ (always assumed to be stable under isomorphism) having the following properties:

(1) For $X \in \mathcal{D}_{\geq 0}$ and $Y \in \mathcal{D}_{\leq -1}$, we have $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$.

(2) $\mathcal{D}_{\geq 0}[1] \subseteq \mathcal{D}_{\geq 0}$, $\mathcal{D}_{\leq 0}[-1] \subseteq \mathcal{D}_{\leq 0}$.

(3) For any $X \in \mathcal{C}$, there exists a distinguished triangle $X' \to X \to X'' \to X'[1]$ where $X' \in \mathcal{D}_{\geq 0}$ and $X'' \in \mathcal{D}_{\leq 0}[-1]$.

Notation 6.2. If $\mathcal{D}$ is a triangulated category equipped with a t-structure, we will write $\mathcal{D}_{\geq n}$ for $\mathcal{D}_{\geq 0}[n]$ and $\mathcal{D}_{\leq n}$ for $\mathcal{D}_{\leq 0}[n]$. Observe that we use a *homological* indexing convention.

Remark 6.3. In Definition 6.1, either of the full subcategories $\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0} \subseteq \mathcal{C}$ determines the other. For example, an object $X \in \mathcal{D}$ belongs to $\mathcal{D}_{\leq -1}$ if and only if $\mathrm{Hom}_{\mathcal{D}}(Y, X)$ vanishes for all $Y \in \mathcal{D}_{\geq 0}$.

Definition 6.4. Let $\mathcal{C}$ be a stable ∞-category. A *t-structure* on $\mathcal{C}$ is a t-structure on the homotopy category $\mathrm{h}\mathcal{C}$. If $\mathcal{C}$ is equipped with a t-structure, we let $\mathcal{C}_{\geq n}$ and $\mathcal{C}_{\leq n}$ denote the full subcategories of $\mathcal{C}$ spanned by those objects which belong to $(\mathrm{h}\mathcal{C})_{\geq \mathrm{n}}$ and $(\mathrm{h}\mathcal{C})_{\leq \mathrm{n}}$, respectively.

Proposition 6.5. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. For each $n \in \mathbf{Z}$, the full subcategory $\mathcal{C}_{\leq n}$ is a localization of $\mathcal{C}$.*

Proof. Without loss of generality, we may suppose $n = -1$. According to Proposition T.5.2.6.7, it will suffice to prove that for each $X \in \mathcal{C}$, there exists a map $f : X \to X''$, where $X'' \in \mathcal{C}_{\leq -1}$ and for each $Y \in \mathcal{C}_{\leq -1}$, the map

$$\mathrm{Map}_{\mathcal{C}}(X'', Y) \to \mathrm{Map}_{\mathcal{C}}(X, Y)$$

is a weak homotopy equivalence. Invoking part (3) of Definition 6.1, we can choose f to fit into a distinguished triangle

$$X' \to X \xrightarrow{f} X'' \to X'[1]$$

where $X' \in \mathcal{C}_{\geq 0}$. According to Whitehead's theorem, we need to show that for every $k \leq 0$, the map

$$\mathrm{Ext}^k_{\mathcal{C}}(X'', Y) \to \mathrm{Ext}^k_{\mathcal{C}}(X, Y)$$

is an isomorphism of abelian groups. Using the long exact sequence associated to the exact triangle above, we are reduced to proving that the groups $\mathrm{Ext}^k_{\mathcal{C}}(X', Y)$ vanish for $k \leq 0$. We now use condition (2) of Definition 6.1 to conclude that $X'[-k] \in \mathcal{C}_{\geq 0}$. Condition (1) of Definition 6.1 now implies that

$$\mathrm{Ext}^k_{\mathcal{C}}(X', Y) \simeq \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X'[-k], Y) \simeq 0.$$

□

Corollary 6.6. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. The full subcategories $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$ are stable under all limits which exist in $\mathcal{C}$. Dually, the full subcategories $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ are stable under all colimits which exist in $\mathcal{C}$.*

Notation 6.7. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. We will let $\tau_{\leq n}$ denote a left adjoint to the inclusion $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$, and $\tau_{\geq n}$ a right adjoint to the inclusion $\mathcal{C}_{\geq n} \subseteq \mathcal{C}$.

Remark 6.8. Fix $n, m \in \mathbf{Z}$, and let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. Then the truncation functors $\tau_{\leq n}$, $\tau_{\geq n}$ map the full subcategory $\mathcal{C}_{\leq m}$ to itself. To prove this, we first observe that $\tau_{\leq n}$ is equivalent to the identity on $\mathcal{C}_{\leq m}$ if $m \leq n$, while if $m \geq n$ the essential image of $\tau_{\leq n}$ is contained in $\mathcal{C}_{\leq n} \subseteq \mathcal{C}_{\leq m}$. To prove the analogous result for $\tau_{\geq n}$, we observe that the proof of Proposition 6.5 implies that for each X, we have a distinguished triangle

$$\tau_{\geq n} X \to X \xrightarrow{f} \tau_{\leq n-1} X \to (\tau_{\geq n} X)[1].$$

If $X \in \mathcal{C}_{\leq m}$, then $\tau_{\leq n-1} X$ also belongs to $\mathcal{C}_{\leq m}$, so that $\tau_{\geq n} X \simeq \ker(f)$ belongs to $\mathcal{C}_{\leq m}$ since $\mathcal{C}_{\leq m}$ is stable under limits.

Warning 6.9. In §T.5.5.5, we introduced for every ∞-category $\mathcal{C}$ a full subcategory $\tau_{\leq n} \mathcal{C}$ of *n-truncated objects* of $\mathcal{C}$. In that context, we used the symbol $\tau_{\leq n}$ to denote a left adjoint to the inclusion $\tau_{\leq n} \mathcal{C} \subseteq \mathcal{C}$. This is *not* compatible with Notation 6.7. In fact, if $\mathcal{C}$ is a stable ∞-category, then it has no nonzero truncated objects at all: if $X \in \mathcal{C}$ is nonzero, then the identity map from X to itself determines a nontrivial homotopy class in $\pi_n \mathrm{Map}_{\mathcal{C}}(X[-n], X)$, for all $n \geq 0$. Nevertheless, the two notations are consistent when restricted to $\mathcal{C}_{\geq 0}$, in view of the following fact:

- Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. An object $X \in \mathcal{C}_{\geq 0}$ is k-truncated (as an object of $\mathcal{C}_{\geq 0}$ if and only if $X \in \mathcal{C}_{\leq k}$.

In fact, we have the following more general statement: for any $X \in \mathcal{C}$ and $k \geq -1$, X belongs to $\mathcal{C}_{\leq k}$ if and only if $\mathrm{Map}_{\mathcal{C}}(Y, X)$ is k-truncated for every $Y \in \mathcal{C}_{\geq 0}$. Because the latter condition is equivalent to the vanishing of $\mathrm{Ext}^n_{\mathcal{C}}(Y, X)$ for $n < -k$, we can use the shift functor to reduce to the case where $n = 0$ and $k = -1$, which is covered by Remark 6.3.

Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure, and let $n, m \in \mathbf{Z}$. Remark 6.8 implies that we have a commutative diagram of simplicial sets

$$\begin{array}{ccc} \mathcal{C}_{\geq n} & \hookrightarrow & \mathcal{C} \\ \downarrow{\scriptstyle \tau_{\leq m}} & & \downarrow{\scriptstyle \tau_{\leq m}} \\ \mathcal{C}_{\geq n} \cap \mathcal{C}_{\leq m} & \hookrightarrow & \mathcal{C}_{\leq m}. \end{array}$$

As explained in §T.7.3.1, we get an induced transformation of functors

$$\theta : \tau_{\leq m} \circ \tau_{\geq n} \to \tau_{\geq n} \circ \tau_{\leq m}.$$

Proposition 6.10. *Let $\mathcal{C}$ be a stable ∞-category equipped with t-structure. Then the natural transformation*

$$\theta : \tau_{\leq m} \circ \tau_{\geq n} \to \tau_{\geq n} \circ \tau_{\leq m}$$

is an equivalence of functors $\mathcal{C} \to \mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n}$.

Proof. This is a classical fact concerning triangulated categories; we include a proof for completeness. Fix $X \in \mathcal{C}$; we wish to show that

$$\theta(X) : \tau_{\leq m}\tau_{\geq n} X \to \tau_{\geq n}\tau_{\leq m} X$$

is an isomorphism in $\mathrm{h}(\mathcal{C}_{\leq \mathrm{m}} \cap \mathcal{C}_{\geq \mathrm{n}})$. If $m < n$, then both sides are zero and there is nothing to prove; let us therefore assume that $m \geq n$. Fix $Y \in \mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n}$; it will suffice to show that composition with $\theta(X)$ induces an isomorphism

$$\mathrm{Ext}^0(\tau_{\geq n}\tau_{\leq m} X, Y) \to \mathrm{Ext}^0(\tau_{\leq m}\tau_{\geq n} X, Y) \simeq \mathrm{Ext}^0(\tau_{\geq n} X, Y).$$

We have a map of long exact sequences

$$\begin{array}{ccc} \mathrm{Ext}^0(\tau_{\leq n-1}\tau_{\leq m} X, Y) & \xrightarrow{f_0} & \mathrm{Ext}^0(\tau_{\leq n-1} X, Y) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^0(\tau_{\leq m} X, Y) & \xrightarrow{f_1} & \mathrm{Ext}^0(X, Y) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^0(\tau_{\geq n}\tau_{\leq m} X, Y) & \xrightarrow{f_2} & \mathrm{Ext}^0(\tau_{\geq n} X, Y) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^1(\tau_{\leq n-1}\tau_{\leq m} X, Y) & \xrightarrow{f_3} & \mathrm{Ext}^1(\tau_{\leq n-1} X, Y) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^1(\tau_{\leq m} X, Y) & \xrightarrow{f_4} & \mathrm{Ext}^1(X, Y). \end{array}$$

Since $m \geq n$, the natural transformation $\tau_{\leq n-1} \to \tau_{\leq n-1}\tau_{\leq m}$ is an equivalence of functors; this proves that f_0 and f_3 are bijective. Since $Y \in \mathcal{C}_{\leq m}$, f_1 is bijective and f_4 is injective. It follows from the "five lemma" that f_2 is bijective, as desired. □

Definition 6.11. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. The *heart* $\mathcal{C}^{\heartsuit}$ of $\mathcal{C}$ is the full subcategory $\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0} \subseteq \mathcal{C}$. For each $n \in \mathbf{Z}$, we let $\pi_0 : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ denote the functor $\tau_{\leq 0} \circ \tau_{\geq 0} \simeq \tau_{\geq 0} \circ \tau_{\leq 0}$, and we let $\pi_n : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ denote the composition of π_0 with the shift functor $X \mapsto X[-n]$.

Remark 6.12. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure, and let $X, Y \in \mathcal{C}^{\heartsuit}$. The homotopy group $\pi_n \operatorname{Map}_{\mathcal{C}}(X,Y) \simeq \operatorname{Ext}^{-n}_{\mathcal{C}}(X,Y)$ vanishes for $n > 0$. It follows that $\mathcal{C}^{\heartsuit}$ is equivalent to (the nerve of) its homotopy category $\mathrm{h}\mathcal{C}^{\heartsuit}$. Moreover, the category $\mathrm{h}\mathcal{C}^{\heartsuit}$ is abelian ([3]).

Let $\mathcal{C}$ be a stable ∞-category. In view of Remark 6.3, t-structures on $\mathcal{C}$ are determined by the corresponding localizations $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$. However, not every localization of $\mathcal{C}$ arises in this way. Recall (see §T.5.5.4) that every localization of $\mathcal{C}$ has the form $S^{-1}\mathcal{C}$, where S is an appropriate collection of morphisms of $\mathcal{C}$. Here $S^{-1}\mathcal{C}$ denotes the full subcategory of $\mathcal{C}$ spanned by S-local objects, where an object $X \in \mathcal{C}$ is said to be S-local if and only if, for each $f : Y' \to Y$ in S, composition with f induces a homotopy equivalence

$$\operatorname{Map}_{\mathcal{C}}(Y,X) \to \operatorname{Map}_{\mathcal{C}}(Y',X).$$

If $\mathcal{C}$ is stable, then we extend the morphism f to a distinguished triangle

$$Y' \to Y \to Y'' \to Y'[1],$$

and we have an associated long exact sequence

$$\ldots \to \operatorname{Ext}^i_{\mathcal{C}}(Y'',X) \to \operatorname{Ext}^i_{\mathcal{C}}(Y,X) \xrightarrow{\theta_i} \operatorname{Ext}^i_{\mathcal{C}}(Y',X) \to \operatorname{Ext}^{i+1}_{\mathcal{C}}(Y'',X) \to \ldots$$

The requirement that X be $\{f\}$-local amounts to the condition that θ_i be an isomorphism for $i \leq 0$. Using the long exact sequence, we see that if X is $\{f\}$-local, then $\operatorname{Ext}^i_{\mathcal{C}}(Y'',X) = 0$ for $i \leq 0$. Conversely, if $\operatorname{Ext}^i_{\mathcal{C}}(Y'',X) = 0$ for $i \leq 1$, then X is $\{f\}$-local. Experience suggests that it is usually more natural to require the vanishing of the groups $\operatorname{Ext}^i_{\mathcal{C}}(Y'',X)$ than it is to require that the maps θ_i to be isomorphisms. Of course, if Y' is a zero object of $\mathcal{C}$, then the distinction between these conditions disappears.

Definition 6.13. Let $\mathcal{C}$ be a category which admits pushouts. We will say that a collection S of morphisms of $\mathcal{C}$ is *weakly saturated* if it satisfies the following conditions:

(1) Every equivalence in $\mathcal{C}$ belongs to S.

(2) Given a 2-simplex $\Delta^2 \to \mathcal{C}$

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle g} & \\ & Y, & \end{array}$$

if any two of f and g belongs to S, then so does the third.

(3) Given a pushout diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y', \end{array}$$

if $f \in S$, then $f' \in S$.

Any intersection of weakly saturated collections of morphisms is weakly saturated. Consequently, for any collection of morphisms S there is a smallest weakly saturated collection $\overline{S}$ containing S. We will say that $\overline{S}$ is the *weakly saturated collection of morphisms generated by* S.

Definition 6.14. Let $\mathcal{C}$ be a stable ∞-category. A full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is *closed under extensions* if, for every distinguished triangle

$$X \to Y \to Z \to X[1]$$

such that X and Z belong to $\mathcal{C}'$, the object Y also belongs to $\mathcal{C}'$.

We observe that if $\mathcal{C}$ is as in Definition 6.13 and $L : \mathcal{C} \to \mathcal{C}$ is a localization functor, then the collection of all morphisms f of $\mathcal{C}$ such that $L(f)$ is an equivalence is weakly saturated.

Proposition 6.15. *Let $\mathcal{C}$ be a stable ∞-category, let $L : \mathcal{C} \to \mathcal{C}$ be a localization functor, and let S be the collection of morphisms f in $\mathcal{C}$ such that $L(f)$ is an equivalence. The following conditions are equivalent:*

(1) *There exists a collection of morphisms $\{f : 0 \to X\}$ which generates S (as a weakly saturated collection of morphisms).*

(2) *The collection of morphisms $\{0 \to X : L(X) \simeq 0\}$ generates S (as a weakly saturated collection of morphisms).*

(3) *The essential image of L is closed under extensions.*

(4) *For any $A \in \mathcal{C}$, $B \in L\,\mathcal{C}$, the natural map $\mathrm{Ext}^1(LA, B) \to \mathrm{Ext}^1(A, B)$ is injective.*

(5) *The full subcategories $\mathcal{C}_{\geq 0} = \{A : LA \simeq 0\}$ and $\mathcal{C}_{\leq -1} = \{A : LA \simeq A\}$ determine a t-structure on $\mathcal{C}$.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious. We next prove that (2) $\Rightarrow$ (3). Suppose given an exact triangle

$$X \to Y \to Z$$

where X and Z are both S-local. We wish to prove that Y is S-local. In view of assumption (2), it will suffice to show that $\mathrm{Map}_{\mathcal{C}}(A, Y)$ is contractible, provided that $L(A) \simeq 0$. In other words, we must show that $\mathrm{Ext}^i_{\mathcal{C}}(A, Y) \simeq 0$ for $i \leq 0$. We now observe that there is an exact sequence

$$\mathrm{Ext}^i_{\mathcal{C}}(A, X) \to \mathrm{Ext}^i_{\mathcal{C}}(A, Y) \to \mathrm{Ext}^i_{\mathcal{C}}(A, Z)$$

where the outer groups vanish, since X and Z are S-local and the map $0 \to A$ belongs to S.

We next show that (3) $\Rightarrow$ (4). Let $B \in L\,\mathcal{C}$, and let $\eta \in \mathrm{Ext}^1_{\mathcal{C}}(LA, B)$ classify a distinguished triangle

$$B \to C \xrightarrow{g} LA \xrightarrow{\eta} B[1].$$

Condition (3) implies that $C \in L\,\mathcal{C}$. If the image of η in $\mathrm{Ext}^1_{\mathcal{C}}(A, B)$ is trivial, then the localization map $A \to LA$ factors as a composition

$$A \xrightarrow{f} C \xrightarrow{g} LA.$$

Applying L to this diagram (and using the fact that C is local) we conclude that the map g admits a section, so that $\eta = 0$.

We now claim that (4) $\Rightarrow$ (5). Assume (4), and define $\mathcal{C}_{\geq 0}$, $\mathcal{C}_{\leq -1}$ as in (5). We will show that the axioms of Definition 6.1 are satisfied:

- If $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq -1}$, then $\mathrm{Ext}^0_{\mathcal{C}}(X, Y) \simeq \mathrm{Ext}^0_{\mathcal{C}}(LX, Y) \simeq \mathrm{Ext}^0_{\mathcal{C}}(0, Y) \simeq 0$.

- Since $\mathcal{C}_{\leq -1}$ is a localization of $\mathcal{C}$, it is stable under limits, so that $\mathcal{C}_{\leq -1}[-1] \subseteq \mathcal{C}_{\leq -1}$. Similarly, since the functor $L : \mathcal{C} \to \mathcal{C}_{\leq -1}$ preserves all colimits which exist in $\mathcal{C}$, the subcategory $\mathcal{C}_{\geq 0}$ is stable under finite colimits, so that $\mathcal{C}_{\geq 0}[1] \subseteq \mathcal{C}_{\geq 0}$.

- Let $X \in \mathcal{C}$, and form a distinguished triangle

$$X' \to X \to LX \to X'[1].$$

We claim that $X' \in \mathcal{C}_{\geq 0}$; in other words, that $LX' = 0$. For this, it suffices to show that for all $Y \in L\,\mathcal{C}$, the morphism space

$$\mathrm{Ext}^0_{\mathcal{C}}(LX', Y) = 0.$$

Since Y is local, we have isomorphisms

$$\mathrm{Ext}^0_{\mathcal{C}}(LX', Y) \simeq \mathrm{Ext}^0_{\mathcal{C}}(X', Y) \simeq \mathrm{Ext}^1_{\mathcal{C}}(X'[1], Y).$$

We now observe that there is a long exact sequence

$$\mathrm{Ext}^0(LX, Y) \xrightarrow{f} \mathrm{Ext}^0(X, Y) \to \mathrm{Ext}^1_{\mathcal{C}}(X'[1], Y) \to \mathrm{Ext}^1_{\mathcal{C}}(LX, Y) \xrightarrow{f'} \mathrm{Ext}^1_{\mathcal{C}}(X, Y).$$

Here f is bijective (since Y is local) and f' is injective (in virtue of assumption (4)).

We conclude by showing that (5) $\Rightarrow$ (1). Let S' be the smallest weakly saturated collection of morphisms which contains the zero map $0 \to A$, for every $A \in \mathcal{C}_{\geq 0}$. We wish to prove that $S = S'$. For this, we choose an arbitrary morphism $u : X \to Y$ belonging to S. Then $Lu : LX \to LY$ is an equivalence, so we have a pushout diagram

$$\begin{array}{ccc} X' & \xrightarrow{u'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{u} & Y, \end{array}$$

where X' and Y' are kernels of the respective localization maps $X \to LX$, $Y \to LY$. Consequently, it will suffice to prove that $u' \in S'$. Since $X', Y' \in \mathcal{C}_{\geq 0}$, this follows from the two-out-of-three property, applied to the diagram

$$\begin{array}{ccccc} & & X' & & \\ & \nearrow & & \searrow^{u'} & \\ 0 & & \longrightarrow & & Y'. \end{array}$$

$\square$

7 Boundedness and Completeness

Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. We let $\mathcal{C}^- = \bigcup \mathcal{C}_{\leq n} \subseteq \mathcal{C}$, $\mathcal{C}^+ = \bigcup \mathcal{C}_{\geq -n}$, and $\mathcal{C}^b = \mathcal{C}^- \cap \mathcal{C}^+$. It is easy to see that $\mathcal{C}^+$, $\mathcal{C}^-$, and $\mathcal{C}^b$ are stable subcategories of $\mathcal{C}$. We will say that $\mathcal{C}$ is *left bounded* if $\mathcal{C} = \mathcal{C}^-$, *right bounded* if $\mathcal{C} = \mathcal{C}^+$, and *bounded* if $\mathcal{C} = \mathcal{C}^b$.

At the other extreme, given a stable ∞-category $\mathcal{C}$ equipped with a t-structure, we define the *left completion* $\widehat{\mathcal{C}}$ of $\mathcal{C}$ to be homotopy limit limit of the tower

$$\ldots \to \mathcal{C}_{\leq 2} \overset{\tau_{\leq 1}}{\to} \mathcal{C}_{\leq 1} \overset{\tau_{\leq 0}}{\to} \mathcal{C}_{\leq 0} \overset{\tau_{\leq -1}}{\to} \ldots$$

Using the results of §T.3.3.4, we can obtain a very concrete description of this inverse limit: it is the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbf{Z}), \mathcal{C})$ spanned by those functors $F : \mathrm{N}(\mathbf{Z}) \to \mathcal{C}$ with the following properties:

(1) For each $n \in \mathbf{Z}$, $F(n) \in \mathcal{C}_{\leq -n}$.

(2) For each $m \leq n \in \mathbf{Z}$, the associated map $F(m) \to F(n)$ induces an equivalence $\tau_{\leq -n} F(m) \to F(n)$.

We will denote this inverse limit by $\widehat{\mathcal{C}}$, and refer to it as the *left completion* of $\mathcal{C}$.

Proposition 7.1. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. Then:*

(1) *The left completion $\widehat{\mathcal{C}}$ is also stable.*

(2) *Let $\widehat{\mathcal{C}}_{\leq 0}$ and $\widehat{\mathcal{C}}_{\geq 0}$ be the full subcategories of $\widehat{\mathcal{C}}$ spanned by those functors $F : \mathrm{N}(\mathbf{Z}) \to \mathcal{C}$ which factor through $\mathcal{C}_{\leq 0}$ and $\mathcal{C}_{\geq 0}$, respectively. Then these subcategories determine a t-structure on $\widehat{\mathcal{C}}$.*

(3) *There is a canonical functor* $\mathcal{C} \to \widehat{\mathcal{C}}$*. This functor is exact, and induces an equivalence* $\mathcal{C}_{\leq 0} \to \widehat{\mathcal{C}}_{\leq 0}$*.*

Proof. We observe that $\widehat{\mathcal{C}}$ can be identified with the homotopy inverse limit of the tower

$$\ldots \to \mathcal{C}_{\leq 0} \stackrel{\tau_{\leq 0}\Sigma}{\to} \mathcal{C}_{\leq 0} \stackrel{\tau_{\leq 0}\Sigma}{\to} \mathcal{C}_{\leq 0}\,.$$

In other words, $\widehat{\mathcal{C}}^{op} \simeq \mathrm{Stab}(\mathcal{C}^{op})$ (see §11). Assertion (1) now follows from Proposition 10.10.

We next prove (2). We begin by observing that, if we identify $\widehat{\mathcal{C}}$ with a full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbf{Z}), \mathcal{C})$, then the shift functors on $\widehat{\mathcal{C}}$ can be defined by the formula

$$(F[n])(m) = F(m-n)[n].$$

This proves immediately that $\widehat{\mathcal{C}}_{\geq 0}[1] \subseteq \widehat{\mathcal{C}}_{\geq 0}$ and $\widehat{\mathcal{C}}_{\leq 0}[-1] \subseteq \widehat{\mathcal{C}}_{\leq 0}$. Moreover, if $X \in \widehat{\mathcal{C}}_{\geq 0}$ and $Y \in \widehat{\mathcal{C}}_{\leq -1} = \widehat{\mathcal{C}}_{\leq 0}[-1]$, then $\mathrm{Map}_{\widehat{\mathcal{C}}}(X, Y)$ can be identified with the homotopy limit of a tower of spaces

$$\ldots \to \mathrm{Map}_{\mathcal{C}}(X(n), Y(n)) \to \mathrm{Map}_{\mathcal{C}}(X(n-1), Y(n-1)) \to \ldots$$

Since each of these spaces is contractible, we conclude that $\mathrm{Map}_{\widehat{\mathcal{C}}}(X, Y) \simeq *$; in particular, $\mathrm{Ext}^0_{\widehat{\mathcal{C}}}(X, Y) = 0$. Finally, we consider an arbitrary $X \in \widehat{\mathcal{C}}$. Let $X'' = \tau_{\leq -1} \circ X : \mathrm{N}(\mathbf{Z}) \to \mathcal{C}$, and let $u : X \to X''$ be the induced map. It is easy to check that $X'' \in \widehat{\mathcal{C}}_{\leq -1}$ and that $\ker(u) \in \widehat{\mathcal{C}}_{\geq 0}$. This completes the proof of (2).

To prove (3), we let $\mathcal{D}$ denote the full subcategory of $\mathrm{N}(\mathbf{Z}) \times \mathcal{C}$ spanned by pairs (n, C) such that $C \in \mathcal{C}_{\leq -n}$. Using Proposition T.5.2.6.7, we deduce that the inclusion $\mathcal{D} \subseteq \mathrm{N}(\mathbf{Z}) \times \mathcal{C}$ admits a left adjoint L. The composition

$$\mathrm{N}(\mathbf{Z}) \times \mathcal{C} \stackrel{L}{\to} \mathcal{D} \subseteq \mathrm{N}(\mathbf{Z}) \times \mathcal{C}) \to \mathcal{C}$$

can be identified with a functor $\theta : \mathcal{C} \to \mathrm{Fun}(\mathrm{N}(\mathbf{Z}), \mathcal{C})$ which factors through $\widehat{\mathcal{C}}$. To prove that θ is exact, it suffices to show that θ is right exact (Proposition 5.1). Since the truncation functors $\tau_{\leq n} : \mathcal{C}_{\leq n+1} \to \mathcal{C}_{\leq n}$ are right exact, finite colimits in $\widehat{\mathcal{C}}$ are computed pointwise. Consequently, it suffices to prove that each of compositions

$$\mathcal{C} \stackrel{\theta}{\to} \widehat{\mathcal{C}} \to \tau_{\leq n}\, \mathcal{C}$$

is right exact. But this composition can be identified with the functor $\tau_{\leq n}$.

Finally, we observe that $\widehat{\mathcal{C}}_{\leq 0}$ can be identified with a homotopy limit of the essentially constant tower

$$\ldots \mathcal{C}_{\leq 0} \stackrel{\mathrm{id}}{\to} \mathcal{C}_{\leq 0} \stackrel{\mathrm{id}}{\to} \mathcal{C}_{\leq 0} \stackrel{\tau_{\leq -1}}{\to} \mathcal{C}_{\leq -1} \to \ldots,$$

and that θ induces an identification of this homotopy limit with $\mathcal{C}_{\leq 0}$. □

If $\mathcal{C}$ is a stable ∞-category equipped with a t-structure, then we will say that $\mathcal{C}$ is *left complete* if the functor $\mathcal{C} \to \widehat{\mathcal{C}}$ described in Proposition 7.1 is an equivalence.

Remark 7.2. Let $\mathcal{C}$ be as in Proposition 7.1. Then the inclusion $\mathcal{C}^{-} \subseteq \mathcal{C}$ induces an equivalence $\widehat{\mathcal{C}^{-}} \to \widehat{\mathcal{C}}$, and the functor $\mathcal{C} \to \widehat{\mathcal{C}}$ induces an equivalence $\mathcal{C}^{-} \to \widehat{\mathcal{C}}^{-}$. Consequently, the constructions

$$\mathcal{C} \mapsto \widehat{\mathcal{C}}$$

$$\mathcal{C} \mapsto \mathcal{C}^{-}$$

furnish an equivalence between the theory of left bounded stable ∞-categories and the theory of left complete stable ∞-categories.

We conclude this section with a useful criterion for establishing left completeness.

Proposition 7.3. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. Suppose that $\mathcal{C}$ admits countable products, and that $\mathcal{C}_{\geq 0}$ is stable under countable products. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is left complete.*

(2) *The full subcategory $\mathcal{C}_{\geq\infty} = \bigcap \mathcal{C}_{\geq n} \subseteq \mathcal{C}$ consists only of zero objects of $\mathcal{C}$.*

Proof. We first observe every tower of objects

$$\ldots \to X_n \to X_{n-1} \to \ldots$$

in $\mathcal{C}$ admits a limit $\varprojlim\{X_n\}$: we can compute this limit as the kernel of an appropriate map

$$\prod X_n \to \prod X_n.$$

Moreover, if each X_n belongs to $\mathcal{C}_{\geq 0}$, then $\varprojlim\{X_n\}$ belongs to $\mathcal{C}_{\geq -1}$.

The functor $F : \mathcal{C} \to \widehat{\mathcal{C}}$ of Proposition 7.1 admits a right adjoint G, given by

$$f \in \widehat{\mathcal{C}} \subseteq \operatorname{Fun}(\mathrm{N}(\mathbf{Z}), \mathcal{C}) \mapsto \varprojlim(f).$$

Assrtion (1) is equivalent to the statement that the unit and counit maps

$$u : F \circ G \to \mathrm{id}_{\widehat{\mathcal{C}}}$$

$$v : \mathrm{id}_{\mathcal{C}} \to G \circ F$$

are equivalences. If v is an equivalence, then any object $X \in \mathcal{C}$ can be recovered as the limit of the tower $\{\tau_{\leq n} X\}$. In particular, this implies that $X = 0$ if $X \in \mathcal{C}_{\geq\infty}$, so that (1) $\Rightarrow$ (2).

Now assume (2); we will prove that u and v are both equivalences. To prove that u is an equivalence, we must show that for every $f \in \widehat{\mathcal{C}}$, the natural map

$$\theta : \varprojlim(f) \to f(n)$$

induces an equivalence $\tau_{\leq -n} \varprojlim(f) \to f(n)$. In other words, we must show that the kernel of θ belongs to $\mathcal{C}_{\geq -n+1}$. To prove this, we first observe that θ factors as a composition

$$\varprojlim(f) \xrightarrow{\theta'} f(n-1) \xrightarrow{\theta''} f(n).$$

The octahedral axiom ($(TR4)$ of Definition 3.1) implies the existence of an exact triangle

$$\ker(\theta') \to \ker(\theta) \to \ker(\theta'').$$

Since $\ker(\theta'')$ clearly belongs to $\mathcal{C}_{\geq -n+1}$, it will suffice to show that $\ker(\theta')$ belongs to $\mathcal{C}_{\geq -n+1}$. We observe that $\ker(\theta')$ can be identified with the limit of a tower $\{\ker(f(m) \to f(n-1))\}_{m<n}$. It therefore suffices to show that each $\ker(f(m) \to f(n-1))$ belongs to $\mathcal{C}_{\geq -n+2}$, which is clear.

We now show prove that v is an equivalence. Let X be an object of $\mathcal{C}$, and $v_X : X \to (G \circ F)(X)$ the associated map. Since u is an equivalence of functors, we conclude that $\tau_{\leq n}(v_X)$ is an equivalence for all $n \in \mathbf{Z}$. It follows that $\operatorname{coker}(v_X) \in \mathcal{C}_{\geq n+1}$ for all $n \in \mathbf{Z}$. Invoking assumption (2), we conclude that $\operatorname{coker}(v_X) \simeq 0$, so that v_X is an equivalence as desired. □

Remark 7.4. The ideas introduced above can be dualized in an obvious way, so that we can speak of *right completions* and *right completeness* for a stable ∞-category equipped with a t-structure.

8 The Stability of Ind-Categories

The purpose of this section is to prove that if $\mathcal{C}$ is a stable ∞-category, then the ∞-category $\mathrm{Ind}(\mathcal{C})$ is also stable. To explain the idea of the proof, let us proceed immediately to the most difficult step: we must show that a square $\sigma : \Delta^1 \times \Delta^1 \to \mathrm{Ind}(\mathcal{C})$ is Cartesian if and only if it is coCartesian. The idea is deduce this from the stability of $\mathcal{C}$, by approximating σ by squares $\sigma_\alpha : \Delta^1 \times \Delta^1 \to \mathcal{C}$. The existence of the desired approximations follows from the following technical result:

Lemma 8.1. *Let $\mathcal{C}$ be a small ∞-category, κ a regular cardinal, and $j : \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$ the Yoneda embedding. Let A be a finite partially ordered set, and let $j' : \mathrm{Fun}(\mathrm{N}(A), \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(A), \mathrm{Ind}_\kappa(\mathcal{C}))$ be the induced map. Then j' induces an equivalence*

$$\mathrm{Ind}_\kappa(\mathrm{Fun}(\mathrm{N}(A), \mathcal{C})) \to \mathrm{Fun}(\mathrm{N}(A), \mathrm{Ind}_\kappa(\mathcal{C})).$$

We will defer a proof until the end of this section.

Warning 8.2. The statement of Lemma 8.1 fails if we replace $\mathrm{N}(A)$ by an arbitrary finite simplicial set. For example, we may identify the category of abelian groups with the category of Ind-objects of the category of finitely generated abelian groups. If $n > 1$, then the map $q \mapsto \frac{q}{n}$ from the group of rational numbers $\mathbf{Q}$ to itself cannot be obtained as a filtered colimit of endomorphisms finitely generated abelian groups.

We now prove our main result.

Proposition 8.3. *Let $\mathcal{C}$ be a (small) stable ∞-category, and let κ be a regular cardinal. Then the ∞-category $\mathrm{Ind}_\kappa(\mathcal{C})$ is stable, and the Yoneda embedding $j : \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$ is exact.*

Proof. The functor j preserves finite limits and colimits (Propositions T.5.1.3.2 and T.5.3.5.14). It follows that $j(0)$ is a zero object of $\mathrm{Ind}_\kappa(\mathcal{C})$, so that $\mathrm{Ind}_\kappa(\mathcal{C})$ is pointed.

We next show that every morphism $f : X \to Y$ in $\mathrm{Ind}_\kappa(\mathcal{C})$ admits a kernel and a cokernel. According to Lemma 8.1, we may assume that f is a κ-filtered colimit of morphisms $f_\alpha : X_\alpha \to Y_\alpha$ which belong to the essential image $\mathcal{C}'$ of j. Since j preserves kernels and cokernels, each of the maps f_α has a kernel and a cokernel in Ind_κ. It follows immediately that f has a cokernel (which can be written as a colimit of the cokernels of the maps f_α). The existence of $\ker(f)$ is slightly more difficult. Choose a κ-filtered diagram $p : \mathcal{I} \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}')$, where each $p(\alpha)$ is a pullback square

$$\begin{array}{ccc} Z_\alpha & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha. \end{array}$$

Let σ be a colimit of the diagram p; we wish to show that σ is a pullback diagram in $\mathrm{Ind}_\kappa(\mathcal{C})$. Since $\mathrm{Ind}_\kappa(\mathcal{C})$ is stable under κ-small limits in $\mathcal{P}(\mathcal{C})$, it will suffice to show that σ is a pullback square in $\mathcal{P}(\mathcal{C})$. Since $\mathcal{P}(\mathcal{C})$ is an ∞-topos, filtered colimits in $\mathcal{P}(\mathcal{C})$ are left exact (Example T.7.3.4.7); it will therefore suffice to show that each $p(\alpha)$ is a pullback diagram in $\mathcal{P}(\mathcal{C})$. This is obvious, since the inclusion $\mathcal{C}' \subseteq \mathcal{P}(\mathcal{C})$ preserves all limits which exist in $\mathcal{C}'$ (Proposition T.5.1.3.2).

To complete the proof, we must show that a triangle in $\mathrm{Ind}_\kappa(\mathcal{C})$ is exact if and only if it is coexact. Suppose given an exact triangle

$$\begin{array}{ccc} Z & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

in $\mathrm{Ind}_\kappa(\mathcal{C})$. The above argument shows that we can write this triangle as a filtered colimit of exact triangles

$$\begin{array}{ccc} Z_\alpha & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ X_\alpha & \longrightarrow & Y_\alpha \end{array}$$

in $\mathcal{C}'$. Since $\mathcal{C}'$ is stable, we conclude that these triangles are also coexact. The original triangle is therefore a filtered colimit of coexact triangles in $\mathcal{C}'$, hence coexact. The converse follows by the same argument. $\square$

Proof of Lemma 8.1. According to Proposition T.5.3.5.11, it will suffice to prove the following:

(i) The functor j' is fully faithful.

(ii) The essential image of j' consists of κ-compact objects of $\mathrm{Fun}(\mathrm{N}(A), \mathrm{Ind}_\kappa(\mathcal{C}))$.

(iii) The essential image of j' generates $\mathrm{Fun}(\mathrm{N}(A), \mathrm{Ind}_\kappa(\mathcal{C}))$ under small, κ-filtered colimits.

Since the Yoneda embedding $j : \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$ satisfies the analogues of these conditions, (i) is obvious and (ii) follows from Proposition T.5.3.4.13. To prove (iii), we fix an object $F \in \mathrm{Fun}(\mathrm{N}(A), \mathrm{Ind}_\kappa(\mathcal{C}))$. Let $\mathcal{C}'$ denote the essential image of j, and form a pullback diagram of simplicial sets

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathrm{Fun}(\mathrm{N}(A), \mathcal{C}') \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathrm{N}(A), \mathrm{Ind}_\kappa(\mathcal{C}))_{/F} & \longrightarrow & \mathrm{Fun}(\mathrm{N}(A), \mathrm{Ind}_\kappa(\mathcal{C})) \end{array} .$$

Since $\mathcal{D}$ is essentially small, (iii) is a consequence of the following assertions:

(a) The ∞-category $\mathcal{D}$ is κ-filtered.

(b) The canonical map $\mathcal{D}^{\triangleright} \to \mathrm{Fun}(\mathrm{N}(A), \mathcal{C})$ is a colimit diagram.

To prove (a), it will to show that $\mathcal{D}$ has the right lifting property with respect to the inclusion $\mathrm{N}(B) \subseteq \mathrm{N}(B \cup \{\infty\})$, for every κ-small partially ordered set B (Remark T.5.3.1.10). Regard $B \cup \{\infty, \infty'\}$ as a partially ordered set with $b < \infty < \infty'$ for each $b \in B$. Unwinding the definitions, we see that (a) is equivalent to the following assertion:

(a') Let $\overline{F} : \mathrm{N}(A \times (B \cup \{\infty'\})) \to \mathrm{Ind}_\kappa(\mathcal{C})$ be such that $\overline{F}|\mathrm{N}(A \times \{\infty'\}) = F$, and $\overline{F}'|\mathrm{N}(A \times B)$ factors through $\mathcal{C}'$. Then there exists a map $\overline{F}' : \mathrm{N}(A \times (B \cup \{\infty, \infty'\})) \to \mathrm{Ind}_\kappa(\mathcal{C})$ which extends $\overline{F}$, such that $\overline{F}'|\mathrm{N}(A \times (B \cup \{\infty\}))$ factors through $\mathcal{C}'$.

To find $\overline{F}'$, we write $A = \{a_1, \ldots, a_n\}$, where $a_i \leq a_j$ implies $i \leq j$. We will construct a compatible sequence of maps

$$\overline{F}_k : \mathrm{N}((A \times (B \cup \{\infty'\})) \cup (\{a_1, \ldots, a_k\} \times \{\infty\})) \to \mathcal{C}$$

with $\overline{F}_0 = \overline{F}$ and $\overline{F}_n = \overline{F}'$. For each $a \in A$, we let $A_{\leq a} = \{a' \in A : a' \leq a\}$, and we define $A_{<a}$, $A_{\geq a}$, $A_{>a}$ similarly. Supposing that $\overline{F}_{k-1}$ has been constructed, we observe that constructing $\overline{F}_k$ amounts to constructing an object of the ∞-category

$$(\mathcal{C}'_{/F|\mathrm{N}(A_{\geq a_k})})_{\overline{F}_{k-1}|M/},$$

where $M = (A_{\leq a_k} \times B) \cup (A_{<a_k} \times \{\infty\})$. The inclusion $\{a_k\} \subseteq \mathrm{N}(A_{\geq a_k})$ is left anodyne. It will therefore suffice to construct an object in the equivalent ∞-category $(\mathcal{C}'_{/F(a_k)})_{\overline{F}_{k-1}|M/}$. Since M is κ-small, it suffices to show that the ∞-category $\mathcal{C}'_{/F(a_k)}$ is κ-filtered. This is simply a reformulation of the fact that $F(a_k) \in \mathrm{Ind}_\kappa(\mathcal{C})$.

We now prove (b). It will suffice to show that for each $a \in A$, the composition

$$\mathcal{D}^{\triangleright} \to \mathrm{Fun}(\mathrm{N}(A), \mathrm{Ind}_{\kappa}(\mathcal{C})) \to \mathrm{Ind}_{\kappa}(\mathcal{C})$$

is a colimit diagram, where the second map is given by evaluation at a. Let $\mathcal{D}(a) = \mathcal{C}' \times_{\mathrm{Ind}_{\kappa}(\mathcal{C})} \mathrm{Ind}_{\kappa}(\mathcal{C})_{/F(a)}$, so that $\mathcal{D}(a)$ is κ-filtered and the associated map $\mathcal{D}(a)^{\triangleright} \to \mathrm{Ind}_{\kappa}(\mathcal{C})$ is a colimit diagram. It will therefore suffice to show that the canonical map $\mathcal{D} \to \mathcal{D}(a)$ is cofinal. According to Theorem T.4.1.3.1, it will suffice to show that for each object $D \in \mathcal{D}(a)$, the fiber product $\mathcal{E} = \mathcal{D} \times_{\mathcal{D}(a)} \mathcal{D}(a)_{D/}$ is weakly contractible. In view of Lemma T.5.3.1.18, it will suffice to show that $\mathcal{E}$ is filtered. This can be established by a minor variation of the argument given above. □

9 The ∞-Category of Spectra

In this section, we will discuss what is perhaps the most important example of a stable ∞-category: the ∞-category of spectra. In classical homotopy theory, one defines a spectrum to be a sequence of pointed spaces $\{X_n\}_{n \geq 0}$, equipped with homotopy equivalences (or homeomorphisms, depending on the author) $X_n \to \Omega(X_{n+1})$ for all $n \geq 0$. Here Ω denotes the functor which associates to each pointed space X the (based) loop space $\Omega(X)$. In some sense, the theory of spectra is obtained from the theory of pointed spaces by formally inverting the functor Ω. The invertibility of Ω forces the ∞-category of spectra to be stable (see Corollary 10.12). We will begin with a slightly weaker version of this statement.

Lemma 9.1. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits. Then $\mathcal{C}$ is stable if and only if the following conditions are satisfied:*

(1) *The ∞-category $\mathcal{C}$ is pointed; that is, initial objects of $\mathcal{C}$ are also final.*

(2) *The suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence of ∞-categories (see §3).*

(3) *Every coexact triangle*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

in $\mathcal{C}$ is exact.

Remark 9.2. We will later show that condition (3) of Lemma 9.1 is superfluous (Corollary 10.12).

Proof. The necessity of conditions (1) and (3) is obvious, and the necessity of (2) was established in §3. For the converse, suppose that conditions (1), (2), and (3) are satisfied. We will show that $\mathcal{C}$ is stable. To prove this, we must show:

(i) Every morphism $f : X \to Y$ in $\mathcal{C}$ admits a kernel.

(ii) Every exact triangle in $\mathcal{C}$ is coexact.

In view of conditions (1) and (2), Theorem 3.11 endows the homotopy category $\mathrm{h}\mathcal{C}$ with the structure of a triangulated category. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then we can extend f to a distinguished triangle

$$Z \to X \xrightarrow{f} Y \to Z[1]$$

in the homotopy category $\mathrm{h}\mathcal{C}$, which arises from a coexact triangle $Z \to X \xrightarrow{f} Y$ in $\mathcal{C}$. Invoking (3), we deduce that $Z \simeq \ker(f)$; this proves (i). It also provides a specific example of an exact triangle $Z \to X \xrightarrow{f} Y$ which is coexact. Since an exact triangle is determined up to equivalence by f, we conclude that every exact triangle in $\mathcal{C}$ is coexact. □

We are now almost ready to introduce the ∞-category of *finite spectra*.

Notation 9.3. Let $\mathcal{S}_*$ denote the ∞-category of pointed objects of $\mathcal{S}$. That is, $\mathcal{S}_*$ denotes the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{S})$ spanned by those morphisms $f : X \to Y$ for which X is a final object of $\mathcal{S}$ (Definition T.7.2.2.1). Let $\mathcal{S}^{\mathrm{fin}}$ denote the smallest full subcategory of $\mathcal{S}$ which contains the final object $*$ and is stable under finite colimits. We will refer to $\mathcal{S}^{\mathrm{fin}}$ as the *∞-category of finite spaces*. We let $\mathcal{S}^{\mathrm{fin}}_* \subseteq \mathcal{S}_*$ denote the ∞-category of pointed objects of $\mathcal{S}^{\mathrm{fin}}$. We observe that the suspension functor $\Sigma : \mathcal{S}_* \to \mathcal{S}_*$ carries $\mathcal{S}^{\mathrm{fin}}_*$ to itself. For each $n \geq 0$, we let $S^n \in \mathcal{S}_*$ denote a representative for the (pointed) n-sphere.

Remark 9.4. It follows from Remark T.5.3.5.9 and Proposition T.4.3.2.15 that $\mathcal{S}^{\mathrm{fin}}$ is characterized by the following universal property: for every ∞-category $\mathcal{D}$ which admits finite colimits, evaluation at $*$ induces an equivalence of ∞-categories $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{S}^{\mathrm{fin}}, \mathcal{D}) \to \mathcal{D}$. Here $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{S}^{\mathrm{fin}}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}, \mathcal{D})$ spanned by the right exact functors.

Lemma 9.5. (1) *Each object of $\mathcal{S}^{\mathrm{fin}}_*$ is compact in $\mathcal{S}_*$.*

(2) *The inclusion $\mathcal{S}^{\mathrm{fin}}_* \subseteq \mathcal{S}_*$ induces an equivalence $\mathrm{Ind}(\mathcal{S}^{\mathrm{fin}}_*) \to \mathcal{S}_*$. In particular, $\mathcal{S}_*$ is compactly generated.*

(3) *The subcategory $\mathcal{S}^{\mathrm{fin}}_* \subseteq \mathcal{S}_*$ is the smallest full subcategory which contains S^0 and is stable under finite colimits.*

Proof. Since $\mathcal{S}^{\mathrm{fin}}$ consists of compact objects of $\mathcal{S}$, Proposition T.5.4.5.15 implies that $\mathcal{S}^{\mathrm{fin}}_*$ consists of compact objects of $\mathcal{S}_*$. This proves (1).

We next observe that $\mathcal{S}^{\mathrm{fin}}_*$ is stable under finite colimits in $\mathcal{S}_*$. Using the proof of Corollary T.4.4.2.4, we may reduce to showing that $\mathcal{S}^{\mathrm{fin}}_*$ is stable under pushouts and contains an initial object of $\mathcal{S}_*$. The second assertion is obvious, and the first follows from the fact that the forgetful functor $\mathcal{S}_* \to \mathcal{S}$ commutes with pushouts (Proposition T.4.4.2.8).

We now prove (3). Let $\mathcal{S}'_*$ be the smallest full subcategory which contains S^0 and is stable under finite colimits. The above argument shows that $\mathcal{S}'_* \subseteq \mathcal{S}^{\mathrm{fin}}_*$. To prove the converse, we let $f : \mathcal{S} \to \mathcal{S}_*$ be a left adjoint to the forgetful functor, so that $f(X) \simeq X \coprod *$. Then f preserves small colimits. Since $f(*) \simeq S^0 \in \mathcal{S}'_*$, we conclude that f carries $\mathcal{S}^{\mathrm{fin}}$ into $\mathcal{S}'_*$. If $x : * \to X$ is a pointed object of $\mathcal{S}$, then x can be written as a coproduct $f(X) \coprod_{S^0} *$. In particular, if $x \in \mathcal{S}^{\mathrm{fin}}_*$, then $X \in \mathcal{S}^{\mathrm{fin}}$, so that $f(X), S^0, * \in \mathcal{S}'_*$. Since $\mathcal{S}'_*$ is stable under pushouts, we conclude that $x \in \mathcal{S}'_*$; this completes the proof of (3).

We now prove (2). Part (1) and Proposition T.5.3.5.11 imply that we have a fully faithful functor $\theta : \mathrm{Ind}(\mathcal{S}^{\mathrm{fin}}_*) \subseteq \mathcal{S}_*$. Let $\mathcal{S}''_*$ be the essential image of θ. Proposition T.5.5.1.9 implies that $\mathcal{S}''_*$ is stable under small colimits. Since $S^0 \in \mathcal{S}''_*$ and f preserves small colimits, we conclude that $f(X) \in \mathcal{S}''_*$ for all $X \in \mathcal{S}$. Since $\mathcal{S}''_*$ is stable under pushouts, we conclude that $\mathcal{S}''_* = \mathcal{S}_*$, as desired. ☐

Warning 9.6. The ∞-category $\mathcal{S}^{\mathrm{fin}}$ does not coincide with the ∞-category of compact objects $\mathcal{S}^{\omega} \subseteq \mathcal{S}$. Instead, there is an inclusion $\mathcal{S}^{\mathrm{fin}} \subseteq \mathcal{S}^{\omega}$, which realizes $\mathcal{S}^{\omega}$ as an idempotent completion of $\mathcal{S}^{\mathrm{fin}}$. An object of $X \in \mathcal{S}^{\omega}$ belongs to $\mathcal{S}^{\mathrm{fin}}$ if and only if its *Wall finiteness obstruction* vanishes.

We define $\mathcal{S}^{\mathrm{fin}}_{\infty}$ to be the direct limit of the sequence of ∞-categories

$$\mathcal{S}^{\mathrm{fin}}_* \xrightarrow{\Sigma} \mathcal{S}^{\mathrm{fin}}_* \xrightarrow{\Sigma} \dots .$$

We will refer to $\mathcal{S}^{\mathrm{fin}}_{\infty}$ as the ∞-category of *finite spectra*.

Remark 9.7. More concretely: the objects of $\mathcal{S}^{\mathrm{fin}}_{\infty}$ can be identified with pairs (X, n), where X is a finite pointed space and n is an integer, with morphisms given by

$$\mathrm{Hom}^{R}_{\mathcal{S}^{\mathrm{fin}}_{\infty}}((X,n),(Y,m)) \simeq \varinjlim_{k} \mathrm{Hom}^{R}_{\mathcal{S}_*}(\Sigma^{k-n}(X), \Sigma^{k-m}(Y)).$$

It follows that the homotopy category $\mathrm{h}\mathcal{S}^{\mathrm{fin}}_{\infty}$ can be identified with the classical stable homotopy category for finite spaces.

Our next goal is to prove that $\mathcal{S}_\infty^{\mathrm{fin}}$ is stable. To prove this, we will need a bit of classical homotopy-theoretic terminology.

Definition 9.8. Let $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{S}$ be a diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y'. \end{array}$$

We will say that σ is *n-Cartesian* if the associated map $X \to X' \times_{Y'} Y$ is n-connected.

We will say that a square $\Delta^1 \times \Delta^1 \to \mathcal{S}_*$ is *n-Cartesian* if the underlying square in $\mathcal{S}$ is n-Cartesian.

Lemma 9.9. (1) *A diagram $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{S}$ is Cartesian if and only if it is n-Cartesian for all n.*

(2) *The collection of n-Cartesian diagrams is stable under filtered colimits in* $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{S})$.

(3) *Suppose that*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

is an n-Cartesian diagram in $\mathcal{S}_*$. Then the associated diagram

$$\begin{array}{ccc} \Omega^k(X) & \longrightarrow & \Omega^k(Y) \\ \downarrow & & \downarrow \\ \Omega^k(X') & \longrightarrow & \Omega^k(Y') \end{array}$$

is $(n-k)$-Cartesian.

Proof. Assertion (1) follows immediately from Whitehead's theorem (a morphism in $\mathcal{S}$ is an equivalence if and only if it is n-connected for all $n \geq 0$). To prove (2), it suffices to observe that filtered colimits in $\mathcal{S}$ are left exact (Example T.7.3.4.4) and the collection of n-connected morphisms in $\mathcal{S}$ is stable under filtered colimits. Finally, (3) follows from the fact that the loop functor Ω preserves pullback diagrams, and carries n-connected morphisms to $(n-1)$-connected morphisms. □

Proposition 9.10. *The ∞-category $\mathcal{S}_\infty^{\mathrm{fin}}$ is stable.*

Proof. We will show that $\mathcal{S}_\infty^{\mathrm{fin}}$ satisfies the hypotheses of Lemma 9.1. Let $\Sigma^{\infty-n} : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}_\infty^{\mathrm{fin}}$ be map from the nth copy of $\mathcal{S}_*^{\mathrm{fin}}$ into the direct limit. Since $\mathcal{S}_*^{\mathrm{fin}}$ admits finite colimits and the functor Σ is right exact, each of the functors $\Sigma^{\infty-n}$ is right exact. In particular, since every finite diagram in $\mathcal{S}_\infty^{\mathrm{fin}}$ factors through $\Sigma^{\infty-n}$ for $n \gg 0$, we conclude that $\mathcal{S}_\infty^{\mathrm{fin}}$ admits finite colimits.

Let $*$ be a zero object of $\mathcal{S}_*^{\mathrm{fin}}$. Since $\Sigma : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}_*^{\mathrm{fin}}$ preserves zero objects, it is easy to see that $\Sigma^\infty(*)$ is a zero object of $\mathcal{S}_\infty^{\mathrm{fin}}$. By construction, the suspension functor on $\mathcal{S}_\infty^{\mathrm{fin}}$ is an equivalence of ∞-categories. To complete the proof, it will suffice to show that every coexact triangle in $\mathcal{S}_\infty^{\mathrm{fin}}$ is exact. Fix a coexact triangle $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{S}_\infty^{\mathrm{fin}}$

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X''. \end{array}$$

We wish to show that σ is a pullback square. In view of Proposition T.5.1.3.2, it will suffice to show that for every for every $Y \in \mathcal{S}_\infty^{\mathrm{fin}}$, the image of σ under the functor co-represented by Y is a pullback square in $\mathcal{S}\mathrm{et}_\Delta$.

The object Y lies in the image of $\Sigma^{\infty-m}$ for $m \gg 0$. Shifting if necessary, we may suppose that $m = 0$. Let $\mathcal{E} \subseteq \mathcal{S}_*^{\text{fin}}$ be the full subcategory of $\mathcal{S}_*^{\text{fin}}$ spanned by those objects Y' such that $\Sigma^\infty(Y')$ corepresents a functor $\mathcal{S}_\infty^{\text{fin}} \to \mathcal{S}$ which carries σ to a pullback square. Since the collection of pullback squares in $\mathcal{S}$ is stable under finite limits, we conclude that $\mathcal{E} \subseteq \mathcal{S}_*^{\text{fin}}$ is stable under retracts and finite colimits. In view of Lemma 9.5, it will suffice to prove that the 0-sphere S^0 belongs to $\mathcal{E}$.

Choose $n \gg 0$, so that σ is the image under $\Sigma^{\infty-n}$ of a pushout square $\overline{\sigma} : \Delta^1 \times \Delta^1 \to \mathcal{S}_*^{\text{fin}}$, which we will denote by

$$\begin{array}{ccc} \overline{X'} & \longrightarrow & \overline{X} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \overline{X}''. \end{array}$$

Let $e_\infty : \mathcal{S}_\infty^{\text{fin}} \to \mathcal{S}$ be the functor corepresented by $\Sigma^\infty(S^0)$, and $e_k : \mathcal{S}_*^{\text{fin}}\, rightarrow\, \mathcal{S}$ the functor corepresented by $S^n = \Sigma^n(S^0)$. We wish to prove that $e_\infty \circ \sigma$ is a pullback diagram in $\mathcal{S}$. According to part (1) of Lemma 9.9, it will suffice to show that $e_\infty \circ \sigma$ is j-Cartesian for each $j \geq 0$. We observe that $e_\infty \circ \sigma$ can be written as a filtered colimit of the system $\{e_{k+n} \circ \Sigma^k \circ \overline{\sigma}\}$. Invoking part (2) of Lemma 9.9, we are reduced to proving that $e_{k+n} \circ \Sigma^k \circ \overline{\sigma}$ is j-Cartesian for $k \gg j$. We now observe that e_{k+n} can be identified with the composite functor

$$\mathcal{S}_* \stackrel{\Omega^{k+n}}{\to} \mathcal{S}_* \to \mathcal{S}\,.$$

In view of part (3) of Lemma 9.9, it will suffice to show that the square

$$\begin{array}{ccc} \Sigma^k(\overline{X'}) & \longrightarrow & \Sigma^k(\overline{X}) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma^k(\overline{X}'') \end{array}$$

is $(j+k+n)$-Cartesian, for all sufficiently large k. This follows immediately from the homotopy excision theorem ([7], p. 360). □

Definition 9.11. Let $\Omega : \mathcal{S}_* \to \mathcal{S}_*$ be a loop space functor. The ∞-category $\mathcal{S}_\infty$ of *spectra* is defined to be a homotopy inverse limit of the tower of ∞-categories

$$\ldots \stackrel{\Omega}{\to} \mathcal{S}_* \stackrel{\Omega}{\to} \mathcal{S}_*\,.$$

We let $\Omega_*^\infty : \mathcal{S}_\infty \to \mathcal{S}_*$ denote the map which corresponds to the final entry in the tower, and $\Omega^\infty : \mathcal{S}_\infty \to \mathcal{S}$ the composition of Ω_*^∞ with the forgetful map $\mathcal{S}_* \to \mathcal{S}$.

Remark 9.12. Definition 9.11 involves a slight abuse of terminology: the loop functor $\Omega : \mathcal{S}_* \to \mathcal{S}_*$ and the homotopy inverse limit in question are really only well-defined up to equivalence. We will later give a very explicit model for the ∞-category $\mathcal{S}_\infty$, as an ∞-category of *homology theories* (Corollary 10.16).

Proposition 9.13.

(1) *There is an equivalence of ∞-categories* $\text{Ind}(\mathcal{S}_\infty^{\text{fin}}) \to \mathcal{S}_\infty$.

(2) *The ∞-category $\mathcal{S}_\infty$ is stable and compactly generated.*

(3) *Let $(\mathcal{S}_\infty)_{\leq -1}$ denote the full subcategory of $\mathcal{S}_\infty$ spanned by those objects X such that $\Omega^\infty(X) \in \mathcal{S}$ is contractible. Then $(\mathcal{S}_\infty)_{\leq -1}$ determines an accessible t-structure on $\mathcal{S}_\infty$ (see §16).*

(4) *The t-structure on $\mathcal{S}_\infty$ is both left complete and right complete, and the heart $\mathcal{S}_\infty^\heartsuit$ is canonically equivalent to the (nerve of the) category of abelian groups.*

Proof. Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\omega}$, $\mathcal{P}\mathrm{r}^{\mathrm{R}}_{\omega}$, and $\mathcal{C}\mathrm{at}^{\mathrm{Rex}}_{\infty}$ be defined as §T.5.5.6. According to Proposition T.5.5.6.8, we can view the construction of Ind-categories as determining a localization functor $\mathrm{Ind} : \mathcal{C}\mathrm{at}^{\mathrm{Rex}}_{\infty} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\omega}$. Proposition T.5.5.6.9 shows that we can view $\mathcal{S}^{\mathrm{fin}}_{\infty}$ as the colimit of the sequence

$$\mathcal{S}^{\mathrm{fin}}_{*} \xrightarrow{\Sigma} \mathcal{S}^{\mathrm{fin}}_{*} \xrightarrow{\Sigma} \dots$$

in $\mathcal{C}\mathrm{at}^{\mathrm{Rex}}_{\infty}$. Since $\mathcal{S}_{*} \simeq \mathrm{Ind}(\mathcal{S}^{\omega}_{*})$ (Lemma 9.5) and the functor Ind preserves colimits, we conclude that $\mathrm{Ind}(\mathcal{S}^{\mathrm{fin}}_{\infty})$ can be identified with the colimit of the sequence

$$\mathcal{S}_{*} \xrightarrow{\Sigma} \mathcal{S}_{*} \xrightarrow{\Sigma} \dots$$

in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\omega}$. Invoking the equivalence between $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\omega}$ and $(\mathcal{P}\mathrm{r}^{\mathrm{R}}_{\omega})^{op}$ (see Notation T.5.5.6.5), we can identify $\mathrm{Ind}(\mathcal{S}^{\mathrm{fin}}_{\infty})$ with the *limit* of the tower

$$\dots \xrightarrow{\Omega} \mathcal{S}_{*} \xrightarrow{\Omega} \mathcal{S}_{*}$$

in $\mathcal{P}\mathrm{r}^{\mathrm{R}}_{\omega}$. Since the inclusion functor $\mathcal{P}\mathrm{r}^{\mathrm{R}}_{\omega} \subseteq \widehat{\mathcal{C}\mathrm{at}}_{\infty}$ preserves limits (Proposition T.5.5.6.4), we conclude that there is an equivalence $\mathrm{Ind}(\mathcal{S}^{\mathrm{fin}}_{\infty}) \simeq \mathcal{S}_{\infty}$. This proves (1), and also shows that $\mathcal{S}_{\infty}$ is compactly generated. The stability of $\mathcal{S}_{\infty}$ follows from Propositions 8.3 and 9.10.

Assertion (3) is a special case of Proposition 16.4, which will be established in §16. A spectrum X can be identified with a sequence of pointed spaces $\{X(n)\}$, equipped with equivalences $X(n) \simeq \Omega X(n+1)$ for all $n \geq 0$. We observe that $X \in (\mathcal{S}_{\infty})_{\leq m}$ if and only if each $X(n)$ is $(n+m)$-truncated. In general, the sequence $\{\tau_{\leq n+m} X(n)\}$ itself determines a spectrum, which we can identify with the truncation $\tau_{\leq m} X$. It follows that $X \in (\mathcal{S}_{\infty})_{\geq m+1}$ if and only if each $X(n)$ is $(n+m)$-connected. In particular, X lies in the heart of $\mathcal{S}_{\infty}$ if and only if each $X(n)$ is an Eilenberg-MacLane object of $\mathcal{S}$ of degree n (see Definition T.7.2.2.1). It follows that $\mathcal{S}_{\infty}{}^{\heartsuit}$ can be identified with the homotopy inverse limit of the tower of ∞-categories

$$\dots \xrightarrow{\Omega} \mathcal{EM}_1(\mathcal{S}) \xrightarrow{\Omega} \mathcal{EM}_0(\mathcal{S}),$$

where $\mathcal{EM}_n(\mathcal{S})$ denotes the full subcategory of $\mathcal{S}_{*}$ spanned by the Eilenberg-MacLane object of degree n. Proposition T.7.2.2.12 asserts that after the second term, this tower is equivalent to the constant diagram taking the value $\mathrm{N}(\mathcal{A}b)$, where $\mathcal{A}b$ is category of abelian groups.

It remains to prove that $\mathcal{S}_{\infty}$ is both right and left complete. We begin by observing that if $X \in \mathcal{S}_{\infty}$ is such that $\pi_n X \simeq 0$ for all $n \in \mathbf{Z}$, then X is a zero object of $\mathcal{S}_{\infty}$ (since each $X(n) \in \mathcal{S}$ has vanishing homotopy groups, and is therefore contractible by Whitehead's theorem). Consequently, both $\bigcap (\mathcal{S}_{\infty})_{\leq -n}$ and $\bigcap (\mathcal{S}_{\infty})_{\geq n}$ coincide with the collection of zero objects of $\mathcal{S}_{\infty}$. It follows that

$$(\mathcal{S}_{\infty})_{\geq 0} = \{X \in \mathcal{S}_{\infty} : (\forall n < 0)[\pi_n X \simeq 0]\}$$

$$(\mathcal{S}_{\infty})_{\leq 0} = \{X \in \mathcal{S}_{\infty} : (\forall n > 0)[\pi_n X \simeq 0]\}.$$

According to Proposition 7.3, to prove that $\mathcal{S}_{\infty}$ is left and right complete it will suffice to show that the subcategories $(\mathcal{S}_{\infty})_{\geq 0}$ and $(\mathcal{S}_{\infty})_{\leq 0}$ are stable under products and coproducts. In view of the above formulas, it will suffice to show that the homotopy group functors $\pi_n : \mathcal{S}_{\infty} \to \mathrm{N}(\mathcal{A}b)$ preserve products and coproducts. Since π_n obviously commutes with finite coproducts, it will suffice to show that π_n commutes with products and filtered colimits. Shifting if necessary, we may reduce to the case $n = 0$. Since products and filtered colimits in the category of abelian groups can be computed at the level of the underlying sets, we are reduced to proving that the composition

$$\mathcal{S}_{\infty} \xrightarrow{\Omega^{\infty}} \mathcal{S} \xrightarrow{\pi_0} \mathrm{N}(\mathcal{S}\mathrm{et})$$

preserves products and filtered colimits. This is clear, since each of the factors individually preserves products and filtered colimits. □

Remark 9.14. Part (1) of Proposition 9.13 implies that we can identify $\mathcal{S}^{\mathrm{fin}}_{\infty}$ with a full subcategory of the compact objects of $\mathcal{S}_{\infty}$. In fact, every compact object of $\mathcal{S}_{\infty}$ belongs to this full subcategory. The proof of this is not completely formal (especially in view of Warning 9.6); it relies on the fact that the ring of integers $\mathbf{Z}$ is a principal ideal domain, so that every finitely generated projective $\mathbf{Z}$-module is free.

Remark 9.15. Let $\mathcal{A}b$ denote the category of abelian groups. For each $n \in \mathbf{Z}$, we let $\pi_n : \mathcal{S}_\infty \to \mathrm{N}(\mathcal{A}b)$ be the associated functor (note that, if $n \geq 2$, then π_n can be identified with the composition

$$\mathcal{S}_\infty \xrightarrow{\Omega^\infty_*} \mathcal{S}_* \xrightarrow{\pi_n} \mathrm{N}(\mathcal{A}b)$$

where the second map is the usual homotopy group functor. Since $\mathcal{S}_\infty$ is both left and right complete, we conclude that a map $f : X \to Y$ of spectra is an equivalence if and only if it induces isomorphisms $\pi_n X \to \pi_n Y$ for all $n \in \mathbf{Z}$.

Proposition 9.16. *The functor $\Omega^\infty : (\mathcal{S}_\infty)_{\geq 0} \to \mathcal{S}$ preserves geometric realizations of simplicial objects.*

Proof. Since the simplicial set $\mathrm{N}(\mathbf{\Delta})$ is weakly contractible, the forgetful functor $\mathcal{S}_* \to \mathcal{S}$ preserves geometric realizations of simplicial objects (Proposition T.4.4.2.8). It will therefore suffice to prove that the functor $\Omega^\infty_* | (\mathcal{S}_\infty)_{\geq 0} \to \mathcal{S}_*$ preserves geometric realizations of simplicial objects.

For each $n \geq 0$, let $\mathcal{S}^{\geq n}$ denote the full subcategory of $\mathcal{S}$ spanned by the $(n-1)$-connected objects, and let $\mathcal{S}^{\geq n}_*$ be the ∞-category of pointed objects of $\mathcal{S}^{\geq n}$. We observe that $(\mathcal{S}_\infty)_{\geq 0}$ can be identified with the homotopy inverse limit of the tower

$$\ldots \xrightarrow{\Omega} \mathcal{S}^{\geq 1}_* \xrightarrow{\Omega} \mathcal{S}^{\geq 0}_* .$$

It will therefore suffice to prove that for every $n \geq 0$, the loop functor $\Omega : \mathcal{S}^{\geq n+1}_* \to \mathcal{S}^{\geq n}_*$ preserves geometric realizations of simplicial objects.

The ∞-category $\mathcal{S}^{\geq n}$ is the preimage (under $\tau_{\leq n-1}$) of the full subcategory of $\tau_{\leq n-1}\mathcal{S}$ spanned by the final objects. Since this full subcategory is stable under geometric realizations of simplicial objects and since $\tau_{\leq n-1}$ commutes with all colimits, we conclude that $\mathcal{S}^{\geq n} \subseteq \mathcal{S}$ is stable under geometric realizations of simplicial objects.

According to Lemmas T.7.2.2.11 and T.7.2.2.10, there is an equivalence of $\mathcal{S}^{\geq 1}_*$ with the ∞-category of group objects $\mathcal{G}\mathrm{rp}(\mathcal{S}_*)$. This restricts to an equivalence of $\mathcal{S}^{\geq n+1}_*$ with $\mathcal{G}\mathrm{rp}(\mathcal{S}^{\geq n}_*)$ for all $n \geq 0$. Moreover, under this equivalence, the loop functor Ω can be identified with the composition

$$\mathcal{G}\mathrm{rp}(\mathcal{S}^{\geq n}_*) \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{S}^{\geq n}_*) \to \mathcal{S}^{\geq n}_*,$$

where the second map is given by evaluation at the object $[1] \in \mathbf{\Delta}$. This evaluation map commutes with geometric realizations of simplicial objects (Proposition T.5.1.2.2). Consequently, it will suffice to show that $\mathcal{G}\mathrm{rp}(\mathcal{S}^{\geq n}_*) \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{S}^{\geq n}_*)$ is stable under geometric realizations of simplicial objects.

Without loss of generality, we may suppose $n = 0$; now we are reduced to showing that $\mathcal{G}\mathrm{rp}(\mathcal{S}_*) \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{S}_*)$ is stable under geometric realizations of simplicial objects. In view of Lemma T.7.2.2.10, it will suffice to show that $\mathcal{G}\mathrm{rp}(\mathcal{S}) \subseteq \mathcal{S}_\Delta$ is stable under geometric realizations of simplicial objects. Invoking Proposition T.7.2.2.4, we are reduced to proving that the formation of geometric realizations in $\mathcal{S}$ commutes with finite products, which follows from Lemma 12.4. □

10 Stabilization and Excisive Functors

The ∞-category of spectra can be regarded as obtained from the ∞-category $\mathcal{S}$ of spaces by formally inverting the loop space functor. In this section, we will discuss a generalization of this construction, in which $\mathcal{S}$ is replaced by a more general ∞-category $\mathcal{C}$.

Definition 10.1. Let $\mathcal{C}$ be an ∞-category which admits finite limits, and let $\mathcal{C}_*$ be the ∞-category of pointed objects of $\mathcal{C}$. We define the *stabilization of* $\mathcal{C}$ to be a homotopy limit of the tower

$$\ldots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* .$$

We will denote the stabilization of $\mathcal{C}$ by $\mathrm{Stab}(\mathcal{C})$. Let $\Omega^\infty_* : \mathrm{Stab}(\mathcal{C}) \to \mathcal{C}_*$ be the functor corresponding to evaluation at the bottom of the tower, and $\Omega^\infty : \mathrm{Stab}(\mathcal{C}) \to \mathcal{C}$ the functor obtained by composing Ω^∞_* with the forgetful functor from $\mathcal{C}_*$ to $\mathcal{C}$.

Remark 10.2. Definition 10.1 determines the ∞-category $\mathrm{Stab}(\mathcal{C})$ only up to (canoncial) equivalence. At the end of this section, we will introduce a model for $\mathrm{Stab}(\mathcal{C})$ which is well-defined up to isomorphism; see Corollary 10.15 and Remark 10.17.

Example 10.3. The ∞-category of spectra can be identified with $\mathrm{Stab}(\mathcal{S})$.

Example 10.4. Let $\mathbf{Q}$ be the ring of rational numbers, let $\mathbf{A}$ be the category of simplicial commutative $\mathbf{Q}$-algebras, viewed as simplicial model category (see Proposition 13.1), and let $\mathcal{C} = \mathrm{N}(\mathbf{A}^{\circ})$ be the underlying ∞-category. Suppose that R is a commutative $\mathbf{Q}$-algebra, regarded as an object of $\mathcal{C}$. Then $\mathrm{Stab}(\mathcal{C}_{/R})$ is a stable ∞-category, whose homotopy category is equivalent to the (unbounded) derived category of R-modules. The loop functor $\Omega^{\infty} : \mathrm{Stab}(\mathcal{C}_{/R}) \rightarrow \mathcal{C}_{/R}$ admits a left adjoint $\Sigma^{\infty} : \mathcal{C}_{/R} \rightarrow \mathrm{Stab}(\mathcal{C}_{/R})$ (Proposition 15.4). This left adjoint assigns to each morphism of commutative rings $S \xrightarrow{\phi} R$ an object $\Sigma^{\infty}(\phi) \in \mathrm{Stab}(\mathcal{C}_{/R})$, which can be identified with $L_S \otimes_S R$, where L_S denotes the (absolute) *cotangent complex* of S. We will discuss this example in greater detail in [15].

In order to study the relationship between an ∞-category $\mathcal{C}$ and its stabilization $\mathrm{Stab}(\mathcal{C})$, we need to introduce a bit of terminology.

Definition 10.5. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between ∞-categories.

(i) If $\mathcal{C}$ has an initial object $\emptyset$, then we will say that F is *weakly excisive* if $F(\emptyset)$ is a final object of $\mathcal{D}$. We let $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the weakly excisive functors.

(ii) If $\mathcal{C}$ admits finite colimits, then we will say that F is *excisive* if it is weakly excisive, and F carries pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$. We let $\mathrm{Exc}(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the excisive functors.

Remark 10.6. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between ∞-categories, and suppose that $\mathcal{C}$ admits finite colimits. If $\mathcal{C}$ is stable, then F is excisive if and only if it is left exact (Proposition 4.4). If instead $\mathcal{D}$ is stable, then F is excisive if and only if it is right exact. In particular, if both $\mathcal{C}$ and $\mathcal{D}$ are stable, then F is excisive if and only if it is exact (Proposition 5.1).

Lemma 10.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and assume that $\mathcal{C}$ has an initial object. Then:*

(1) *The forgetful functor $\theta : \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}_*) \rightarrow \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ is a trivial fibration of simplicial sets.*

(2) *If $\mathcal{C}$ admits finite colimits, then the forgetful functor $\theta' : \mathrm{Exc}(\mathcal{C}, \mathcal{D}_*) \rightarrow \mathrm{Exc}(\mathcal{C}, \mathcal{D})$ is a trivial fibration of simplicial sets.*

Remark 10.8. If the ∞-category $\mathcal{D}$ does not have a final object, then the conclusion of Lemma 10.7 is valid, but degenerate: both of the relevant ∞-categories of functors are empty.

Proof. To prove (1), we first observe that objects of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D}_*)$ can be identified with maps $F : \mathcal{C} \times \Delta^1 \rightarrow \mathcal{D}$ with the following properties:

(a) For every initial object $C \in \mathcal{C}$, $F(C, 1)$ is a final object of $\mathcal{D}$.

(b) For every object $C \in \mathcal{C}$, $F(C, 0)$ is a final object of $\mathcal{D}$.

Assume for the moment that (a) is satisfied, and let $\mathcal{C}' \subseteq \mathcal{C} \times \Delta^1$ be the full subcategory spanned by those objects (C, i) for which either $i = 1$, or C is an initial object of $\mathcal{C}$. We observe that (b) is equivalent to the following pair of conditions:

(b') The functor $F | \mathcal{C}'$ is a right Kan extension of $F | \mathcal{C} \times \{1\}$.

(b'') The functor F is a left Kan extension of $F | \mathcal{C}'$.

Let $\mathcal{E}$ be the full subcategory of $\mathrm{Fun}(\mathcal{C} \times \Delta^1, \mathcal{D})$ spanned by those functors which satisfy conditions (b') and (b''). Using Proposition T.4.3.2.15, we deduce that the projection $\overline{\theta} : \mathcal{E} \rightarrow \mathrm{Fun}(\mathcal{C} \times \{1\}, \mathcal{D})$ is a trivial Kan fibration. Since θ is a pullback of $\overline{\theta}$, we conclude that θ is a trivial Kan fibration. This completes the proof of (1).

To prove (2), we observe that θ' is a pullback of θ (since Proposition T.1.2.13.8 asserts that a square in $\mathcal{D}_*$ is a pullback if and only if the underlying square in $\mathcal{D}$ is a pullback). □

Remark 10.9. Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits, and $\mathcal{D}$ a pointed ∞-category which admits finite limits. Let $F : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be given by composition with the suspension functor $\mathcal{C} \rightarrow \mathcal{C}$, and let $G : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be given by composition with the loop functor $\Omega : \mathcal{D} \rightarrow \mathcal{D}$. Then F and G restrict to give homotopy inverse equivalences

$$\mathrm{Exc}(\mathcal{C}, \mathcal{D}) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Exc}(\mathcal{C}, \mathcal{D}).$$

Let $\mathcal{C}$ be a (small) pointed ∞-category. We define $\mathcal{P}_*(\mathcal{C})$ to be the full subcategory of $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ spanned by those functors which carry zero objects of $\mathcal{C}$ to contractible objects of $\mathcal{S}$. It is clear that the Yoneda embedding $j : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{C})$ factors through $\mathcal{P}_*(\mathcal{C})$. Lemma 10.7 implies that the canonical map $\mathrm{Fun}_*(\mathcal{C}^{op}, \mathcal{S}_*) \rightarrow \mathcal{P}_*(\mathcal{C})$ is a trivial fibration. Consequently, the Yoneda embedding lifts to a fully faithful functor $j' : \mathcal{C} \rightarrow \mathrm{Fun}_*(\mathcal{C}^{op}, \mathcal{S}_*)$, which we will refer to as the *pointed Yoneda embedding*. Our terminology is slightly abusive: the functor j' is only well-defined up to a contractible space of choices; we will ignore this ambiguity.

Proposition 10.10. *Let $\mathcal{C}$ be an ∞-category which admits finite limits. Then:*

(1) *The ∞-category* $\mathrm{Stab}(\mathcal{C})$ *is stable.*

(2) *The functor* $\Omega^\infty : \mathrm{Stab}(\mathcal{C}) \rightarrow \mathcal{C}$ *is left exact.*

(3) *Let $\mathcal{D}$ be a pointed ∞-category which admits finite colimits. Then composition with Ω^∞ induces an equivalence of ∞-categories*
$$\mathrm{Exc}(\mathcal{D}, \mathrm{Stab}(\mathcal{C})) \rightarrow \mathrm{Exc}(\mathcal{D}, \mathcal{C}).$$

Proof. We begin with the proof of (1). Let $j : \mathcal{C}_* \rightarrow \mathrm{Fun}_*(\mathcal{C}_*^{op}, \mathcal{S}_*)$ denote the pointed Yoneda embedding, and let $\mathcal{D} \subseteq \mathrm{Fun}(\mathcal{C}_*^{op}, \mathcal{S}_*) \subseteq \mathrm{Fun}_*(\mathcal{C}_*, \mathcal{S}_*)$ be the essential image of j. Using Propositions T.5.1.3.2 and T.1.2.13.8, we conclude that j preserves finite limits. Moreover, $\mathrm{Fun}_*(\mathcal{C}_*, \mathcal{S}_*)$ is stable under limits in $\mathrm{Fun}(\mathcal{C}_*, \mathcal{S}_*)$. Consequently, $\mathrm{Stab}(\mathcal{C})$ can be identified with the homotopy inverse limit of the tower

$$\ldots \xrightarrow{\Omega} \mathcal{D} \xrightarrow{\Omega} \mathcal{D},$$

which is in turn a full subcategory of the homotopy inverse limit of the tower

$$\ldots \xrightarrow{\Omega} \mathrm{Fun}(\mathcal{C}_*, \mathcal{S}_*) \xrightarrow{\Omega} \mathrm{Fun}(\mathcal{C}_*, \mathcal{S}_*).$$

Using Proposition T.5.1.2.2, we can identify the loop functor on $\mathrm{Fun}(\mathcal{C}_*, \mathcal{S}_*)$ with the functor given by composition with the loop functor $\Omega_{\mathcal{S}_*} : \mathcal{S}_* \rightarrow \mathcal{S}_*$. It follows that the homotopy inverse limit of the latter tower can be identified with $\mathrm{Fun}(\mathcal{C}_*, \mathcal{S}_\infty)$. Proposition 4.1 asserts that $\mathrm{Fun}(\mathcal{C}_*, \mathcal{S}_\infty)$ is stable. The above argument shows that $\mathrm{Stab}(\mathcal{C})$ can be identified with the full subcategory of $\mathrm{Fun}(\mathcal{C}_*, \mathcal{S}_\infty)$ spanned by those functors F with the property that for each $n \geq 0$, the composite functor $\mathcal{C}_* \xrightarrow{F} \mathcal{S}_\infty \xrightarrow{\Omega^{\infty-n}} \mathcal{S}$ is representable by an object of $\mathcal{C}_*$. This subcategory is obviously stable under translations. Moreover, since $\Omega^{\infty-n}$ and the Yoneda embedding are both left exact functors, we deduce that $\mathrm{Stab}(\mathcal{C})$ is stable under finite limits in $\mathrm{Fun}(\mathcal{C}_*, \mathcal{S}_\infty)$. Applying Lemma 4.3, we conclude that $\mathrm{Stab}(\mathcal{C})$ is a stable subcategory of $\mathrm{Fun}(\mathcal{C}_*, \mathcal{S}_\infty)$; this proves (1).

To prove (2), we observe that Ω^∞ can be obtained as a composition

$$\mathrm{Stab}(\mathcal{C}) \stackrel{\Omega^\infty_*}{\to} \mathcal{C}_* \to \mathcal{C}.$$

The second map is left exact by Proposition T.1.2.13.8, and the first is left exact because it is the homotopy inverse limit of a tower of left exact functors.

We now prove (3). Lemma 10.7 implies that the forgetful functor $\mathrm{Exc}(\mathcal{C}, \mathcal{D}_*) \to \mathrm{Exc}(\mathcal{C}, \mathcal{D})$ is a trivial fibration of simplicial sets. It will therefore suffice to show that composition with Ω^∞_* induces an equivalence $\theta : \mathrm{Exc}(\mathcal{D}, \mathrm{Stab}(\mathcal{C})) \to \mathrm{Exc}(\mathcal{D}, \mathcal{C}_*)$. Since the loop functor $\Omega : \mathcal{C}_* \to \mathcal{C}_*$ is left exact, the domain of θ can be identified with a homotopy limit of the tower

$$\ldots \stackrel{\circ\Omega}{\to} \mathrm{Exc}(\mathcal{D}, \mathcal{C}_*) \stackrel{\circ\Omega}{\to} \mathrm{Exc}(\mathcal{D}, \mathcal{C}_*).$$

Remark 10.9 implies that this tower is essentially constant. □

Example 10.11. Let $\mathcal{C}$ be an ∞-category which admits finite limits, and K an arbitrary simplicial set. Then $\mathrm{Fun}(K, \mathcal{C})$ admits finite limits (Proposition T.5.1.2.2). We have a canonical isomorphism $\mathrm{Fun}(K, \mathcal{C})_* \simeq \mathrm{Fun}(K, \mathcal{C}_*)$, and the loop functor on $\mathrm{Fun}(K, \mathcal{C})_*$ can be identified with the functor given by composition with $\Omega : \mathcal{C}_* \to \mathcal{C}_*$. It follows that there is a canonical equivalence of ∞-categories

$$\mathrm{Stab}(\mathrm{Fun}(K, \mathcal{C})) \simeq \mathrm{Fun}(K, \mathrm{Stab}(\mathcal{C})).$$

In particular, $\mathrm{Stab}(\mathcal{P}(K))$ can be identified with $\mathrm{Fun}(K, \mathcal{S}_\infty)$.

Corollary 10.12. *Let $\mathcal{C}$ be a pointed ∞-category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is stable.*

(2) *The ∞-category $\mathcal{C}$ admits finite colimits and the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence.*

(3) *The ∞-category $\mathcal{C}$ admits finite limits and the loop functor $\Omega : \mathcal{C} \to \mathcal{C}$ is an equivalence.*

Proof. We will show that (1) ⇔ (3); the dual argument will prove that (1) ⇔ (2). The implication (1) ⇒ (3) is clear. Conversely, suppose that $\mathcal{C}$ admits finite limits and that Ω is an equivalence. Lemma T.7.2.2.9 asserts that the forgetful functor $\mathcal{C}_* \to \mathcal{C}$ is a trivial fibration. Consequently, $\mathrm{Stab}(\mathcal{C})$ can be identified with the homotopy inverse limit of the tower

$$\ldots \stackrel{\Omega}{\to} \mathcal{C} \stackrel{\Omega}{\to} \mathcal{C}.$$

By assumption, the functor Ω is an equivalence, so this tower is essentially constant. It follows that $\Omega^\infty : \mathrm{Stab}(\mathcal{C}) \to \mathcal{C}$ is an equivalence of ∞-categories. Since $\mathrm{Stab}(\mathcal{C})$ is stable (Proposition 10.10), so is $\mathcal{C}$. □

We can apply Proposition 10.10 to give another description of the ∞-category $\mathrm{Stab}(\mathcal{C})$.

Lemma 10.13. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits, let $f : \mathcal{C} \to \mathcal{C}_*$ be a left adjoint to the forgetful functor, and let $\mathcal{D}$ be a stable ∞-category. Then composition with f induces an equivalence of functor ∞-categories $\phi : \mathrm{Exc}(\mathcal{C}_*, \mathcal{D}) \to \mathrm{Exc}(\mathcal{C}, \mathcal{D})$.*

Proof. Consider the composition

$$\theta : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times \mathcal{C}_* \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{D}) \stackrel{\mathrm{coker}}{\to} \mathcal{D}.$$

We can identify θ with a map $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_*, \mathcal{D})$. Since the collection of pullback squares in $\mathcal{D}$ is a stable subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{D})$, we conclude θ restricts to a map $\psi : \mathrm{Exc}(\mathcal{C}, \mathcal{D}) \to \mathrm{Exc}(\mathcal{C}_*, \mathcal{D})$. It is not difficult to verify that ψ is a homotopy inverse to ϕ. □

Proposition 10.14. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits, and let $\mathcal{D}$ be an ∞-category which admits finite limits. Then there is a canonical isomorphism* $\mathrm{Exc}(\mathcal{C}_*, \mathcal{D}) \simeq \mathrm{Exc}(\mathcal{C}, \mathrm{Stab}(\mathcal{D}))$ *in the homotopy category of ∞-categories.*

Proof. Combining Lemma 10.13 and part (3) of Proposition 10.10, we obtain a diagram of equivalences

$$\mathrm{Exc}(\mathcal{C}_*, \mathcal{D}) \leftarrow \mathrm{Exc}(\mathcal{C}_*, \mathrm{Stab}(\mathcal{D})) \rightarrow \mathrm{Exc}(\mathcal{C}, \mathrm{Stab}(\mathcal{D})).$$

□

Corollary 10.15. *Let $\mathcal{D}$ be an ∞-category which admits finite limits. Then there is a canonical isomorphism* $\mathrm{Stab}(\mathcal{D}) \simeq \mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{D})$ *in the homotopy category of ∞-categories.*

Proof. Combine Proposition 10.14, Remark 9.4, and Remark 10.6. □

Corollary 10.16. *The ∞-category of spectra is equivalent to the ∞-category* $\mathrm{Exc}(\mathcal{S}_*^{f}, \mathcal{S})$.

Remark 10.17. The ∞-category $\mathcal{S}_\infty$ is only determined up to equivalence by Definition 9.11. However, Corollary 10.16 provides a very explicit model for spectra. Namely, we can identify a spectrum with an excisive functor $F : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$. We should think of F as a *homology theory* A. More precisely, given a pair of finite spaces $X_0 \subseteq X$, we can define the relative homology group $A_n(X, X_0)$ to be $\pi_n F(X/X_0)$, where X/X_0 denotes the pointed space obtained from X by collapsing X_0 to a point (here the homotopy group is taken with base point provided by the map $* \simeq F(*) \to F(X/X_0)$). The assumption that F is excisive is precisely what is needed to guarantee the existence of the usual excision exact sequences for the homology theory A.

11 Homological Algebra

Let $\mathcal{A}$ be an abelian category. In classical homological algebra, it is customary to associate to $\mathcal{A}$ a certain triangulated category, called the *derived category* of $\mathcal{A}$, the objects of which are chain complexes with values in $\mathcal{A}$. In this section, we will review the theory of derived categories from the perspective of higher category theory. To simplify the discussion, we primarily consider only abelian categories $\mathcal{A}$ which have enough projective objects (the dual case of abelian categories with enough injective objects can be understood by passing to the opposite category).

We begin by considering an arbitrary additive category $\mathcal{A}$. Let $\mathrm{Ch}(\mathcal{A})$ denote the category whose objects are chain complexes

$$\ldots \to A_1 \to A_0 \to A_{-1} \to \ldots$$

with values in $\mathcal{A}$. The category $\mathrm{Ch}(\mathcal{A})$ is naturally enriched over simplicial sets. For $A_\bullet, B_\bullet \in \mathrm{Ch}(\mathcal{A})$, the simplicial set $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)$ is characterized by the property that for every finite simplicial set K there is a natural bijection

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(K, \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)) \simeq \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet \otimes C_\bullet(K), B_\bullet).$$

Here $C_\bullet(K)$ denotes the normalized chain complex for computing the homology of K, so that $C_n(K)$ is a free abelian group whose generators are in bijection with the nondegenerate n-simplices of K. Unwinding the definitions, we see that the vertices of $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)$ are just the maps of chain complexes from $A_\bullet$ to $B_\bullet$. An edge e of $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)$ is determined by three pieces of data:

(*i*) A vertex $d_0(e)$, corresponding to a chain map $f : A_\bullet \to B_\bullet$.

(*ii*) A vertex $d_1(e)$, corresponding to a chain map $g : A_\bullet \to B_\bullet$.

(*iii*) A map $h : A_\bullet \to B_{\bullet+1}$, which determines a chain homotopy from f to g.

Remark 11.1. Let $\mathcal{A}b$ be the category of abelian groups, and let $\mathrm{Ch}_{\geq 0}(\mathcal{A}b)$ denote the full subcategory of $\mathrm{Ch}(\mathcal{A}b)$ spanned by those complexes $A_\bullet$ such that $A_n \simeq 0$ for all $n < 0$. The *Dold-Kan correspondence* (see [23]) asserts that $\mathrm{Ch}_{\geq 0}(\mathcal{A}b)$ is equivalent to the category of *simplicial* abelian groups. In particular, there is a forgetful functor $\theta : \mathrm{Ch}_{\geq 0}(\mathcal{A}b) \to \mathcal{S}\mathrm{et}_\Delta$.

Given a pair of complexes $A_\bullet, B_\bullet \in \mathrm{Ch}(\mathcal{A})$, the mapping space $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)$ can be defined as follows:

(1) First, we extract the mapping complex
$$[A_\bullet, B_\bullet] \in \mathrm{Ch}(\mathcal{A}b),$$
where $[A_\bullet, B_\bullet]_n = \prod \mathrm{Hom}_{\mathcal{A}}(A_m, B_{n+m})$.

(2) The inclusion $\mathrm{Ch}_{\geq 0}(\mathcal{A}b) \subseteq \mathrm{Ch}(\mathcal{A}b)$ has a right adjoint, which associates to an arbitrary chain complex $M_\bullet$ the truncated complex
$$\ldots \to M_1 \to \ker(M_0 \to M_{-1}) \to 0 \to \ldots$$
Applying this functor to $[A_\bullet, B_\bullet]$, we obtain a new complex $[A_\bullet, B_\bullet]_{\geq 0}$, whose degree zero term coincides with the set of chain maps from $A_\bullet$ to $B_\bullet$.

(3) Applying the Dold-Kan correspondence θ, we can convert the chain complex $[A_\bullet, B_\bullet]_{\geq 0}$ into a simplicial set $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)$.

Because every simplicial abelian group is a Kan complex, the simplicial category $\mathrm{Ch}(\mathcal{A})$ is automatically *fibrant*.

Remark 11.2. Let $\mathcal{A}$ be an additive category, and let $A_\bullet, B_\bullet \in \mathrm{Ch}(\mathcal{A})$. The homotopy group
$$\pi_n \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)$$
can be identified with the group of chain-homotopy classes of maps from $A_\bullet$ to $B_{\bullet+n}$.

Example 11.3. Let $\mathcal{A}$ be an abelian category, and let $A_\bullet, B_\bullet \in \mathrm{Ch}(\mathcal{A})$. Suppose that $A_n \simeq 0$ for $n < 0$, and that $B_n \simeq 0$ for $n > 0$. Then the simplicial set $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)$ is constant, with value $\mathrm{Hom}_{\mathcal{A}}(\mathrm{H}_0(A_\bullet), \mathrm{H}_0(B_\bullet))$.

Lemma 11.4. *Let $\mathcal{A}$ be an additive category. Then:*

(1) *Let*
$$\begin{array}{ccc} A_\bullet & \xrightarrow{f} & B_\bullet \\ \downarrow & & \downarrow \\ A'_\bullet & \longrightarrow & B'_\bullet \end{array}$$
be a pushout diagram in the (ordinary) category $\mathrm{Ch}(\mathcal{A})$*, and suppose that f is* degreewise split *(so that each $B_n \simeq A_n \oplus C_n$, for some $C_n \in \mathcal{A}$). Then the above diagram determines a homotopy pushout square in the ∞-category* $\mathrm{N}(\mathrm{Ch}(\mathcal{A}))$.

(2) *The ∞-category* $\mathrm{N}(\mathrm{Ch}(\mathcal{A}))$ *is stable.*

Proof. To prove (1), it will suffice (Theorem T.4.2.4.1) to show that for every $D_\bullet \in \mathrm{Ch}(\mathcal{A})$, the associated diagram of simplicial sets
$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(B'_\bullet, D_\bullet) & \longrightarrow & \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A'_\bullet, D_\bullet) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(B_\bullet, D_\bullet) & \xrightarrow{f'} & \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, D_\bullet) \end{array}$$

is homotopy Cartesian. The above diagram is obviously a pullback, it will suffice to prove that f' is a Kan fibration. This follows from the fact that f' is the map of simplicial sets associated (under the Dold-Kan correspondence) to a map between complexes of abelian groups which is surjective in positive (homological) degrees.

It follows from (1) that the ∞-category $\mathrm{N}(\mathrm{Ch}(\mathcal{A}))$ admits pushouts: it suffices to observe that any morphism $f : A_\bullet \to B_\bullet$ is chain homotopy-equivalent to a morphism which is degreewise split (replace $B_\bullet$ by the mapping cylinder of f). It is obvious that $\mathrm{N}(\mathrm{Ch}(\mathcal{A}))$ has a zero object (since $\mathrm{Ch}(\mathcal{A})$ has a zero object). Moreover, we can use (1) to describe the suspension functor on $\mathrm{Ch}(\mathcal{A})$: for each $A_\bullet \in \mathrm{Ch}(\mathcal{A})$, let $C(A_\bullet)$ denote the cone of $A_\bullet$, so that $C(A_\bullet) \simeq 0$ and there is a pushout diagram

$$\begin{array}{ccc} A_\bullet & \longrightarrow & C(A_\bullet) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & A_{\bullet-1}. \end{array}$$

It follows that the suspension functor Σ can be identified with the shift functor

$$A_\bullet \mapsto A_{\bullet-1}.$$

In particular, we conclude that Σ is an equivalence of ∞-categories, so that $\mathcal{C}$ is stable (Proposition 10.12). □

Remark 11.5. Let $\mathcal{A}$ be an additive category, and let $\mathrm{Ch}'(\mathcal{A})$ be a full subcategory of $\mathrm{Ch}(\mathcal{A})$. Suppose that $\mathrm{Ch}'(\mathcal{A})$ is stable under translations and the formation of mapping cones. Then the proof of Lemma 11.4 shows that $\mathrm{N}(\mathrm{Ch}'(\mathcal{A}))$ is a stable subcategory of $\mathrm{N}(\mathrm{Ch}(\mathcal{A}))$. In particular, if $\mathrm{Ch}^+(\mathcal{A})$ denotes the full subcategory of $\mathrm{Ch}(\mathcal{A})$ spanned by those complexes $A_\bullet$ such that $A_n \simeq 0$ for $n \ll 0$, then $\mathrm{N}(\mathrm{Ch}^+(\mathcal{A}))$ is a stable subcategory of $\mathrm{N}(\mathrm{Ch}(\mathcal{A}))$.

Definition 11.6. Let $\mathcal{A}$ be an abelian category with enough projective objects. We let $\mathcal{D}^+(\mathcal{A})$ denote the nerve of the simplicial category $\mathrm{Ch}^+(\mathcal{A}_0)$, where $\mathcal{A}_0 \subseteq \mathcal{A}$ is the full subcategory spanned by the projective objects of $\mathcal{A}$. We will refer to $\mathcal{D}^+(\mathcal{A})$ as the *derived ∞-category of $\mathcal{A}$*.

Remark 11.7. The homotopy category $\mathrm{h}\mathcal{D}^+(\mathcal{A})$ can be described as follows: objects are given by (bounded above) chain complexes of projective objects of $\mathcal{A}$, and morphisms are given by homotopy classes of chain maps. Consequently, $\mathrm{h}\mathcal{D}^+(\mathcal{A})$ can be identified with the derived category of $\mathcal{A}$ studied in classical homological algebra (with appropriate boundedness conditions imposed).

Lemma 11.8. *Let $\mathcal{A}$ be an abelian category, and let $P_\bullet \in \mathrm{Ch}(\mathcal{A})$ be a complex of projective objects of $\mathcal{A}$ such that $P_n \simeq 0$ for $n \ll 0$. Let $Q_\bullet \to Q'_\bullet$ be a quasi-isomorphism in $\mathrm{Ch}(\mathcal{A})$. Then the induced map*

$$\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(P_\bullet, Q_\bullet) \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(P_\bullet, Q'_\bullet)$$

is a homotopy equivalence.

Proof. We observe that $P_\bullet$ is a homotopy colimit of its naive truncations

$$\ldots \to 0 \to P_n \to P_{n-1} \to \ldots.$$

It therefore suffices to prove the result for each of these truncations, so we may assume that $P_\bullet$ is concentrated in finitely many degrees. Working by induction, we can reduce to the case where $P_\bullet$ is concentrated in a single degree. Shifting, we can reduce to the case where $P_\bullet$ consists of a single projective object P concentrated in degree zero. Since P is projective, we have isomorphisms

$$\mathrm{Ext}^i_{\mathrm{N}(\mathrm{Ch}(\mathcal{A}))}(P_\bullet, Q_\bullet) \simeq \mathrm{Hom}_{\mathcal{A}}(P, \mathrm{H}_{-i}(Q_\bullet)) \simeq \mathrm{Hom}_{\mathcal{A}}(P, \mathrm{H}_{-i}(Q'_\bullet)) \simeq \mathrm{Ext}^i_{\mathrm{N}(\mathrm{Ch}(\mathcal{A}))}(P_\bullet, Q'_\bullet).$$

□

Lemma 11.9. *Let $\mathcal{A}$ be an abelian category. Suppose that $P_\bullet, Q_\bullet \in \mathrm{Ch}(\mathcal{A})$ have the following properties:*

(1) *Each P_n is projective, and $P_n \simeq 0$ for $n < 0$.*

(2) *The homologies $\mathrm{H}_n(Q_\bullet)$ vanish for $n > 0$.*

Then the space $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(P_\bullet, Q_\bullet)$ is discrete, and we have a canonical isomorphism of abelian groups

$$\mathrm{Ext}^0(P_\bullet, Q_\bullet) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}_0(P_\bullet), \mathrm{H}_0(Q_\bullet)).$$

Proof. Let $Q'_\bullet$ be the complex

$$\ldots \to 0 \to \mathrm{coker}(Q_1 \to Q_0) \to Q_{-1} \to \ldots.$$

Condition (2) implies that the canonical map $Q_\bullet \to Q'_\bullet$ is a quasi-isomorphism. In view of (1) and Lemma 11.8, it will suffice to prove the result after replacing $Q_\bullet$ by $Q'_\bullet$. The result now follows from Example 11.3. ☐

Proposition 11.10. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Then:*

(1) *The ∞-category $\mathcal{D}^+(\mathcal{A})$ is stable.*

(2) *Let $\mathcal{D}^+_{\geq 0}(\mathcal{A})$ be the full subcategory of $\mathcal{D}^+(\mathcal{A})$ spanned by those complexes $A_\bullet$ such that the homology objects $\mathrm{H}_n(A_\bullet) \in \mathcal{A}$ vanish for $n < 0$, and let $\mathcal{D}^+_{\leq 0}(\mathcal{A})$ be defined similarly. Then $(\mathcal{D}^+_{\geq 0}(\mathcal{A}), \mathcal{D}^+_{\leq 0}(\mathcal{A}))$ determine a t-structure on $\mathcal{D}^+(\mathcal{A})$.*

(3) *The heart of $\mathcal{D}^+(\mathcal{A})$ is equivalent to (the nerve of) the abelian category $\mathcal{A}$.*

Proof. Assertion (1) follows from Remark 11.5.

To prove (2), we first make the following observation:

($*$) For any object $A_\bullet \in \mathrm{Ch}(\mathcal{A})$, there exists a map $f : P_\bullet \to A_\bullet$ where each P_n is projective, $P_n \simeq 0$ for $n < 0$, and the induced map $\mathrm{H}_k(P_\bullet) \to \mathrm{H}_k(A_\bullet)$ is an isomorphism for $k \geq 0$.

This is proven by a standard argument in homological algebra, using the assumption that $\mathcal{A}$ has enough projectives. We also note that if $A_\bullet \in \mathcal{D}^+(\mathcal{A})$ and the homologies $\mathrm{H}_n(A_\bullet)$ vanish for $n < 0$, then f is a quasi-isomorphism between projective complexes and therefore a chain homotopy equivalence.

It is obvious that $\mathcal{D}^+_{\leq 0}(\mathcal{A})[-1] \subseteq \mathcal{D}^+_{\leq 0}(\mathcal{A})$ and $\mathcal{D}^+_{\geq 0}(\mathcal{A})[1] \subseteq \mathcal{D}^+_{\geq 0}(\mathcal{A})$. Suppose now that $A_\bullet \in \mathcal{D}^+_{\geq 0}(\mathcal{A})$ and $B_\bullet \in \mathcal{D}^+_{\leq -1}(\mathcal{A})$; we wish to show that $\mathrm{Ext}^0_{\mathcal{D}^+(\mathcal{A})}(A_\bullet, B_\bullet) \simeq 0$. Using ($*$), we may reduce to the case where $A_n \simeq 0$ for $n < 0$. The desired result now follows immediately from Lemma 11.9. Finally, choose an arbitrary object $A_\bullet \in \mathcal{D}^+(\mathcal{A})$, and let $f : P_\bullet \to A_\bullet$ be as in ($*$). It is easy to see that $\mathrm{coker}(f) \in \mathcal{D}^+_{\leq -1}(\mathcal{A})$. This completes the proof of (2).

To prove (3), we begin by observing that the functor $A_\bullet \mapsto \mathrm{H}_0(A_\bullet)$ determines a functor $\theta : \mathrm{N}(\mathrm{Ch}(\mathcal{A})) \to \mathrm{N}(\mathcal{A})$. Let $\mathcal{C} \subseteq \mathrm{N}(\mathrm{Ch}(\mathcal{A}))$ be the full subcategory spanned by complexes $P_\bullet$ such that each P_n is projective, $P_n \simeq 0$ for $n < 0$, and $\mathrm{H}_n(P_\bullet) \simeq 0$ for $n \neq 0$. Assertion ($*$) implies that the inclusion $\mathcal{C} \subseteq \mathcal{D}^+(\mathcal{A})^{\heartsuit}$ is an equivalence of ∞-categories. Lemma 11.9 implies that $\theta | \mathcal{C}$ is fully faithful. Finally, we can apply ($*$) in the case where $A_\bullet$ is concentrated in degree zero to deduce that $\theta | \mathcal{C}$ is essentially surjective. This proves (3). ☐

Remark 11.11. Let $\mathcal{A}$ be an abelian category with enough projective objects. Then $\mathcal{D}^+(\mathcal{A})$ is a colocalization of $\mathrm{N}(\mathrm{Ch}^+(\mathcal{A}))$. To prove this, it will suffice to show that for every $A_\bullet \in \mathrm{Ch}^+(\mathcal{A})$, there exists a map of chain complexes $f : P_\bullet \to A_\bullet$ where $P_\bullet \in \mathcal{D}^+(\mathcal{A})$, and such that f induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(Q_\bullet, P_\bullet) \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(Q_\bullet, A_\bullet)$$

for every $Q_\bullet \in \mathcal{D}^+(\mathcal{A})$ (Proposition T.5.2.6.7). According Lemma 11.8, it will suffice to choose f to be a quasi-isomorphism; the existence now follows from ($*$) in the proof of Proposition 11.10.

Let $L : \mathrm{N}(\mathrm{Ch}^+(\mathcal{A})) \rightarrow \mathcal{D}^+(\mathcal{A})$ be a right adjoint to the inclusion. Roughly speaking, the functor L associates to each complex $A_\bullet$ a projective resolution $P_\bullet$ as above. We observe that, if $f : A_\bullet \rightarrow B_\bullet$ is a map of complexes, then Lf is a chain homotopy equivalence if and only if f is a quasi-isomorphism. Consequently, we may regard $\mathcal{D}^+(\mathcal{A})$ as the ∞-category obtained from $\mathrm{N}(\mathrm{Ch}^+(\mathcal{A}))$ by inverting all quasi-isomorphism.

12 Nonabelian Derived Categories

According to Corollary T.4.2.3.11, we can analyze arbitrary colimits in an ∞-category $\mathcal{C}$ in terms of finite colimits and filtered colimits. In particular, if $\mathcal{C}$ admits finite colimits, then we can construct a new ∞-category $\mathrm{Ind}(\mathcal{C})$ by formally adjoining filtered colimits to $\mathcal{C}$. Then $\mathrm{Ind}(\mathcal{C})$ admits all small colimits (Theorem T.5.5.1.1), and the Yoneda embedding $\mathcal{C} \rightarrow \mathrm{Ind}(\mathcal{C})$ preserves small colimits (Proposition T.5.3.5.14). Moreover, we can identify $\mathrm{Ind}(\mathcal{C})$ with the ∞-category of functors $\mathcal{C}^{op} \rightarrow \mathcal{S}$ which carry finite colimits in $\mathcal{C}$ to finite limits in $\mathcal{S}$. In this section, we will introduce a variation on the same theme. Instead of assuming $\mathcal{C}$ admits *all* finite colimits, we will only assume that $\mathcal{C}$ admits finite coproducts. We will construct a coproduct-preserving embedding of $\mathcal{C}$ into a larger ∞-category $\mathcal{P}_\Sigma(\mathcal{C})$ which admits all small colimits. Moreover, we can characterize $\mathcal{P}_\Sigma(\mathcal{C})$ as the ∞-category obtained from $\mathcal{C}$ by formally adjoining filtered colimits and geometric realizations (Proposition 12.7). In §14, we will apply these ideas to study the derived ∞-category $\mathcal{D}^+(\mathcal{A})$ associated to an abelian category $\mathcal{A}$.

Definition 12.1. Let $\mathcal{C}$ be a small ∞-category which admits finite coproducts. We let $\mathcal{P}_\Sigma(\mathcal{C})$ denote the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by those functors $\mathcal{C}^{op} \rightarrow \mathcal{S}$ which preserve finite products.

Remark 12.2. The ∞-categories of the form $\mathcal{P}_\Sigma(\mathcal{C})$ have been studied in [19], where they are called *homotopy varieties*. Many of the results proven below can also be found in [19].

Proposition 12.3. *Let $\mathcal{C}$ be a small ∞-category which admits finite coproducts. Then:*

(1) *The ∞-category $\mathcal{P}_\Sigma(\mathcal{C})$ is an accessible localization of $\mathcal{P}(\mathcal{C})$. In particular, $\mathcal{P}_\Sigma(\mathcal{C})$ is presentable.*

(2) *The Yoneda embedding $j : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{C})$ factors through $\mathcal{P}_\Sigma(\mathcal{C})$. Moreover, j carries finite coproducts in $\mathcal{C}$ to finite coproducts in $\mathcal{P}_\Sigma(\mathcal{C})$.*

(3) *Let $\mathcal{D}$ be a presentable ∞-category, and let*

$$\mathcal{P}(\mathcal{C}) \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

be a pair of adjoint functors. Then G factors through $\mathcal{P}_\Sigma(\mathcal{C})$ if and only if $F \circ j : \mathcal{C} \rightarrow \mathcal{D}$ preserves finite coproducts.

(4) *The full subcategory $\mathcal{P}_\Sigma(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ is stable under filtered colimits and geometric realizations of simplicial objects.*

(5) *Let $L : \mathcal{P}(\mathcal{C}) \rightarrow \mathcal{P}_\Sigma(\mathcal{C})$ be a left adjoint to the inclusion. Then L preserves filtered colimits and geometric realizations of simplicial objects.*

(6) *The ∞-category $\mathcal{P}_\Sigma(\mathcal{C})$ is compactly generated.*

Before giving the proof, we need a preliminary result concerning the interactions between products and geometric realizations.

Lemma 12.4. *Let $\mathcal{X}$ be an ∞-topos, and let $f : \mathcal{X}_\Delta \rightarrow \mathcal{X}$ be a geometric realization functor (which associates to every simplicial object $U_\bullet : \mathrm{N}(\mathbf{\Delta})^{op} \rightarrow \mathcal{X}$ its colimit $|U_\bullet|$). Then f commutes with finite products.*

Proof. Since the simplicial set $\mathrm{N}(\mathbf{\Delta})$ is weakly contractible, Corollary T.4.4.4.9 implies that f preserves final objects. To complete the proof, it will suffice to show that f preserves pairwise products. Let $X_\bullet$ and $Y_\bullet$ be simplicial objects of $\mathcal{X}$. We wish to prove that the canonical map

$$f(X_\bullet \times Y_\bullet) \to f(X_\bullet) \times f(Y_\bullet)$$

is an equivalence. Since colimits in $\mathcal{X}$ are universal, the right hand side can be identified with the colimit of the bisimplicial object

$$U_{\bullet,\bullet} : \mathrm{N}(\mathbf{\Delta} \times \mathbf{\Delta})^{op} \to \mathcal{S}$$

$$U_{n,m} = X_n \times Y_m,$$

while the left hand side can be identified with the colimit of the diagonal of $U_{\bullet,\bullet}$. We conclude by observing that the diagonal map $\mathrm{N}(\mathbf{\Delta})^{op} \to \mathrm{N}(\mathbf{\Delta} \times \mathbf{\Delta})^{op}$ is cofinal (Lemma T.6.5.3.6). □

Proof of Proposition 12.3. Part (1) follows immediately from Lemmas T.5.5.4.20, T.5.5.4.21, and T.5.5.4.22. To prove (2), it suffices to show that for every representable functor $e : \mathcal{P}_\Sigma(\mathcal{C})^{op} \to \mathcal{S}$, the composition

$$\mathcal{C}^{op} \stackrel{j^{op}}{\to} \mathcal{P}_\Sigma(\mathcal{C})^{op} \stackrel{e}{\to} \mathcal{S}$$

is preserves finite products (Proposition T.5.1.3.2). This is obvious, since the composition can be identified with the object of $\mathcal{P}_\Sigma(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ representing f.

We next prove (3). We note that f preserves finite coproducts if and only if, for every object $D \in \mathcal{D}$, the composition

$$\mathcal{C}^{op} \stackrel{f^{op}}{\to} \mathcal{D}^{op} \stackrel{e_D}{\to} \mathcal{S}$$

preserves finite products, where e_D denotes the functor represented by D. This composition can be identified with $G(D)$, so that f preserves finite coproducts if and only if G factors through $\mathcal{P}_\Sigma(\mathcal{C})$.

We now consider (4). The stability of $\mathcal{P}_\Sigma(\mathcal{C})$ under filtered colimits follows from the compatibility of filtered colimits with finite limits (Proposition T.5.3.3.3). The stability under geometric realizations follows from Lemma 12.4.

Assertion (5) follows formally from (4). To prove (6), we first observe that $\mathcal{P}(\mathcal{C})$ is compactly generated (Proposition T.5.3.5.12). Let $\mathcal{E} \subseteq \mathcal{P}(\mathcal{C})$ be the full subcategory spanned by the compact objects, and let $L : \mathcal{P}(\mathcal{C}) \to \mathcal{P}_\Sigma(\mathcal{C})$ be a localization functor. Since $\mathcal{E}$ generates $\mathcal{P}(\mathcal{C})$ under filtered colimits, $L(\mathcal{D})$ generates $\mathcal{P}_\Sigma(\mathcal{C})$ under filtered colimits. Consequently, it will suffice to show that for each $E \in \mathcal{E}$, the object $LE \in \mathcal{P}_\Sigma(\mathcal{C})$ is compact. Let $f : \mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{S}$ be the functor co-represented by LE, and let $f' : \mathcal{P}(\mathcal{C}) \to \mathcal{S}$ be the functor co-represented by E. Then the map $E \to LE$ induces an equivalence $f \to f' | \mathcal{P}_\Sigma(\mathcal{C})$. Since f' is continuous and $\mathcal{P}_\Sigma(\mathcal{C})$ is stable under filtered colimits in $\mathcal{P}(\mathcal{C})$, we conclude that f is continuous, so that LE is a compact object of $\mathcal{P}_\Sigma(\mathcal{C})$ as desired. □

Our next goal is to prove a partial converse to part (4) of Proposition 12.3. Namely, we will show that $\mathcal{P}_\Sigma(\mathcal{C})$ is generated by the essential image of the Yoneda embedding under filtered colimits and geometric realizations (Lemma 12.6). The proof is based on the following technical result:

Lemma 12.5. *Let $\mathcal{C}$ be a small ∞-category, and let X be an object of $\mathcal{P}(\mathcal{C})$. Then there exists a simplicial object $Y_\bullet : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{P}(\mathcal{C})$ with the following properties:*

(1) *The colimit of $Y_\bullet$ is equivalent to X.*

(2) *For each $n \geq 0$, the object $Y_n \in \mathcal{P}(\mathcal{C})$ is equivalent to a small coproduct of objects lying in the image of the Yoneda embedding $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$.*

We will defer the proof until the end of this section.

Lemma 12.6. *Let $\mathcal{C}$ be a small ∞-category which admits finite coproducts, and let $X \in \mathcal{P}(\mathcal{C})$. The following conditions are equivalent:*

(1) *The object X belongs to $\mathcal{P}_\Sigma(\mathcal{C})$.*

(2) *There exists a simplicial object $U_\bullet : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathrm{Ind}(\mathcal{C})$ whose colimit in $\mathcal{P}(\mathcal{C})$ is X.*

Proof. The full subcategory $\mathcal{P}_\Sigma(\mathcal{C})$ contains the essential image of the Yoneda embedding and is stable under filtered colimits and geometric realizations (Proposition 12.3); thus (2) $\Rightarrow$ (1). We will prove that (1) $\Rightarrow$ (2).

We first choose a simplicial object $Y_\bullet$ of $\mathcal{P}(\mathcal{C})$ which satisfies the conclusions of Lemma 12.5. Let L be a left adjoint to the inclusion $\mathcal{P}_\Sigma(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$. Since X is a colimit of $Y_\bullet$, $LX \simeq X$ is a colimit of $LY_\bullet$ (part (5) of Proposition 12.3. It will therefore suffice to prove that each LY_n belongs to $\mathrm{Ind}(\mathcal{C})$. By hypothesis, each Y_n can be written as a small coproduct $\coprod_{\alpha \in A} j(C_\alpha)$, where $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ denotes the Yoneda embedding. Using the results of §T.4.2.3, we see that Y_n can be obtained also as a filtered colimit of coproducts $\coprod_{\alpha \in A_0} j(C_\alpha)$, where A_0 ranges over the finite subsets of A. Since L preserves filtered colimits (Proposition 12.3), it will suffice to show that each of the objects

$$L(\coprod_{\alpha \in A_0} j(C_\alpha))$$

belongs to $\mathrm{Ind}(\mathcal{C})$. We now invoke part (2) of Proposition 12.3 to identify this object with $j(\coprod_{\alpha \in A_0} C_\alpha)$. □

Proposition 12.7. *Let $\mathcal{C}$ be a small ∞-category which admits finite coproducts, and let $\mathcal{D}$ be an ∞-category which admits filtered colimits and geometric realizations. Let $\mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D})$ denote the full subcategory spanned by those functors $\mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{D}$ which preserve filtered colimits and geometric realizations. Then:*

(1) *Composition with the Yoneda embedding $j : \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C})$ induces an equivalence of categories*

$$\theta : \mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

(2) *Assume that $\mathcal{D}$ admits finite coproducts. A functor $g \in \mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D})$ preserves finite coproducts if and only if $g \circ j$ preserves finite coproducts.*

Proof. Lemma 12.6 and Proposition 12.3 imply that $\mathcal{P}_\Sigma(\mathcal{C})$ is the smallest full subcategory of $\mathcal{P}(\mathcal{C})$ which is closed under filtered colimits, closed under geometric realizations, and contains the essential image of the Yoneda embedding. Consequently, assertion (1) follows from Remark T.5.3.5.9 and Proposition T.4.3.2.15.

The "only if" direction of (2) is immediate, since the Yoneda embedding $j : \mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{C})$ preserves finite coproducts (Proposition 12.3). To prove the converse, we first apply Lemma T.5.3.5.7 to reduce to the case where $\mathcal{D}$ is a full subcategory of an ∞-category $\mathcal{D}'$, with the following properties:

(i) The ∞-category $\mathcal{D}'$ admits all small colimits.

(ii) A small diagram $K^\triangleright \to \mathcal{D}$ is a colimit if and only if the induced diagram $K^\triangleright \to \mathcal{D}'$ is a colimit.

Using Lemma T.5.1.5.5, we conclude that there exists functor $G : \mathcal{P}(\mathcal{C}) \to \mathcal{D}'$ which is a left Kan extension of $G|\,\mathcal{C}' = g|\,\mathcal{C}'$, such that G preserves small colimits. Without loss of generality, we may suppose that $g = G|\,\mathcal{P}_\Sigma(\mathcal{C})$. Since $G \circ j = g \circ j$ preserves finite coproducts, $g = G|\,\mathcal{P}_\Sigma(\mathcal{C})$ is a colimit preserving functor $\mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{D}'$ (part (3) of Proposition 12.3). □

Under the hypotheses of Proposition 12.7, every functor $f : \mathcal{C} \to \mathcal{D}$ extends (up to homotopy) to a functor $F : \mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{D}$, which preserves filtered colimits and geometric realizations. We will sometimes refer to F as the *left derived functor* of f. In §14, we will explain how to specialize this concept to recover the classical theory of left derived functors in homological algebra.

We conclude this section by giving the proof of Lemma 12.5. First, we need a few preliminaries.

Remark 12.8 (Anatomy of a Simplicial Object). Let $\mathcal{X}$ be an ∞-category, and suppose we wish to construct a simplicial object $\psi : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{X}$. For each $n \geq -1$, we let $\mathbf{\Delta}^{\leq n}$ denote the full subcategory of $\mathbf{\Delta}$ spanned by the objects $\{[k]\}_{k \leq n}$. Suppose that we have already constructed a functor $\psi(n-1) : \mathrm{N}(\mathbf{\Delta}^{\leq n-1})^{op} \to \mathcal{X}$ and we would like to find a compatible extension $\psi(n) : \mathrm{N}(\mathbf{\Delta}^{\leq n})^{op} \to \mathcal{X}$.

Let $K^{(k)}$ denote the simplicial subset of $\mathrm{N}(\mathbf{\Delta}^{\leq n})^{op}$ spanned by those nondegenerate simplices which include the vertex $[n] \in \mathbf{\Delta}^{\leq n}$ no more than k times. Then we have a filtration

$$\mathrm{N}(\mathbf{\Delta}^{\leq n-1})^{op} \simeq K^{(0)} \subseteq K^{(1)} \subseteq \ldots$$

which exhausts $\mathrm{N}(\mathbf{\Delta}^{\leq n})^{op}$. An easy calculation shows that the inclusions $K^{(k)} \subseteq K^{(k+1)}$ are categorical equivalences for $k \geq 1$. It follows that $\psi(n)$ is determined, up to equivalence, by the restriction $\psi(n)|K^{(1)}$. Moreover, we have a pushout diagram of simplicial sets

$$\begin{array}{ccc} \mathrm{N}(\mathbf{\Delta}^{\leq n-1})^{op}_{/[n]} \star \mathrm{N}(\mathbf{\Delta}^{\leq n-1})^{op}_{[n]/} & \longrightarrow & \mathrm{N}(\mathbf{\Delta}^{\leq n-1}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{N}(\mathbf{\Delta}^{\leq n-1})^{op}_{/[n]} \star \{[n]\} \star \mathrm{N}(\mathbf{\Delta}^{\leq n-1})^{op}_{[n]/} & \longrightarrow & K^{(1)}. \end{array}$$

If $\mathcal{X}$ admits finite limits and colimits, then we can define a *matching object* M_n to be a limit of the composition

$$\mathrm{N}(\mathbf{\Delta}^{\leq n-1})_{[n]/} \rightarrow \mathrm{N}(\mathbf{\Delta}^{\leq n-1}) \stackrel{\psi(n-1)}{\rightarrow} \mathcal{X}$$

and a *latching object* L_n to be a colimit of the composition

$$\mathrm{N}(\mathbf{\Delta}^{\leq n-1})_{/[n]} \rightarrow \mathrm{N}(\mathbf{\Delta}^{\leq n-1}) \stackrel{\psi(n-1)}{\rightarrow} \mathcal{X}.$$

The diagram $\psi(n-1)$ determines a morphism $\phi : L_n \rightarrow M_n$ (well-defined up to homotopy), and constructing $\psi(n)$ is equivalent to producing a factorization

$$\begin{array}{ccccc} & & Y_n & & \\ & \nearrow & & \searrow & \\ L_n & & \xrightarrow{\phi} & & M_n. \end{array}$$

Proof of Lemma 12.5. The ∞-topos $\mathcal{P}(\mathcal{C})$ has enough points (given by evaluation at objects of $\mathcal{C}$), and is therefore hypercomplete (Remark T.6.5.4.7). To achieve (1), it will suffice to construct $Y_\bullet$ as the simplicial object underlying a hypercovering $\psi : \mathrm{N}(\mathbf{\Delta})^{op} \rightarrow \mathcal{P}(\mathcal{C})_{/X}$. We will construct the restrictions $\psi(n) = \psi|\,\mathrm{N}(\mathbf{\Delta}^{\leq n})^{op}$ by induction on n. Assume that $\psi(n-1)$ has already been constructed. Then $\psi(n-1)$ determines latching and matching objects L_n and M_n, and a map $\phi : L_n \rightarrow M_n$ in $\mathcal{P}(\mathcal{C})_{/X}$ (see Remark 12.8). We will abuse notation by identifying L_n and M_n with their images in $\mathcal{P}(\mathcal{C})$. Choose an effective epimorphism $Z_n \rightarrow M_n$ in $\mathcal{P}(\mathcal{C})$, where Z_n is a small coproduct of objects lying in the essential image of j. We then have a factorization

$$\begin{array}{ccccc} & & L_n \coprod Z_n & & \\ & \nearrow & & \searrow^{\phi'} & \\ L_n & & \xrightarrow{\phi} & & M_n \end{array}$$

which determines the desired extension $\phi(n)$. By construction, each of the maps ϕ' is an effective epimorphism, so that $\psi : \mathrm{N}(\mathbf{\Delta})^{op} \rightarrow \mathcal{P}(\mathcal{C})_{/X}$ is a hypercovering as desired; let $Y_\bullet$ be the underlying simplicial object of $\mathcal{P}(\mathcal{C})$. Then Y satisfies (1). We now observe that each Y_n can be obtained as the coproduct of objects Z_k, indexed by the collection of surjective maps $[n] \rightarrow [k]$. This proves (2). ☐

13 Quillen's Model for $\mathcal{P}_\Sigma(\mathcal{C})$

Let $\mathcal{C}$ be a small category which admits finite products. Then $\mathrm{N}(\mathcal{C})^{op}$ is an ∞-category which admits finite coproducts. In §12, we studied the ∞-category $\mathcal{P}_\Sigma(\mathrm{N}(\mathcal{C})^{op})$, which we can view as the full subcategory of

$\mathrm{Fun}(\mathrm{N}(\mathcal{C}), \mathcal{S})$ spanned by the product-preserving functors. According to Proposition T.A.3.6.1, $\mathrm{Fun}(\mathrm{N}(\mathcal{C}), \mathcal{S})$ is equivalent to the underlying ∞-category of the model category of diagrams $\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}$ (which we will endow with the *projective* model structure described in §T.A.3.1). It follows that every functor $f : \mathrm{N}(\mathcal{C}) \to \mathcal{S}$ is equivalent to the (simplicial) nerve of a functor $F : \mathcal{C} \to \mathcal{K}\mathrm{an}$. Moreover, f belongs to $\mathcal{P}_{\Sigma}(\mathrm{N}(\mathcal{C})^{op})$ if and only if the functor F is *weakly* product preserving, in the sense that for any finite collection of objects $\{C_i \in \mathcal{C}\}_{1 \leq i \leq n}$, the natural map

$$F(C_1 \times \ldots C_n) \to F(C_1) \times \ldots \times F(C_n)$$

is a homotopy equivalence of Kan complexes. Our goal in this section is to prove a refinement of Proposition T.A.3.6.1: if f preserves finite products, then it is possible to arrange that F preserves finite products (up to isomorphism, rather than up to homotopy equivalence). Our result is most naturally phrased as an equivalence between model categories (Proposition 13.2).

Proposition 13.1 (Quillen). *Let $\mathcal{C}$ be a category which admits finite products, and let $\mathbf{A}$ denote the category of functors $F : \mathcal{C} \to \mathcal{S}\mathrm{et}_{\Delta}$ which preserve finite products. Then $\mathbf{A}$ has the structure of a simplicial model category, where:*

- (W) *A natural transformation $\alpha : F \to F'$ of functors is a weak equivalence in $\mathbf{A}$ if and only if $\alpha(C) : F(C) \to F'(C)$ is a weak homotopy equivalence of simplicial sets, for every $C \in \mathcal{C}$.*
- (F) *A natural transformation $\alpha : F \to F'$ of functors is a fibration in $\mathbf{A}$ if and only if $\alpha(C) : F(C) \to F'(C)$ is a Kan fibration of simplicial sets, for every $C \in \mathcal{C}$.*

For a proof, we refer the reader to [18].

Suppose that $\mathcal{C}$ and $\mathbf{A}$ are as in the statement of Proposition 13.1. Then we may regard $\mathbf{A}$ as a full subcategory of the category $\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}$ of *all* functors from $\mathcal{C}$ to $\mathcal{S}\mathrm{et}_{\Delta}$, which we regard as endowed with the projective model structure (so that fibrations and weak equivalences are given pointwise). The inclusion $G : \mathbf{A} \subseteq \mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}$ preserves fibrations and trivial fibrations, and therefore determines a Quillen adjunction

$$\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{A}.$$

(A more explicit description of the adjoint functor F will be given below.) Our main result can be stated as follows:

Proposition 13.2. *Let $\mathcal{C}$ be a small category which admits finite products, and let*

$$\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{A}$$

be as above. Then the induced map of homotopy categories

$$RG : \mathrm{h}\mathbf{A} \to \mathrm{h}\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}$$

is fully faithful, and an object $f \in \mathrm{h}\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}$ belongs to the essential image of RG if and only if f is compatible with finite products up to weak homotopy equivalence.

Corollary 13.3. *Let $\mathcal{C}$ be a small category which admits finite products, and let $\mathbf{A}$ be as in Proposition 13.2. Then the natural map*

$$\mathrm{N}(\mathbf{A}^{o}) \to \mathcal{P}_{\Sigma}(\mathrm{N}(\mathcal{C})^{op})$$

is an equivalence of ∞-categories.

The proof of Proposition 13.8 is somewhat technical and will occupy the rest of this section. We begin by introducing some preliminary ideas.

Notation 13.4. Let $\mathcal{C}$ be a small category. We define a pair of categories $\mathcal{U}(\mathcal{C}) \subseteq \mathcal{U}^+(\mathcal{C})$ as follows:

(*i*) An object of $\mathcal{U}^+(\mathcal{C})$ is a pair $C = (J, \{C_j\}_{j\in J})$, where J is a finite set and each C_j is an object of $\mathcal{C}$. The object C belongs to $\mathcal{U}(\mathcal{C})$ if and only if J is nonempty.

(*ii*) Given objects $C = (J, \{C_j\}_{j\in J})$ and $C' = (J', \{C'_{j'}\}_{j'\in J'})$ of $\mathcal{U}^+(\mathcal{C})$, a morphism $C \to C'$ consists of the following data:

(*a*) A map $f : J' \to J$ of finite sets.

(*b*) For each $j' \in J'$, a morphism $C_{f(j')} \to C'_{j'}$ in the category $\mathcal{C}$.

Such a morphism belongs to $\mathcal{U}(\mathcal{C})$ if and only if J and J' are nonempty, and f is surjective.

There is a fully faithful embedding functor $\theta : \mathcal{C} \to \mathcal{U}(\mathcal{C})$, given by $C \mapsto (*, \{C\})$. We can view $\mathcal{U}^+(\mathcal{C})$ as the category obtained from $\mathcal{C}$ by freely adjoining finite products. In particular, if $\mathcal{C}$ admits finite products, then θ admits a (product-preserving) left inverse $\phi^+_{\mathcal{C}}$, given by the formula $(J, \{C_j\}_{j\in J}) \mapsto \prod_{j\in J} C_j$.. We let $\phi_{\mathcal{C}}$ denote the restricton $\phi^+_{\mathcal{C}} | \mathcal{U}(\mathcal{C})$.

Given a functor $\mathcal{F} \in \mathcal{S}\mathrm{et}_\Delta^{\mathcal{D}}$, we let $E^+(\mathcal{F}) \in \mathcal{S}\mathrm{et}_\Delta^{\mathcal{U}^+(\mathcal{C})}$ denote the composition

$$\mathcal{U}^+(\mathcal{C}) \stackrel{\mathcal{U}^+(\mathcal{F})}{\to} \mathcal{U}^+(\mathcal{S}\mathrm{et}_\Delta) \stackrel{\phi^+_{\mathcal{S}\mathrm{et}_\Delta}}{\to} \mathcal{S}\mathrm{et}_\Delta$$

$$(J, \{C_j\}_{j\in J}) \mapsto \prod f(C_j).$$

We let $E(\mathcal{F})$ denote the restriction $E^+(\mathcal{F}) | \mathcal{U}(\mathcal{C}) \in \mathcal{S}\mathrm{et}_\Delta^{\mathcal{U}(\mathcal{C})}$.

If the category $\mathcal{C}$ admits finite products, then we let $L, L^+ : \mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}} \to \mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}}$ denote the compositions

$$\mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}} \stackrel{E}{\to} \mathcal{S}\mathrm{et}_\Delta^{\mathcal{U}(\mathcal{C})} \stackrel{(\phi_{\mathcal{C}})_!}{\to} \mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}}$$

$$\mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}} \stackrel{E^+}{\to} \mathcal{S}\mathrm{et}_\Delta^{\mathcal{U}^+(\mathcal{C})} \stackrel{(\phi^+_{\mathcal{C}})_!}{\to} \mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}},$$

where $(\phi_{\mathcal{C}})_!$ and $(\phi^+_{\mathcal{C}})_!$ indicate left Kan extension functors. There is a canonical isomorphism $\theta^* \circ E \simeq \mathrm{id}$, which induces a natural transformation $\alpha : \mathrm{id} \to L$. Let $\beta : L \to L^+$ indicate the natural transformation induced by the inclusion $\mathcal{U}(\mathcal{C}) \subseteq \mathcal{U}^+(\mathcal{C})$.

Remark 13.5. Let $\mathcal{C}$ be a small category. The functor $E^+ : \mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}} \to \mathcal{S}\mathrm{et}_\Delta^{\mathcal{U}^+(\mathcal{C})}$ is fully faithful, and has a left adjoint given by θ^*.

We begin by constructing the left adjoint which appears in the statement of Proposition 13.2.

Lemma 13.6. *Let $\mathcal{C}$ be a simplicial category which admits finite products, and let $\mathcal{F} \in \mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}}$. Then:*

(1) *The object $L^+(\mathcal{F}) \in \mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}}$ is product-preserving.*

(2) *If $\mathcal{F}' \in \mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}}$ is product-preserving, then composition with $\beta \circ \alpha$ induces an isomorphism of simplicial sets*

$$\mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}}}(L^+(\mathcal{F}), \mathcal{F}') \to \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}}}(\mathcal{F}, \mathcal{F}').$$

Proof. Suppose given a finite collection of objects $\{C_1, \ldots, C_n\}$ in $\mathcal{C}$, and let

$$u : L^+(\mathcal{F})(C_1 \times \ldots \times C_n) \to L^+(\mathcal{F})(C_1) \times \ldots \times L^+(\mathcal{F})(C_n)$$

be the product of the projection maps. We wish to show that u is an isomorphism of simplicial sets. We will given an explicit construction of a left inverse to u. For $C \in \mathcal{C}$, we let $\mathcal{U}^+(\mathcal{C})_{/C}$ denote the fiber product $\mathcal{U}^+(\mathcal{C}) \times_{\mathcal{D}} \mathcal{C}_{/C}$. For $1 \leq i \leq n$, let $\mathcal{G}_i$ denote the restriction of $E^+(\mathcal{F})$ to $\mathcal{U}^+(\mathcal{C})_{/C_i}$, and let

$$\mathcal{G} : \prod \mathcal{U}^+(\mathcal{D})_{/C_i} \to \mathcal{S}\mathrm{et}_\Delta$$

be the product of the functors $\mathcal{G}_i$. We observe that $L^+(\mathcal{F})(C_i) \simeq \varinjlim(\mathcal{G}_i)$, so that the product $\prod L^+(\mathcal{F})(C_i) \simeq \varinjlim(\mathcal{G})$. We now observe that the formation of products in $\mathcal{E}^+(\mathcal{C})$ gives an identification of $\mathcal{G}$ with the composition

$$\prod \mathcal{U}^+(\mathcal{C})_{/C_i} \to \mathcal{U}^+(\mathcal{D})_{/C_1 \times \ldots \times C_n} \stackrel{E^+(\mathcal{F})}{\to} \mathcal{S}\mathrm{et}_{\Delta}.$$

We thereby obtain a morphism

$$v : \varinjlim(\mathcal{G}) \to \varinjlim(E^+(\mathcal{F})| \mathcal{U}^+(\mathcal{D})_{/C_1 \times \ldots \times C_n} \simeq L^+(\mathcal{F})(C_1 \times \ldots \times C_n).$$

It is not difficult to check that v is an inverse to u.

We observe that (2) is equivalence to the assertion that composition with θ^* induces an isomorphism

$$\mathrm{Map}_{\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{U}^+(\mathcal{C})}}(E^+(\mathcal{F}), (\phi^+_{\mathcal{C}})^*(\mathcal{F}')) \to \mathrm{Map}_{\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}}(\mathcal{F}, \mathcal{F}').$$

Because $\mathcal{G}$ is product-preserving, there is a natural isomorphism $(\phi^+_{\mathcal{C}})^*(\mathcal{F}') \simeq E^+(\mathcal{F}')$. The desired result now follows from Remark 13.5. □

It follows that the functor $L^+ : \mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}} \to \mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}$ factors through $\mathbf{A}$, and can be identified with a left adjoint to the inclusion $\mathbf{A} \subseteq \mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}$. In order to prove Proposition 13.2, we need to be able to compute the functor L^+. We will do this in two steps: first, we show that (under mild hypotheses), the natural transformation $L \to L^+$ is a weak equivalence. Second, we will see that the colimit defining L is actually a homotopy colimit, and therefore has good properties. More precisely, we have the following pair of lemmas, whose proofs will be given at the end of this section.

Lemma 13.7. *Let $\mathcal{C}$ be a small category which admits finite products, and let $\mathcal{F} \in \mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}$ be a functor which carries the final object of $\mathcal{C}$ to a contractible Kan complex K. Then the canonical map $\beta : L(\mathcal{F}) \to L^+(\mathcal{F})$ is a weak equivalence in $\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}$.*

Lemma 13.8. *Let $\mathcal{C}$ be a small simplicial category. If $\mathcal{F}$ is a strongly cofibrant object of $\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}$, then $E(\mathcal{F})$ is a strongly cofibrant object of $\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{U}(\mathcal{C})}$.*

We are now almost ready to give the proof of Proposition 13.2. The essential step is contained in the following result:

Lemma 13.9. *Let $\mathcal{C}$ be a simplicial category finite admits finite products, and let*

$$\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{A}$$

be as in the statement of Proposition 13.2. Then:

(1) *The functors F and G are Quillen adjoints.*

(2) *If $\mathcal{F} \in \mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}$ is strongly cofibrant and weakly product preserving, then the unit map $\mathcal{F} \to (G \circ F)(\mathcal{F})$ is a weak equivalence.*

Proof. Assertion (1) is obvious, since G preserves fibrations and trivial cofibrations. It follows that F preserves weak equivalences between strongly cofibrant objects. Let $K \in \mathcal{S}\mathrm{et}_{\Delta}$ denote the image under $\mathcal{F}$ of the final object of $\mathcal{D}$. In proving (2), we are free to replace $\mathcal{F}$ by any weakly equivalent diagram which is also strongly cofibrant. Choosing a fibrant replacement for $\mathcal{F}$, we may suppose that K is a Kan complex. Since $\mathcal{F}$ is weakly product preserving, K is contractible.

In view of Lemma 13.6, we can identify the composition $G \circ F$ with L^+ and the unit map with the composition

$$\mathcal{F} \stackrel{\alpha}{\to} L(\mathcal{F}) \stackrel{\beta}{\to} L^+(\mathcal{F}).$$

Lemma 13.7 implies that β is a weak equivalence. Consequently, it will suffice to show that α is a weak equivalence.

We recall the construction of α. Let $\theta : \mathcal{C} \to \mathcal{U}(\mathcal{C})$ be as in Notation 13.4, so that there is a canonical isomorphism $\mathcal{F} \simeq \theta^* E(\mathcal{F})$. This isomorphism induces a natural transformation $\overline{\alpha} : \theta_! \mathcal{F} \to E(\mathcal{F})$. The functor α is obtained from $\overline{\alpha}$ by applying the functor $(\phi_{\mathcal{C}})_!$, and identifying $((\phi_{\mathcal{C}})_! \circ \theta_!)(\mathcal{F})$ with $\mathcal{F}$. We observe that $(\phi_{\mathcal{C}})_!$ preserves weak equivalences between strongly cofibrant objects. Since $\theta_!$ preserves strong cofibrations, $\theta_! \mathcal{F}$ is strongly cofibrant. Lemma 13.8 asserts that $E(\mathcal{F})$ is strongly cofibrant. Consequently, it will suffice to prove that $\overline{\alpha}$ is a weak equivalence in $\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{U}(\mathcal{C})}$. Unwinding the definitions, this reduces to the condition that $\mathcal{F}$ be weakly compatible with (nonempty) products. □

Proof of Proposition 13.2. Lemma 13.9 shows that (F, G) is a Quillen adjunction. To complete the proof, we must show:

- (i) The counit transformation $LF \circ RG \to \mathrm{id}$ is an isomorphism of functors from the homotopy category $\mathrm{h}\mathbf{A}$ to itself.
- (ii) The essential image of $RG : \mathrm{h}\mathbf{A} \to \mathrm{h}\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}$ consists precisely of those functors which are weakly product-preserving.

We observe that G preserves weak equivalences, so we can identify RG with G. Since G also detects weak equivalences, (i) will follow if we can show that the induced transformation $\theta : G \circ LF \circ G \to G$ is an isomorphism of functors. This transformation has a right inverse, given by composing the unit transformation $\mathrm{id} \to G \circ LF$ with G. Consequently, (i) follows immediately from Lemma 13.9.

The image of G consists precisely of the product-preserving diagrams $\mathcal{C} \to \mathcal{S}\mathrm{et}_{\Delta}$; it follows immediately that every diagram in the essential image of G is weakly product preserving. Lemma 13.9 implies the converse: every weakly product preserving functor belongs to the essential image of G. This proves (ii). □

Remark 13.10. Proposition 13.2 can be generalized to the situation where $\mathcal{C}$ is a *simplicial* category which admits finite products. We leave the necessary modifications to the reader.

It remains to prove Lemmas 13.8 and 13.7.

Proof of Lemma 13.8. For every object $C \in \mathcal{C}$ and every simplicial set K, we let $\mathcal{F}_C^K \in \mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}$ denote the functor given by the formula $\mathcal{F}_C^K(D) = \mathrm{Map}_{\mathcal{C}}(C, D) \times K$. A cofibration $K \to K'$ induces a strong cofibration $\mathcal{F}_C^K \to \mathcal{F}_C^{K'}$. We will refer to a strong cofibration of this form as a *generating strong cofibration.*

The small object argument implies that if $\mathcal{F} \in \mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}$, then there is a transfinite sequence

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_\alpha$$

with the following properties:

- (a) The functor $\mathcal{F}_0 : \mathcal{D} \to \mathcal{S}\mathrm{et}_{\Delta}$ is constant, with value $\emptyset$.
- (b) If $\lambda \leq \alpha$ is a limit ordinal, then $\mathcal{F}_\lambda = \bigcup_{\beta < \lambda} \mathcal{F}_\beta$.
- (c) For each $\beta < \alpha$, the inclusion $\mathcal{F}_\beta \subseteq \mathcal{F}_{\beta+1}$ is a pushout of a generating strong cofibration.
- (d) The functor $\mathcal{F}$ is a retract of $\mathcal{F}_\alpha$.

The functor $\mathcal{G} \mapsto E(\mathcal{G})$ preserves initial objects, filtered colimits, and retracts. Consequently, to show that $E(\mathcal{F})$ is strongly cofibrant, it will suffice to prove the following assertion:

$(*)$ Suppose given a cofibration $K \to K'$ of simplicial sets and a pushout diagram

$$\begin{array}{ccc} \mathcal{F}_C^K & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{F}_C^{K'} & \longrightarrow & \mathcal{G}' \end{array}$$

in $\mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}}$. If $E(\mathcal{G})$ is strongly cofibrant, then $E(\mathcal{G}')$ is strongly cofibrant.

To prove this, we will need to analyze the structure of $E(\mathcal{G}')$. Given an object $C' = (J, \{C'_j\}_{j \in J})$ of $\mathcal{U}(\mathcal{C})$, we have

$$E(\mathcal{G}')(C') = \prod_{j \in J} (\mathcal{G}(C'_j) \coprod_{K \times \mathrm{Map}_{\mathcal{C}}(C, C'_j)} (K' \times \mathrm{Map}_{\mathcal{C}}(C, C'_j)).$$

Let $\sigma : \Delta^n \to E(\mathcal{G}')(C')$ be a simplex, and let $J_\sigma \subseteq J$ be the collection of all indices j for which the corresponding simplex $\sigma(j) : \Delta^n \to \mathcal{G}'(C'_j)$ does not factor through $\mathcal{G}(C'_j)$. In this case, we can identify $\sigma(j)$ with an n-simplex of K' which does not belong to K. We will say that σ is of *index* $\leq k$ if the set $\{\sigma(j) : j \in J_\sigma\}$ has cardinality $\leq k$. Note that σ can be of index smaller than the cardinality of J_σ, since it is possible for $\sigma(j) = \sigma(j') \in \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^n, K')$ even if $j \neq j'$.

Let $E(\mathcal{G}')^{(k)}(C')$ be the full simplicial subset of $E(\mathcal{G}')(C')$ spanned by those simplices which are of index $\leq k$. It is easy to see that that $E(\mathcal{G}')^{(k)}(C')$ depends functorially in C', so we can view $E(\mathcal{G}')^{(k)}$ as an object of $\mathcal{S}\mathrm{et}_\Delta^{\mathcal{U}(\mathcal{C})}$. We observe that

$$E(\mathcal{G}) \simeq E(\mathcal{G}')^{(0)} \subseteq E(\mathcal{G}')^{(1)} \subseteq \ldots$$

and that the union of this sequence is $E(\mathcal{G}')$. Consequently, it will suffice to prove that each of the inclusions $E(\mathcal{G}')^{(k-1)} \subseteq E(\mathcal{G}')^{(k)}$ is a strong cofibration.

First, we need a bit of notation. Let us say that a simplex of ${K'}^k$ is *new* if it consists of k distinct simplices of K', none of which belong to K. We will say that a simplex of ${K'}^k$ is *old* if it is not new. The collection of old simplices of ${K'}^k$ determine a simplicial subset which we will denote by ${K'}^{(k)}$. We define a functor $\psi : \mathcal{U}(\mathcal{C}) \to \mathcal{U}(\mathcal{C})$ by the formula

$$\psi(J, \{C'_j\}_{j \in J}) = (J \cup \{1, \ldots, k\}, \{C'_j\}_{j \in J} \cup \{C\}_{\{1 \ldots k\}}).$$

Let $\psi^* : \mathcal{S}\mathrm{et}_\Delta^{\mathcal{U}(\mathcal{C})} \to \mathcal{S}\mathrm{et}_\Delta^{\mathcal{U}(\mathcal{C})}$ be given by composition with ψ, and let $\psi_! : \mathcal{S}\mathrm{et}_\Delta^{\mathcal{U}(\mathcal{C})} \to \mathcal{S}\mathrm{et}_\Delta^{\mathcal{U}(\mathcal{C})}$ be a left adjoint to ψ^* (a functor of left Kan extension). Since ψ^* preserves weak fibrations and weak equivalences, $\psi_!$ preserves strong cofibrations.

Recall that $\mathcal{S}\mathrm{et}_\Delta^{\mathcal{U}(\mathcal{C})}$ is tensored over the category of simplicial sets: Given an object $\mathcal{M} \in \mathcal{S}\mathrm{et}_\Delta^{\mathcal{U}(\mathcal{C})}$ and a simplicial set A, we let $\mathcal{M} \otimes A \in \mathcal{S}\mathrm{et}_\Delta^{\mathcal{U}(\mathcal{C})}$ be defined by the formula $(\mathcal{M} \otimes A)(D') = \mathcal{M}(D') \times A$. If $\mathcal{M}$ is strongly cofibrant, then the operation $\mathcal{M} \mapsto \mathcal{M} \otimes A$ preserves cofibrations in A.

There is an obvious map $E(\mathcal{G}) \otimes {K'}^k \to \psi^* E(\mathcal{G}')^{(k)}$, which restricts to a map $E(\mathcal{G}) \otimes {K'}^{(k)} \to \psi^* E(\mathcal{G}')^{(k-1)}$. Passing to adjoints, we obtain a commutative diagram

$$\begin{array}{ccc} \psi_!(E(\mathcal{G}) \otimes {K'}^{(k)}) & \longrightarrow & E(\mathcal{G}')^{(k-1)} \\ \downarrow & & \downarrow \\ \psi_!(E(\mathcal{G}) \otimes {K'}^k) & \longrightarrow & E(\mathcal{G}')^{(k)}. \end{array}$$

An easy computation shows that this diagram is coCartesian. Since $E(\mathcal{G})$ is strongly cofibrant, the above remarks imply that the left vertical map is a strong cofibration. It follows that the right vertical map is a strong cofibration as well, which completes the proof. $\square$

The proof of Lemma 13.7 is somewhat more difficult, and will require some preliminaries.

Notation 13.11. Let $\mathcal{M} : \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$ be the covariant functor which associates to each set S the collection of $\mathcal{M}(S)$ of nonempty finite subsets of S. If K is a simplicial set, we let $\mathcal{M}(K)$ denote the composition of K with $\mathcal{M}$, so that an m-simplex of $\mathcal{M}(K)$ is a finite nonempty collection of m-simplices of K.

Lemma 13.12. *Let K be a finite simplicial set, let $X \subseteq \mathcal{M}(K^{\triangleright}) \times \Delta^n$ be a simplicial subset with the following properties:*

(i) *The projection $X \to \Delta^n$ is surjective.*

(ii) *If $\tau = (\tau', \tau'') : \Delta^m \to \mathcal{M}(K^{\triangleright}) \times \Delta^n$ belongs to X, and $\tau' \subseteq \overline{\tau}'$ as subsets of $\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^m, K^{\triangleright})$, then $(\overline{\tau}', \tau'') : \Delta^m \to \mathcal{M}(K^{\triangleright}) \times \Delta^n$ belongs to X.*

Then X is weakly contractible.

Proof. Let $X' \subseteq X$ be the simplicial subset spanned by those simplices $\tau = (\tau', \tau'') : \Delta^m \to \mathcal{M}(K^{\triangleright}) \times \Delta^n$ which factor through X, and for which $\tau' \subseteq \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^m, K^{\triangleright})$ includes the constant simplex at the cone point of $K^{\triangleright}$. Our first step is to show that X' is a deformation retract of X. More precisely, we will construct a map

$$h : \mathcal{M}(K^{\triangleright}) \times \Delta^n \times \Delta^1 \to \mathcal{M}(K^{\triangleright}) \times \Delta^n$$

with the following properties:

(a) The map h carries $X \times \Delta^1$ into X and $X' \times \Delta^1$ into X'.

(b) The restriction $h| \mathcal{M}(K^{\triangleright}) \times \Delta^n \times \{0\}$ is the identity map.

(c) The restriction $h|X \times \{1\}$ factors through X'.

The map h will be the product of a map $h' : \mathcal{M}(K^{\triangleright}) \times \Delta^1 \to \mathcal{M}(K^{\triangleright})$ with the identity map on Δ^n. To define h', we consider an arbitrary simplex $\tau : \Delta^m \to \mathcal{M}(K^{\triangleright}) \times \Delta^1$, corresponding to a subset $S \subseteq \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^m, K^{\triangleright})$ and a decomposition $[m] = \{0, \ldots, i\} \cup \{i+1, \ldots, m\}$. The subset $h'(\tau) \subseteq \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^m, K^{\triangleright})$ is defined as follows: an arbitrary simplex $\sigma : \Delta^m \to K^{\triangleright}$ belongs to $h'(\tau)$ if there exists $\sigma' \in S$, $i < j \leq n$ such that $\sigma'|\Delta^{\{0,\ldots,j-1\}} = \sigma|\Delta^{\{0,\ldots,j-1\}}$, and $\sigma|\Delta^{\{j,\ldots,m\}}$ is constant at the cone point of $K^{\triangleright}$. It is easy to check that h' has the desired properties.

It remains to prove that X' is weakly contractible. At this point, it is convenient to work in the setting of *semisimplicial sets*: that is, we will ignore the degeneracy operations. Let X'' be the semisimplicial subset of $\mathcal{M}(K^{\triangleright}) \times \Delta^n$ spanned by those maps $\tau = (\tau', \tau'') : \Delta^m \to \mathcal{M}(K^{\triangleright}) \times \Delta^n$ for which $\tau' = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^m, K^{\triangleright})$ (we observe that X'' is not stable under the degeneracy operators on $\mathcal{M}(K^{\triangleright}) \times \Delta^n$). Assumptions (i) and (ii) guarantee that $X'' \subseteq X'$. Moreover, the projection $X \to \Delta^n$ induces an isomorphism of semisimplicial sets $X'' \to \Delta^n$. Consequently, it will suffice to prove that X'' is a deformation retract of X'.

The proof now proceeds by a variation on our earlier construction. Namely, we will define a map of semisimplicial sets

$$g : \mathcal{M}(K^{\triangleright}) \times \Delta^n \times \Delta^1 \to \mathcal{M}(K^{\triangleright}) \times \Delta^n$$

with the following properties:

(a) The map g carries $X' \times \Delta^1$ into X' and $X'' \times \Delta^1$ into X''.

(b) The restriction $g|X' \times \{1\}$ is the identity map.

(c) The restriction $g| \mathcal{M}(K^{\triangleright}) \times \Delta^n \times \{0\}$ factors through X'.

As before, g is the product of a map $g' : \mathcal{M}(K^{\triangleright}) \times \Delta^1 \to \mathcal{M}(K^{\triangleright})$ with the identity map on Δ^n. To define g', we consider an arbitrary simplex $\tau : \Delta^m \to \mathcal{M}(K^{\triangleright}) \times \Delta^1$, corresponding to a subset $S \subseteq \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^m, K^{\triangleright})$ and a decomposition $[m] = \{0, \ldots, i\} \cup \{i+1, \ldots, m\}$. We let $g'(\tau) \subseteq \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^m, K^{\triangleright}) = S \cup S'$, where S' is the collection of all simplices $\sigma : \Delta^m \to K^{\triangleright}$ such that $\sigma|\Delta^{\{i+1,\ldots,m\}}$ is the constant map at the cone poine of $K^{\triangleright}$. It is readily checked that g' has the desired properties. □

Lemma 13.13. *Let K be a contractible Kan complex, and let $X \subseteq \mathcal{M}(K) \times \Delta^n$ be a simplicial subset with the following properties:*

(*i*) *The projection $X \rightarrow \Delta^n$ is surjective.*

(*ii*) *If $\tau = (\tau', \tau'') : \Delta^m \rightarrow \mathcal{M}(K) \times \Delta^n$ belongs to X, and $\tau' \subseteq \overline{\tau}'$ as subsets of $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, K)$, then $(\overline{\tau}', \tau'') : \Delta^m \rightarrow \mathcal{M}(K) \times \Delta^n$ belongs to X.*

Then X is weakly contractible.

Proof. It will suffice to show that for every finite simplicial subset $X' \subseteq X$, the inclusion of X' into X is weakly nullhomotopic. Enlarging X' if necessary, we may assume that $X' = (\mathcal{M}(K') \times \Delta^n) \cap X$, where K' is a finite simplicial subset of K. By further enlargement, we may suppose that the map $X' \rightarrow \Delta^n$ is surjective. Since K is a contractible Kan complex, the inclusion $K' \subseteq K$ extends to a map $i : K'^{\triangleright} \rightarrow K$. Let $\overline{X} \subseteq \mathcal{M}(K'^{\triangleright}) \times \Delta^n$ denote the inverse image of X. Then the inclusion $X' \subseteq X$ factors through $\overline{X}$, and Lemma 13.12 implies that $\overline{X}$ is weakly contractible. □

Proof of Lemma 13.7. . Fix an object $C \in \mathcal{C}$. The simplicial set $L(\mathcal{F})(C)$ can be described as follows:

(∗) For every $n \geq 1$, every map $f : C_1 \times \ldots \times C_n \rightarrow C$ in $\mathcal{C}$, and every collection of simplices $\{\sigma_i : \Delta^k \rightarrow \mathcal{F}(C_i)\}$, there is a simplex $f(\{\sigma_i\}) : \Delta^k \rightarrow L(\mathcal{F})(C)$.

The simplices $f(\{\sigma_i\})$ satisfy relations which are determined by morphisms in the simplicial category $\mathcal{U}(\mathcal{C})$.

To every k-simplex $\tau : \Delta^k \rightarrow L(\mathcal{F})(D)$, we can associate a nonempty finite subset $S_\tau \subseteq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^k, K)$. If $\tau = f(\{\sigma_i\})$, we assign the set of images of the simplices σ_i under the canonical maps $\mathcal{F}(C_i) \rightarrow \mathcal{F}(1) = K$. It is easy to see that S_τ is independent of the representation $f(\{\sigma_i\})$ chosen for τ, and depends functorially on τ. Consequently, we obtain a map of simplicial sets $L(\mathcal{F})(C) \rightarrow \mathcal{M}(K)$. Moreover, this map has the following properties:

(*i*) The product map $\beta' : L(\mathcal{F})(C) \rightarrow \mathcal{M}(K) \times L^+(\mathcal{F})(C)$ is a monomorphism of simplicial sets.

(*ii*) If a k-simplex $\tau = (\tau', \tau'') : \Delta^k \rightarrow \mathcal{M}(K) \rightarrow L^+(\mathcal{F})(C)$ belongs to the image of β, and $\tau' \subseteq \overline{\tau}'$ as finite subsets of $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^k, K)$, then $(\overline{\tau}', \tau'') : \Delta^k \rightarrow \mathcal{M}(K) \times L^+(\mathcal{F})(C)$ belongs to the image of β'.

We wish to show that $\beta : L(\mathcal{F})(C) \rightarrow L^+(\mathcal{F})(C)$ is a weak homotopy equivalence. It will suffice to show that for every simplex $\Delta^k \rightarrow L^+(\mathcal{F})(C)$, the fiber product $L(\mathcal{F})(C) \times_{L^+(\mathcal{F})(C)} \Delta^k$ is weakly contractible. In view of (*i*), we can identify this fiber product with a simplicial subset $X \subseteq \Delta^k \times \mathcal{M}(K)$. The surjectivity of β and condition (*ii*) imply that X satisfies the hypotheses of Lemma 13.13, so that X is weakly contractible as desired. □

14 The Universal Property of $\mathcal{D}^+(\mathcal{A})$

In this section, we will apply the results of §12 and §13 to characterize the derived ∞-category $\mathcal{D}^+(\mathcal{A})$ by a universal mapping property. Here $\mathcal{A}$ denotes an abelian category with enough projective objects; to simplify the discussion, we will assume that $\mathcal{A}$ is small.

Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be the full subcategory of $\mathcal{A}$ spanned by the projective objects, and let $\mathbf{A}$ denote the category of product-preserving functors from $\mathcal{A}_0^{op}$ to the category of simplicial sets, as in §13. Let $\mathcal{A}^\vee$ denote the category of product-preserving functors from $\mathcal{A}_0^{op}$ to sets, so that we can identify $\mathbf{A}$ with the category of simplicial objects of $\mathcal{A}^\vee$. Our first goal is to understand the category $\mathcal{A}^\vee$.

Lemma 14.1. *Let $\mathcal{A}$ be an abelian category with enough projective objects, and let $\mathcal{B}$ be an arbitrary category which admits finite limits. Let $\mathcal{C}$ be the category of right exact functors from $\mathcal{A}$ to $\mathcal{B}$, and let $\mathcal{C}'$ be the category of coproduct-preserving functors from $\mathcal{A}_0$ to $\mathcal{B}$. Then the restriction functor $\theta : \mathcal{C} \rightarrow \mathcal{C}'$ is an equivalence of categories.*

Proof. We will describe an explicit construction of an inverse functor. Let $f : \mathcal{A}_0 \to \mathcal{B}$ be a functor which preserves finite coproducts. Let $A \in \mathcal{A}$ be an arbitrary object. Since $\mathcal{A}$ has enough projectives, there exists a projective resolution

$$\ldots \to P_1 \xrightarrow{u} P_0 \to A \to 0.$$

We now define $F(A)$ to be the coequalizer of the map

$$f(P_1) \underset{f(u)}{\overset{f(0)}{\rightrightarrows}} f(P_0) .$$

Of course, this definition appears to depend not only on A but on a choice of projective resolution. However, because any two projective resolutions of A are chain homotopy equivalent to one another, $F(A)$ is well-defined up to canonical isomorphism. It is easy to see that $F : \mathcal{A} \to \mathcal{B}$ is a right exact functor which extends f, and that F is uniquely determined (up to unique isomorphism) by these properties. □

Proposition 14.2. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Then:*

(1) *The category $\mathcal{A}^{\vee}$ can be identified with the category of* Ind-*objects of $\mathcal{A}$.*

(2) *The category $\mathcal{A}^{\vee}$ is abelian.*

(3) *The abelian category $\mathcal{A}^{\vee}$ has enough projective objects.*

Proof. Assertion (1) follows immediately from Lemma 14.1 (taking $\mathcal{B}$ to be the opposite of the category of sets). Part (2) follows formally from (1) and the assumption that $\mathcal{A}$ is an abelian category (see, for example, [2]). We may identify $\mathcal{A}$ with a full subcategory of $\mathcal{A}^{\vee}$ via the Yoneda embedding. Moreover, if P is a projective object of $\mathcal{A}$, then P is also projective when viewed as an object of $\mathcal{A}^{\vee}$. An arbitrary object of $\mathcal{A}^{\vee}$ can be written as a filtered colimit $A = \varinjlim \{A_\alpha\}$, where each $A_\alpha \in \mathcal{A}$. Using the assumption that $\mathcal{A}$ has enough projective objects, we can choose epimorphisms $P_\alpha \to A_\alpha$, where each P_α is projective. We then have an epimorphism $\oplus P_\alpha \to A$. Since $\oplus P_\alpha$ is projective, we conclude that $\mathcal{A}^{\vee}$ has enough projectives. □

Warning 14.3. Let $\mathcal{A}$ be an abelian category with enough projective objects, and let $\mathbf{A}$ be the category of product-preserving functors $\mathcal{A}_0^{op} \to \mathcal{S}\mathrm{et}_{\Delta}$. The Dold-Kan correspondence determines an equivalence of categories $\theta : \mathbf{A} \simeq \mathrm{Ch}_{\geq 0}(\mathcal{A}^{\vee})$. However, this is *not* an equivalence of simplicial categories. Let K be a simplicial set, and let $\mathbf{Z}K$ denote the free simplicial abelian group generated by K (so that the group of n-simplices of $\mathbf{Z}K$ is the free abelian group generated by the set of n-simplices of K, for each $n \geq 0$). Then $\mathbf{A}$ is tensored over the category of simplicial sets in two different ways:

(*i*) Given a simplicial set K and an object $A_\bullet \in \mathbf{A}$ viewed as a simplicial object of $\mathcal{A}^{\vee}$, we can form the tensor product $A_\bullet \otimes K$ given by the formula $(A_\bullet \otimes K)_n = A_n \otimes (\mathbf{Z}K)_n$.

(*ii*) Given a simplicial set K and an object $A_\bullet \in \mathbf{A}$, we can construct a new object $A_\bullet \odot K$, which is characterized by the existence of an isomorphism

$$\theta(A_\bullet \odot K) \simeq \theta(A_\bullet) \otimes \theta'(\mathbf{Z}K)$$

in the category $\mathrm{Ch}(\mathcal{A}^{\vee})$. Here $\theta'(\mathbf{Z}K)$ denotes the object of $\mathrm{Ch}(\mathcal{A}b)$ determined by $\mathbf{Z}K$.

However, it is easy to see that both of these simplicial structures on $\mathbf{A}$ are compatible with the model structure of Proposition 13.1. Moreover, the classical *Alexander-Whitney map* determines a natural transformation $A_\bullet \otimes K \to A_\bullet \odot K$, which endows $\theta^{-1} : \mathrm{Ch}_{\geq 0}(\mathcal{A}^{\vee}) \to \mathbf{A}$ with the structure of a simplicial functor.

We observe that every object of $\mathbf{A}$ is fibrant, and that an object of $\mathbf{A}$ is cofibrant if and only if it corresponds (under θ) to a complex of projective objects of $\mathcal{A}^{\vee}$. Applying Corollary T.A.2.8.3, we obtain an equivalence of ∞-categories $\mathcal{D}^{+}_{\geq 0}(\mathcal{A}^{\vee}) \to \mathrm{N}(\mathbf{A}^{\circ})$. Here $\mathcal{D}^{+}_{\geq 0}(\mathcal{A}^{\vee})$ denotes the full subcategory of $\mathcal{D}^{+}(\mathcal{A}^{\vee})$ spanned by those complexes $P_\bullet$ such that $P_n \simeq 0$ for $n < 0$. Composing with the equivalence of Corollary 13.3, we obtain the following result:

Proposition 14.4. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Then there exists an equivalence of ∞-categories*

$$\psi : \mathcal{D}^+_{\geq 0}(\mathcal{A}^\vee) \to \mathcal{P}_\Sigma(\mathrm{N}(\mathcal{A}_0))$$

whose composition with the inclusion $\mathrm{N}(\mathcal{A}_0) \subseteq \mathcal{D}^+_{\geq}(\mathcal{A}^\vee)$ is equivalent to the Yoneda embedding $\mathrm{N}(\mathcal{A}_0) \to \mathcal{P}_\Sigma(\mathrm{N}(\mathcal{A}_0))$.

Remark 14.5. We can identify $\mathcal{D}^+(\mathcal{A})$ with a full subcategory of $\mathcal{D}^+(\mathcal{A}^\vee)$. Moreover, an object $P_\bullet \in \mathcal{D}^+(\mathcal{A}^\vee)$ belongs to the essential image of $\mathcal{D}^+(\mathcal{A})$ if and only if the homologies $\mathrm{H}_n(P_\bullet)$ belong to $\mathcal{A}$, for all $n \in \mathbf{Z}$.

Proposition 14.6. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Then the t-structure on $\mathcal{D}^+(\mathcal{A})$ is right bounded and left complete.*

Proof. The right boundedness of $\mathcal{D}^+(\mathcal{A})$ is obvious. To prove the left completeness, we must show that $\mathcal{D}^+(\mathcal{A})$ is a homotopy inverse limit of the tower of ∞-categories

$$\ldots \to \mathcal{D}^+(\mathcal{A})_{\leq 1} \to \mathcal{D}^+(\mathcal{A})_{\leq 0} \to \ldots$$

Invoking the right boundedness of $\mathcal{D}^+(\mathcal{A})$, we may reduce to proving that for each $n \in \mathbf{Z}$, $\mathcal{D}^+(\mathcal{A})_{\geq n}$ is a homotopy inverse limit of the tower

$$\ldots \to \mathcal{D}^+(\mathcal{A})_{\leq 1, \geq n} \to \mathcal{D}^+(\mathcal{A})_{\leq 0, \geq n} \to \ldots$$

Shifting if necessary, we may suppose that $n = 0$. Using Remark 14.5, we can replace $\mathcal{A}$ by $\mathcal{A}^\vee$. For each $k \geq 0$, we let $\mathcal{P}^{\leq k}_\Sigma(\mathrm{N}(\mathcal{A}_0))$ denote the ∞-category of product-preserving functors from $\mathrm{N}(\mathcal{A}_0)^{op}$ to $\tau_{\leq k}\,\mathcal{S}$; equivalently, we can define $\mathcal{P}^{\leq k}_\Sigma(\mathrm{N}(\mathcal{A}_0))$ to be the ∞-category of k-truncated objects of $\mathcal{P}_\Sigma(\mathrm{N}(\mathcal{A}_0))$. We observe that the equivalence ψ of Proposition 14.4 restricts to an equivalence

$$\psi(k) : \mathcal{D}^+_{\geq 0}(\mathcal{A}^\vee)_{\leq k} \to \mathcal{P}^{\leq k}_\Sigma(\mathrm{N}(\mathcal{A}_0)).$$

Consequently, it will suffice to show that $\mathcal{P}_\Sigma(\mathrm{N}(\mathcal{A}_0))$ is a homotopy inverse limit for the tower

$$\ldots \to \mathcal{P}^{\leq 1}_\Sigma(\mathrm{N}(\mathcal{A}_0)) \to \mathcal{P}^{\leq 0}_\Sigma(\mathrm{N}(\mathcal{A}_0)).$$

Since the truncation functors on $\mathcal{S}$ commute with finite products (Lemma T.6.5.1.2), we may reduce to the problem of showing that $\mathcal{S}$ is a homotopy inverse limit of the tower

$$\ldots \to \tau_{\leq 1}\,\mathcal{S} \to \tau_{\leq 0}\,\mathcal{S}\,.$$

This amounts to the classical fact that every space X can be recovered as the limit of its Postnikov tower (see for example §T.7.2.1).

□

Our goal is to characterize the derived ∞-category $\mathcal{D}^+(\mathcal{A})$ by a universal mapping property. Propositions 14.4 and 12.7 give a characterization of $\mathcal{D}^+_{\geq 0}(\mathcal{A}^\vee)$ of the right flavor. The next step is to understand the embedding of $\mathcal{D}^+_{\geq 0}(\mathcal{A})$ into $\mathcal{D}^+_{\geq 0}(\mathcal{A}^\vee)$.

Definition 14.7. Let $\mathcal{C}$ and $\mathcal{C}'$ be stable ∞-categories equipped with t-structures. We will say that a functor $f : \mathcal{C} \to \mathcal{C}'$ is *right t-exact* if it is exact, and carries $\mathcal{C}_{\geq 0}$ into $\mathcal{C}'_{\geq 0}$.

Lemma 14.8. (1) *Let $\mathcal{C}$ be an ∞-category which admits finite coproducts and geometric realizations. Then $\mathcal{C}$ admits all finite colimits. Conversely, if $\mathcal{C}$ is an n-category which admits finite colimits, then $\mathcal{C}$ admits geometric realizations.*

(2) *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories which admit finite coproducts and geometric realizations. If F preserves finite coproducts and geometric realizations, then F is right exact. The converse holds if $\mathcal{C}$ and $\mathcal{D}$ are n-categories.*

Proof. We will prove (1); the proof of (2) follows by the same argument. Now suppose that $\mathcal{C}$ admits finite coproducts and geometric realizations of simplicial objects. We wish to show that $\mathcal{C}$ admits all finite colimits. According to Proposition T.4.4.3.2, it will suffice to show that $\mathcal{C}$ admits coequalizers. Let $\mathbf{\Delta}_s^{\leq 1}$ be the full subcategory of $\mathbf{\Delta}$ spanned by the objects [0] and [1], and *injective* maps between them, so that a coequalizer diagram in $\mathcal{C}$ can be identified with a functor $\mathrm{N}(\mathbf{\Delta}_s^{\leq 1})^{op} \to \mathcal{C}$. Let $j : \mathrm{N}(\mathbf{\Delta}_s^{\leq 1})^{op} \to \mathrm{N}(\mathbf{\Delta})^{op}$ be the inclusion functor. We claim that every diagram $f : \mathrm{N}(\mathbf{\Delta}_s^{\leq 1})^{op} \to \mathcal{C}$ has a left Kan extension along j. To prove this, it suffices to show that for each $n \geq 0$, the associated diagram

$$\mathrm{N}(\mathbf{\Delta}_s^{\leq 1})^{op} \times_{\mathrm{N}(\mathbf{\Delta})^{op}} (\mathrm{N}(\mathbf{\Delta})^{op})_{[n]/} \to \mathcal{C}$$

has a colimit. We now observe that this last colimit is equivalent to a coproduct: more precisely, we have $(j_! f)([n]) \simeq f([0]) \coprod f([1]) \coprod \ldots \coprod f([1])$, where there are precisely n summands equivalent to $f([1])$. Since $\mathcal{C}$ admits finite coproducts, the desired Kan extension $j_! f$ exists. We now observe that $\varinjlim(f)$ can be identified with $\varinjlim(j_! f)$, and the latter exists in virtue of our assumption that $\mathcal{C}$ admits geometric realizations for simplicial objects.

Now suppose that $\mathcal{C}$ is an n-category which admits finite colimits; we wish to show that $\mathcal{C}$ admits geometric realizations. Passing to a larger universe if necessary, we may suppose that $\mathcal{C}$ is small. Let $\mathcal{D} = \mathrm{Ind}(\mathcal{C})$, and let $\mathcal{C}' \subseteq \mathcal{D}$ denote the essential image of the Yoneda embedding $j : \mathcal{C} \to \mathcal{D}$. Then $\mathcal{D}$ admits small colimits (Theorem T.5.5.1.1) and j is fully faithful (Proposition T.5.1.3.1); it will therefore suffice to show that $\mathcal{C}'$ is stable under geometric realization of simplicial objects in $\mathcal{D}$.

Fix a simplicial object $U_\bullet : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{C}' \subseteq \mathcal{D}$. Let $V_\bullet : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{D}$ be a left Kan extension of $U_\bullet | \mathrm{N}(\mathbf{\Delta}^{\leq n})^{op}$, and $\alpha_\bullet : V_\bullet \to U_\bullet$ the induced map. The geometric realization of $V_\bullet$ can be identified with the colimit of $U_\bullet | \mathrm{N}(\mathbf{\Delta}^{\leq n})^{op}$, and therefore belongs to $\mathcal{C}'$ since $\mathcal{C}'$ is stable under finite colimits in $\mathcal{D}$ (Proposition T.5.3.5.14). Consequently, it will suffice to prove that $\alpha_\bullet$ induces an equivalence from the geometric realization of $V_\bullet$ to the geometric realization of $U_\bullet$.

Let $L : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ be a left adjoint to the inclusion. Let $|U_\bullet|$ and $|V_\bullet|$ be colimits of $U_\bullet$ and $V_\bullet$ in the ∞-category $\mathcal{P}(\mathcal{C})$, and let $|\alpha_\bullet| : |V_\bullet| \to |U_\bullet|$ be the induced map. We wish to show that $L|\alpha_\bullet|$ is an equivalence in $\mathcal{D}$. Since $\mathcal{C}$ is an n-category, we have inclusions $\mathrm{Ind}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, \tau_{\leq n-1}\, \mathcal{S}) \subseteq \mathcal{P}(\mathcal{C})$. It follows that L factors through the truncation functor $\tau_{\leq n-1} : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$. Consequently, it will suffice to prove that $\tau_{\leq n-1}|\alpha_\bullet|$ is an equivalence in $\mathcal{P}(\mathcal{C})$. For this, it will suffice to show that the morphism $|\alpha_\bullet|$ is $(n-1)$-connected (in the sense of Definition T.6.5.1.10). This follows from Lemma T.6.5.3.11, since $\alpha_k : V_k \to U_k$ is an equivalence for $k \leq n$. □

Lemma 14.9. *Let $\mathcal{C}$ and $\mathcal{C}'$ be stable ∞-categories equipped with t-structures. Let $\theta : \mathrm{Fun}(\mathcal{C}, \mathcal{C}') \to \mathrm{Fun}(\mathcal{C}_{\geq 0}, \mathcal{C}'_{\geq 0})$ be the restriction map. Then:*

(1) *If $\mathcal{C}$ is right-bounded, then θ induces an equivalence from the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ spanned by the right t-exact functors to the full subcategory of $\mathrm{Fun}(\mathcal{C}_{\geq 0}, \mathcal{C}'_{\geq 0})$ spanned by the right exact functors.*

(2) *Let $\mathcal{C}$ and $\mathcal{C}'$ be left complete. Then the ∞-categories $\mathcal{C}_{\geq 0}$ and $\mathcal{C}'_{\geq 0}$ admit geometric realizations of simplicial objects. Furthermore, a functor $F : \mathcal{C}_{\geq 0} \to \mathcal{C}'_{\geq 0}$ is right exact if and only if if it preserves finite coproducts and geometric realizations of simplicial objects.*

Proof. We first prove (1). If $\mathcal{C}$ is right bounded, then $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ is the (homotopy) inverse limit of the tower

$$\ldots \to \mathrm{Fun}(\mathcal{C}_{\geq -1}, \mathcal{C}') \to \mathrm{Fun}(\mathcal{C}_{\geq 0}, \mathcal{C}'),$$

where the functors are given by restriction. The full subcategory of right t-exact functors is then given by the homotopy inverse limit

$$\ldots \to \mathrm{Fun}'(\mathcal{C}_{\geq -1}, \mathcal{C}'_{\geq -1}) \stackrel{\theta(0)}{\to} \mathrm{Fun}(\mathcal{C}_{\geq 0}, \mathcal{C}'_{\geq 0})$$

where $\mathrm{Fun}'(\mathcal{C},\mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ spanned by the right exact functors. To complete the proof, it will suffice to show that this tower is essentially constant; in other words, that each $\theta(n)$ is an equivalence of ∞-categories. Without loss of generality, we may suppose that $n = 0$. Choose shift functors on the ∞-categories $\mathcal{C}$ and $\mathcal{C}'$, and define

$$\psi : \mathrm{Fun}'(\mathcal{C}_{\geq 0}, \mathcal{C}'_{\geq 0}) \to \mathrm{Fun}'(\mathcal{C}_{\geq -1}, \mathcal{C}'_{\geq -1})$$

by the formula $\psi(F) = \Sigma^{-1} \circ F \circ \Sigma$. We claim that ψ is a homotopy inverse to $\theta(0)$. To prove this, we observe that the right exactness of $F \in \mathrm{Fun}'(\mathcal{C}_{\geq 0}, \mathcal{C}'_{\geq 0})$, $G \in \mathrm{Fun}'(\mathcal{C}_{\geq -1}, \mathcal{C}'_{\geq -1})$ determines canonical equivalences

$$(\theta(0) \circ \psi)(F) \simeq F$$

$$(\psi \circ \theta(0))(G) \simeq G.$$

We now prove (2). Since $\mathcal{C}$ is left complete, $\mathcal{C}_{\geq 0}$ is the (homotopy) inverse limit of the tower of ∞-categories $\{(\mathcal{C}_{\geq 0})_{\leq n}\}$ with transition maps given by right exact truncation functors. Lemma 14.8 implies that each $(\mathcal{C}_{\geq 0})_{\leq n}$ admits geometric realizations of simplicial objects, and that each of the truncation functors preserves geometric realizations of simplicial objects. It follows that $\mathcal{C}_{\geq 0}$ admits geometric realizations for simplicial objects. Similarly, $\mathcal{C}'_{\geq 0}$ admits geometric realizations for simplicial objects.

If F preserves finite coproducts and geometric realizations of simplicial objects, then F is right exact (Lemma 14.8). Conversely, suppose that F is right exact; we wish to prove that F preserves geometric realizations of simplicial objects. It will suffice to show that each composition

$$\mathcal{C}_{\geq 0} \xrightarrow{F} \mathcal{C}'_{\geq 0} \xrightarrow{\tau_{\leq n}} (\mathcal{C}'_{\geq 0})_{\leq n}$$

preserves geometric realizations of simplicial objects. We observe that, in virtue of the right exactness of F, this functor is equivalent to the composition

$$\mathcal{C}_{\leq 0} \xrightarrow{\tau_{\leq n}} (\mathcal{C}_{\geq 0})_{\leq n} \xrightarrow{\tau_{\leq n} \circ F} (\mathcal{C}'_{\geq 0})_{\leq n}.$$

It will therefore suffice to prove that $\tau_{\leq n} \circ F$ preserves geometric realizations of simplicial objects, which follows from Lemma 14.8 since both the source and target are equivalent to n-categories. □

Lemma 14.10. *Let $\mathcal{A}$ be a small abelian category with enough projective objects, and let $\mathcal{C} \subseteq \mathcal{P}_\Sigma(\mathrm{N}(\mathcal{A}_0))$ be the essential image of $\mathcal{D}^+_{\geq 0}(\mathcal{A}) \subseteq \mathcal{D}^+_{\geq 0}(\mathcal{A}^\vee)$ under the equivalence $\psi : \mathcal{D}^+_{\geq 0}(\mathcal{A}^\vee) \to \mathcal{P}_\Sigma(\mathrm{N}(\mathcal{A}_0))$ of Proposition 14.4. Then $\mathcal{C}$ is the smallest full subcategory of $\mathcal{P}(\mathrm{N}(\mathcal{A}_0))$ which is closed under geometric realization and contains the essential image of the Yoneda embedding.*

Proof. It is clear that $\mathcal{C}$ contains the essential image of the Yoneda embedding. Lemma 14.9 implies that $\mathcal{D}^+_{\geq 0}(\mathcal{A})$ admits geometric realizations and that the inclusion $\mathcal{D}^+_{\geq 0}(\mathcal{A}) \subseteq \mathcal{D}^+_{\geq 0}(\mathcal{A}^\vee)$ preserves geometric realizations. It follows that $\mathcal{C}$ is closed under geometric realizations in $\mathcal{P}(\mathrm{N}(\mathcal{A}_0))$.

To complete the proof, we will show that every object of $X \in \mathcal{D}^+_{\geq 0}(\mathcal{A})$ can be obtained as the geometric realization, in $\mathcal{D}^+_{\geq 0}(\mathcal{A}^\vee)$, of a simplicial object $P_\bullet$ such that each $P_n \in \mathcal{D}^+_{\geq 0}(\mathcal{A}^\vee)$ consists of a projective object of $\mathcal{A}$, concentrated in degree zero. In fact, we can take $P_\bullet$ to be the simplicial object of $\mathcal{A}_0$ which corresponds to $X \in \mathrm{Ch}_{\geq 0}(\mathcal{A}_0)$ under the Dold-Kan correspondence. It follows from Theorem T.4.2.4.1 and Proposition A.1 that X can be identified with the geometric realization of $P_\bullet$. □

We are now ready to establish our characterization of $\mathcal{D}^+_{\geq 0}(\mathcal{A})$.

Theorem 14.11. *Let $\mathcal{A}$ be an abelian category with enough projective objects, $\mathcal{A}_0 \subseteq \mathcal{A}$ the full subcategory spanned by the projective objects, and $\mathcal{C}$ an arbitrary ∞-category which admits geometric realizations. Let $\mathrm{Fun}'(\mathcal{D}^+_{\geq 0}(\mathcal{A}), \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{D}^+_{\geq 0}(\mathcal{A}), \mathcal{C})$ spanned by those functors which preserve geometric realizations. Then:*

(1) *The restriction map*

$$\mathrm{Fun}'(\mathcal{D}^+_{\geq 0}(\mathcal{A}), \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(\mathcal{A}_0), \mathcal{C})$$

is an equivalence of ∞*-categories.*

(2) *A functor* $F \in \mathrm{Fun}'(\mathcal{D}^+_{\geq 0}(\mathcal{A}), \mathcal{C})$ *preserves preserves finite coproducts if and only if the restriction* $F|\,\mathrm{N}(\mathcal{A}_0)$ *preserves finite coproducts.*

Proof. Part (1) follows from Lemma 14.10, Remark T.5.3.5.9, and Proposition T.4.3.2.15. The "only if" direction of (2) is obvious. To prove the "if" direction, let us suppose that $F|\,\mathrm{N}(\mathcal{A}_0)$ preserves finite coproducts. We may assume without loss of generality that $\mathcal{C}$ admits filtered colimits (Lemma T.5.3.5.7), so that F extends to a functor $F' : \mathcal{D}^+_{\geq 0}(\mathcal{A}^\vee)$ which preserves filtered colimits and geometric realizations (Propositions 14.4 and 12.7). It follows from Proposition 12.7 that F' preserves finite coproducts, so that $F = F'|\,\mathcal{D}^+_{\geq 0}(\mathcal{A})$ also preserves finite coproducts. □

Corollary 14.12. *Let* $\mathcal{A}$ *be an abelian category with enough projective objects, and let* $\mathcal{C}$ *be a stable* ∞*-category equipped with a left complete t-structure. Then the restriction functor*

$$\mathrm{Fun}(\mathcal{D}^+(\mathcal{A}), \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(\mathcal{A}_0), \mathcal{C})$$

induces an equivalence from the full subcategory of $\mathrm{Fun}(\mathcal{D}^+(\mathcal{A}), \mathcal{C})$ *spanned by the right t-exact functors to the full subcategory of* $\mathrm{Fun}(\mathrm{N}(\mathcal{A}_0), \mathcal{C}_{\geq 0})$ *spanned by functors which preserve finite coproducts (here* $\mathcal{A}_0$ *denotes the full subcategory of* $\mathcal{A}$ *spanned by the projective objects).*

Proof. Let $\mathrm{Fun}'(\mathcal{D}^+(\mathcal{A}), \mathcal{C})$ be the full subcategory of $\mathrm{Fun}(\mathcal{D}^+(\mathcal{A}), \mathcal{C})$ spanned by the right t-exact functors. Lemma 14.9 implies that $\mathrm{Fun}'(\mathcal{D}^+(\mathcal{A}), \mathcal{C})$ is equivalent (via restriction) to the full subcategory

$$\mathrm{Fun}'(\mathcal{D}^+_{\geq 0}(\mathcal{A}), \mathcal{C}_{\geq 0}) \subseteq \mathrm{Fun}(\mathcal{D}^+_{\geq 0}(\mathcal{A}), \mathcal{C}_{\geq 0})$$

spanned by those functors which preserve finite coproducts and geometric realizations of simplicial objects. Theorem 14.11 and Proposition 12.7 allow us to identify $\mathrm{Fun}'(\mathcal{D}^+_{\geq 0}(\mathcal{A}), \mathcal{C}_{\geq 0})$ with the ∞-category of finite-coproduct preserving functors from $\mathrm{N}(\mathcal{A}_0)$ into $\mathcal{C}_{\geq 0}$. □

Corollary 14.13. *Let* $\mathcal{A}$ *be an abelian category with enough projective objects, let* $\mathcal{C}$ *be a stable* ∞*-category equipped with a left complete t-structure, and let* $\mathcal{E} \subseteq \mathrm{Fun}(\mathcal{D}^+(\mathcal{A}), \mathcal{C})$ *be the full subcategory spanned by those right t-exact functors which carry projective objects of* $\mathcal{A}$ *into the heart of* $\mathcal{C}$*. Then* $\mathcal{E}$ *is equivalent to (the nerve of) the ordinary category of right exact functors from* $\mathcal{A}$ *to the heart of* $\mathcal{C}$*.*

Proof. Corollary 14.12 implies that the restriction map

$$\mathcal{E} \to \mathrm{Fun}(\mathrm{N}(\mathcal{A}_0), \mathcal{C}^\heartsuit)$$

is fully faithful, and that the essential image of θ consists of the collection of coproduct-preserving functors from $\mathrm{N}(\mathcal{A}_0)$ to $\mathcal{C}^\heartsuit$. Lemma 14.1 allows us to identify the latter ∞-category with the nerve of the category of right exact functors from $\mathcal{A}$ to the heart of $\mathcal{C}$. □

If $\mathcal{A}$ and $\mathcal{C}$ are as in Proposition 14.12, then any right exact functor from $\mathcal{A}$ to $\mathcal{C}^\heartsuit$ can be extended (in an essentially unique way) to a functor $\mathcal{D}^+(\mathcal{A}) \to \mathcal{C}$. In particular, if the abelian category $\mathcal{C}^\heartsuit$ has enough projective objects, then we obtain an induced map $\mathcal{D}^+(\mathcal{C}^\heartsuit) \to \mathcal{C}$.

Example 14.14. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories equipped with enough projective objects. Then any right-exact functor $f : \mathcal{A} \to \mathcal{B}$ extends to a right t-exact functor $F : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$. One typically refers to F as the *left derived functor* of f.

Example 14.15. Let $\mathcal{S}_\infty$ be the stable ∞-category of spectra (see §9), with its natural t-structure. Then the heart of $\mathcal{S}_\infty$ is equivalent to the category $\mathcal{A}$ of abelian groups. We therefore obtain a functor $\mathcal{D}^+(\mathcal{A}) \to \mathcal{S}_\infty$, which carries a complex of abelian groups to the corresponding *generalized Eilenberg-MacLane spectrum.*

15 Presentable Stable ∞-Categories

In this section, we will study the class of *presentable* stable ∞-categories: that is, stable ∞-categories which admit small colimits and are generated (under colimits) by a set of small objects. In the stable setting, the condition of presentability can be formulated in reasonably simple terms.

Proposition 15.1. (1) *A stable ∞-category $\mathcal{C}$ admits small colimits if and only if $\mathcal{C}$ admits small coproducts.*

(2) *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable ∞-categories satisfying* (1)*. Then F preserves small colimits if and only if F preserves small coproducts.*

(3) *Let $\mathcal{C}$ be a stable ∞-category satisfying* (1)*, and let X be an object of $\mathcal{C}$. Then X is compact if and only if the following condition is satisfied:*

($*$) *For every map $f : X \to \coprod_{\alpha \in A} Y_\alpha$ in $\mathcal{C}$, there exists a finite subset $A_0 \subseteq A$ such that f factors (up to homotopy) through $\coprod_{\alpha \in A_0} Y_\alpha$.*

Proof. The "only if" direction of (1) is obvious, and the converse follows from Proposition T.4.4.3.2. Assertion (2) can be proven in the same way.

The "only if" direction of (3) follows from the fact that an arbitrary coproduct $\coprod_{\alpha \in A} Y_\alpha$ can be obtained as a filtered colimit of finite coproducts $\coprod_{\alpha \in A_0} Y_\alpha$ (see §T.4.2.3). Conversely, suppose that an object $X \in \mathcal{C}$ satisfies ($*$); we wish to show that X is compact. Let $f : \mathcal{C} \to \widehat{\mathcal{S}}$ be the functor corepresented by C. Proposition T.5.1.3.2 implies that f is left exact. According to part (3) of Proposition 10.10, we can assume that $f = \Omega^\infty \circ F$, where $F : \mathcal{C} \to \widehat{\mathcal{S}_\infty}$ is an exact functor; here $\widehat{\mathcal{S}_\infty}$ denotes the ∞-category of spectra which are not necessarily small. We wish to prove that f preserves filtered colimits. Since Ω^∞ preserves filtered colimits, it will suffice to show that F preserves *all* colimits. In view of (2), it will suffice to show that F preserves coproducts. In virtue of Remark 9.15, we are reduced to showing that each of the induced functors

$$\mathcal{C} \xrightarrow{F} \widehat{\mathcal{S}_\infty} \xrightarrow{\pi_n} \mathrm{N}(\mathcal{A}b)$$

preserves coproducts, where $\mathcal{A}b$ denotes the category of (not necessarily small) abelian groups. Shifting if necessary, we may suppose $n = 0$. In other words, we must show that for any collection of objects $\{Y_\alpha\}_{\alpha \in A}$, the natural map

$$\theta : \bigoplus \mathrm{Ext}^0_{\mathcal{C}}(X, Y_\alpha) \to \mathrm{Ext}^0_{\mathcal{C}}(X, \coprod Y_\alpha)$$

is an isomorphism of abelian groups. The surjectivity of θ amounts to the assumption ($*$), while the injectivity follows from the observations that each Y_α is a retract of the coproduct $\coprod Y_\alpha$ and that the natural map $\bigoplus \mathrm{Ext}^0_{\mathcal{C}}(X, Y_\alpha) \to \prod \mathrm{Ext}^0_{\mathcal{C}}(X, Y_\alpha)$ is injective. □

If $\mathcal{C}$ is a stable ∞-category, then we will say that an object $X \in \mathcal{C}$ *generates* $\mathcal{C}$ if the condition $\pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y) \simeq *$ implies that Y is a zero object of $\mathcal{C}$.

Corollary 15.2. *Let $\mathcal{C}$ be a stable ∞-category. Then $\mathcal{C}$ is presentable if and only if the following conditions are satisfied:*

(1) *The ∞-category $\mathcal{C}$ admits small coproducts.*

(2) *The homotopy category* $\mathrm{h}\mathcal{C}$ *is locally small.*

(3) *There exists regular cardinal κ and a κ-compact generator $X \in \mathcal{C}$.*

Proof. Suppose first that $\mathcal{C}$ is presentable. Conditions (1) and (2) are obvious. To establish (3), we may assume without loss of generality that $\mathcal{C}$ is an accessible localization of $\mathcal{P}(\mathcal{D})$, for some small ∞-category $\mathcal{D}$. Let $F : \mathcal{P}(\mathcal{D}) \to \mathcal{C}$ be the localization functor and G its right adjoint. Let $j : \mathcal{D} \to \mathcal{P}(\mathcal{D})$ be the Yoneda embedding, and let X be a coproduct of all suspensions (see §3) of objects of the form $F(j(D))$, where $D \in \mathcal{D}$. Since $\mathcal{C}$ is presentable, X is κ-compact provided that κ is sufficiently large. We claim that X generates $\mathcal{C}$. To prove this, we consider an arbitrary $Y \in \mathcal{C}$ such that $\pi_0 \operatorname{Map}_{\mathcal{C}}(X, Y) \simeq *$. It follows that the space

$$\operatorname{Map}_{\mathcal{C}}(F(j(D)), Y) \simeq \operatorname{Map}_{\mathcal{P}(\mathcal{D})}(j(D), G(Y)) \simeq G(Y)(D)$$

is contractible for all $D \in \mathcal{D}$, so that $G(Y)$ is a final object of $\mathcal{P}(\mathcal{D})$. Since G is fully faithful, we conclude that Y is a final object of $\mathcal{C}$, as desired.

Conversely, suppose that (1), (2), and (3) are satisfied. We first claim that $\mathcal{C}$ is itself locally small. It will suffice to show that for every morphism $f : X \to Y$ in $\mathcal{C}$ and every $n \geq 0$, the homotopy group $\pi_n(\operatorname{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y), f)$ is small. We note that $\operatorname{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)$ is equivalent to the loop space of $\operatorname{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y[1])$; the question is therefore independent of base point, so we may assume that f is the zero map. We conclude that the relevant homotopy group can identified with $\operatorname{Hom}_{\mathrm{h}\mathcal{C}}(X[n], Y)$, which is small in virtue of assumption (2).

Fix a regular cardinal κ and a κ-compact object X which generates $\mathcal{C}$. We now define a transfinite sequence of full subcategories

$$\mathcal{C}(0) \subseteq \mathcal{C}(1) \subseteq \ldots$$

as follows. Let $\mathcal{C}(0)$ be the full subcategory of $\mathcal{C}$ spanned by the objects $\{X[n]\}_{n \in \mathbf{Z}}$. If λ is a limit ordinal, let $\mathcal{C}(\lambda) = \bigcup_{\beta < \lambda} \mathcal{C}(\beta)$. Finally, let $\mathcal{C}(\alpha + 1)$ be the full subcategory of $\mathcal{C}$ spanned by all objects which can be obtained as the colimit of κ-small diagrams in $\mathcal{C}(\alpha)$. Since $\mathcal{C}$ is locally small, it follows that each $\mathcal{C}(\alpha)$ is essentially small. It follows by induction that each $\mathcal{C}(\alpha)$ consists of κ-compact objects of $\mathcal{C}$ and is stable under translation. Finally, we observe that $\mathcal{C}(\kappa)$ is stable under κ-small colimits. It follows from Lemma 4.3 that $\mathcal{C}(\kappa)$ is a stable subcategory of $\mathcal{C}$. Choose a small ∞-category $\mathcal{D}$ and an equivalence $f : \mathcal{D} \to \mathcal{C}(\kappa)$. According to Proposition T.5.3.5.11, we may suppose that f factors as a composition

$$\mathcal{D} \xrightarrow{j} \operatorname{Ind}_{\kappa}(\mathcal{D}) \xrightarrow{F} \mathcal{C}$$

where j is the Yoneda embedding and F is a κ-continuous, fully faithful functor. We will complete the proof by showing that F is an equivalence.

Proposition T.5.5.1.9 implies that F preserves small colimits. It follows that F admits a right adjoint $G : \mathcal{C} \to \operatorname{Ind}_{\kappa}(\mathcal{D})$ (Remark T.5.5.2.10). We wish to show that the counit map $u : F \circ G \to \mathrm{id}_{\mathcal{C}}$ is an equivalence of functors. Choose an object $Z \in \mathcal{C}$, and let Y be a cokernel for the induced map $u_Z : (F \circ G)(Z) \to Z$. Since F is fully faithful, $G(u_Z)$ is an equivalence. Because G is an exact functor, we deduce that $G(Y) = 0$. It follows that $\operatorname{Map}_{\mathcal{C}}(F(D), Y) \simeq \operatorname{Map}_{\operatorname{Ind}_{\kappa}(\mathcal{D})}(D, G(Y)) \simeq *$ for all $D \in \operatorname{Ind}_{\kappa}(\mathcal{D})$. In particular, we conclude that $\pi_0 \operatorname{Map}_{\mathcal{C}}(X, Y) \simeq *$. Since X generates $\mathcal{C}$, we deduce that $Y \simeq 0$. Thus u_Z is an equivalence as desired. □

Remark 15.3. In view of Proposition 15.1 and Corollary 15.2, the hypothesis that a stable ∞-category $\mathcal{C}$ be compactly generated can be formulated entirely in terms of the homotopy category $\mathrm{h}\mathcal{C}$. Consequently, one can study this condition entirely in the setting of triangulated categories, without making reference to (or assuming the existence of) an underlying stable ∞-category. We refer to reader to [17] for further discussion.

The following result gives a good class of examples of presentable ∞-categories.

Proposition 15.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories, and suppose that $\mathcal{D}$ is stable.*

(1) *The ∞-category $\operatorname{Stab}(\mathcal{C})$ is presentable.*

(2) *The functor $\Omega^{\infty} : \operatorname{Stab}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\Sigma^{\infty} : \mathcal{C} \to \operatorname{Stab}(\mathcal{C})$.*

(3) *An exact functor* $G : \mathcal{D} \to \mathrm{Stab}(\mathcal{C})$ *admits a left adjoint if and only if* $\Omega^\infty \circ G : \mathcal{D} \to \mathcal{C}$ *admits a left adjoint.*

Proof. We first prove (1). Assume that $\mathcal{C}$ is presentable, and let 1 be a final object of $\mathcal{C}$. Then $\mathcal{C}_*$ is equivalent to $\mathcal{C}_{1/}$, and therefore presentable (Proposition T.5.5.3.11). The loop functor $\Omega : \mathcal{C}_* \to \mathcal{C}_*$ admits a left adjoint $\Sigma : \mathcal{C}_* \to \mathcal{C}_*$. Consequently, we may view the tower

$$\ldots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*$$

as a diagram in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{R}}$. Invoking Theorem T.5.5.3.18, we deduce (1) and the following modified versions of (2) and (3):

($2'$) The functor $\Omega^\infty_* : \mathrm{Stab}(\mathcal{C}) \to \mathcal{C}_*$ admits a left adjoint $\Sigma^\infty_* : \mathcal{C}_* \to \mathrm{Stab}(\mathcal{C})$.

($3'$) An exact functor $G : \mathcal{D} \to \mathrm{Stab}(\mathcal{C})$ admits a left adjoint if and only if $\Omega^\infty_* \circ G : \mathcal{D} \to \mathcal{C}_*$ admits a left adjoint.

To complete the proof, it will suffice to verify the following:

($2''$) The forgetful functor $\mathcal{C}_* \to \mathcal{C}$ admits a left adjoint $\Sigma^\infty : \mathcal{C} \to \mathrm{Stab}(\mathcal{C})$.

($3''$) A functor $G : \mathcal{D} \to \mathcal{C}_*$ admits a left adjoint if and only if the composition $\mathcal{D} \xrightarrow{G} \mathcal{C}_* \to \mathcal{C}$ admits a left adjoint.

To prove ($2''$) and ($3''$), we recall that a functor G between presentable ∞-categories admits a left adjoint if and only if G preserves small limits and small, κ-filtered colimits, for some regular cardinal κ (Corollary T.5.5.2.9). The desired results now follow from Propositions T.4.4.2.8 and T.1.2.13.8. □

Corollary 15.5. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be presentable* ∞*-categories, and suppose that* $\mathcal{D}$ *is stable. Then composition with* $\Sigma^\infty : \mathcal{C} \to \mathrm{Stab}(\mathcal{C})$ *induces an equivalence*

$$\mathcal{P}\mathrm{r}^{\mathrm{L}}(\mathrm{Stab}(\mathcal{C}), \mathcal{D}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}).$$

Proof. This is equivalent to the assertion that composition with Ω^∞ induces an equivalence

$$\mathcal{P}\mathrm{r}^{\mathrm{R}}(\mathcal{D}, \mathrm{Stab}(\mathcal{C})) \to \mathcal{P}\mathrm{r}^{\mathrm{R}}(\mathcal{D}, \mathcal{C}),$$

which follows from Propositions 10.10 and 15.4. □

Using Corollary 15.5, we obtain another characterization of the ∞-category of spectra. Let $S^0 \in \mathcal{S}_\infty$ denote the image under $\Sigma^\infty : \mathcal{S} \to \mathcal{S}_\infty$ of the final object $* \in \mathcal{S}$. We will refer to S^0 as the *sphere spectrum.*

Corollary 15.6. *Let* $\mathcal{D}$ *be a stable, presentable* ∞*-category. Then evaluation on the sphere spectrum induces an equivalence of* ∞*-categories*

$$\theta : \mathcal{P}\mathrm{r}^{\mathrm{L}}(\mathcal{S}_\infty, \mathcal{D}) \to \mathcal{D}\,.$$

In other words, we may regard the ∞-category $\mathcal{S}_\infty$ as the stable ∞-category which is freely generated, under colimits, by a single object.

Proof. We can factor the evaluation map θ as a composition

$$\mathcal{P}\mathrm{r}^{\mathrm{L}}(\mathrm{Stab}(\mathcal{S}), \mathcal{D}) \xrightarrow{\theta'} \mathcal{P}\mathrm{r}^{\mathrm{L}}(\mathcal{S}, \mathcal{D}) \xrightarrow{\theta''} \mathcal{D}$$

where θ' is given by composition with Σ^∞ and θ'' by evaluation at the final object of $\mathcal{S}$. We now observe that θ' and θ'' are both equivalences of ∞-categories (Corollary 15.5 and Theorem T.5.1.5.6). □

We conclude this section by establishing a characterization of the class of stable, presentable ∞-categories.

Lemma 15.7. *Let $\mathcal{C}$ be a stable ∞-category, and let $\mathcal{C}' \subseteq \mathcal{C}$ a localization of $\mathcal{C}$. Let $L : \mathcal{C} \to \mathcal{C}'$ be a left adjoint to the inclusion. Then L is left exact if and only if $\mathcal{C}'$ is stable.*

Proof. The "if" direction follows from Proposition 5.1, since L is right exact. Conversely, suppose that L is left exact. Since $\mathcal{C}'$ is a localization of $\mathcal{C}$, it is closed under finite limits. In particular, it is closed under the formation of kernels and contains a zero object of $\mathcal{C}$. To complete the proof, it will suffice to show that $\mathcal{C}'$ is stable under the formation of pushouts in $\mathcal{C}$. Choose a pushout diagram $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

in $\mathcal{C}$, where $X, X', Y \in \mathcal{C}'$. Proposition 4.4 implies that σ is also a pullback square. Let $L : \mathcal{C} \to \mathcal{C}'$ be a left adjoint to the inclusion. Since L is left exact, we obtain a pullback square $L(\sigma)$:

$$\begin{array}{ccc} LX & \longrightarrow & LX' \\ \downarrow & & \downarrow \\ LY & \longrightarrow & LY'. \end{array}$$

Applying Proposition 4.4 again, we deduce that $L(\sigma)$ is a pushout square in $\mathcal{C}$. The natural transformation $\sigma \to L(\sigma)$ is an equivalence when restricted to Λ^2_0, and therefore induces an equivalence $Y' \to LY'$. It follows that Y' belongs to the essential image of $\mathcal{C}'$, as desired. □

Lemma 15.8. *Let $\mathcal{C}$ be a stable ∞-category, $\mathcal{D}$ an ∞-category which admits finite limits, and $G : \mathcal{C} \to \mathrm{Stab}(\mathcal{D})$ an exact functor. Suppose that $g = \Omega^\infty \circ G : \mathcal{C} \to \mathcal{D}$ is fully faithful. Then G is fully faithful.*

Proof. It will suffice to show that each of the composite maps

$$g_n : \mathcal{C} \to \mathrm{Stab}(\mathcal{D}) \stackrel{\Omega_*^{\infty-n}}{\to} \mathcal{D}_*$$

is fully faithful. Since g_n can be identified with $g_{n+1} \circ \Omega$, where $\Omega : \mathcal{C} \to \mathcal{C}$ denotes the loop functor, we can reduce to the case $n = 0$. Fix objects $C, C' \in \mathcal{C}$; we will show that the map $\mathrm{Map}_{\mathcal{C}}(C, C') \to \mathrm{Map}_{\mathcal{D}_*}(g_0(C), g_0(C'))$ is a homotopy equivalence. We have a homotopy fiber sequence

$$\mathrm{Map}_{\mathcal{D}_*}(g_0(C), g_0(C')) \stackrel{\theta}{\to} \mathrm{Map}_{\mathcal{D}}(g(C), g(C')) \to \mathrm{Map}_{\mathcal{D}}(*, g(C')).$$

Here $*$ denotes a final object of $\mathcal{D}$. Since g is fully faithful, it will suffice to prove that θ is a homotopy equivalence. For this, it suffices to show that $\mathrm{Map}_{\mathcal{D}}(*, g(C'))$ is contractible. Since g is left exact, this space can be identified with $\mathrm{Map}_{\mathcal{D}}(g(*), g(C'))$, where $*$ is the final object of $\mathcal{C}$. Invoking once again our assumption that g is fully faithful, we are reduced to proving that $\mathrm{Map}_{\mathcal{C}}(*, C')$ is contractible. This follows from the assumption that $\mathcal{C}$ is pointed (since $*$ is also an initial object of $\mathcal{C}$). □

Proposition 15.9. *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is presentable and stable.*

(2) *There exists a presentable, stable ∞-category $\mathcal{D}$ and an accessible left-exact localization $L : \mathcal{D} \to \mathcal{C}$.*

(3) *There exists a small ∞-category $\mathcal{E}$ such that $\mathcal{C}$ is equivalent to an accessible left-exact localization of $\mathrm{Fun}(\mathcal{E}, \mathcal{S}_\infty)$.*

Proof. The ∞-category $\mathcal{S}_\infty$ is stable and presentable(Proposition 9.13), so that any functor category $\mathrm{Fun}(\mathcal{E}, \mathcal{S}_\infty)$ is also stable (Proposition 4.1) and presentable (Proposition T.5.5.3.6). This proves (3) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (1) follows from Lemma 15.7. We will complete the proof by showing that (1) $\Rightarrow$ (3).

Since $\mathcal{C}$ is presentable, there exists a small ∞-category $\mathcal{E}$ and a fully faithful embedding $g : \mathcal{C} \to \mathcal{P}(\mathcal{E})$, which admits a left adjoint (Theorem T.5.5.1.1). Proposition 10.10 implies that g is equivalent to a composition

$$\mathcal{C} \xrightarrow{G} \mathrm{Stab}(\mathcal{P}(\mathcal{E})) \xrightarrow{\Omega^\infty} \mathcal{P}(\mathcal{E}),$$

where the functor G is exact, accessible, and limit-preserving. Lemma 15.8 implies that G is fully faithful. Using Corollary T.5.5.2.9 and Lemma 15.7. we conclude that $\mathcal{C}$ is an (accessible) left exact localization of $\mathrm{Stab}(\mathcal{P}(\mathcal{E}))$. We now invoke Example 10.11 to identify $\mathrm{Stab}(\mathcal{P}(\mathcal{E}))$ with $\mathrm{Fun}(\mathcal{E}, \mathcal{S}_\infty)$. □

Remark 15.10. Proposition 15.9 can be regarded as an analogue of Giraud's characterization of topoi as left exact localizations of presheaf categories ([1]). Other variations on this theme include the ∞-categorical version of Giraud's theorem (Theorem T.6.1.0.6) and the Gabriel-Popesco theorem for abelian categories (see [16]).

16 Accessible t-Structures

Let $\mathcal{C}$ be a stable ∞-category. If $\mathcal{C}$ is presentable, then it is reasonably easy to construct t-structures on $\mathcal{C}$: for any small collection of objects $\{X_\alpha\}$ of $\mathcal{C}$, there exists a t-structure *generated* by the objects X_α. More precisely, we have the following result:

Proposition 16.1. *Let $\mathcal{C}$ be a presentable stable ∞-category.*

(1) *If $\mathcal{C}' \subseteq \mathcal{C}$ is a full subcategory which is presentable, closed under small colimits, and closed under extensions, then there exists a t-structure on $\mathcal{C}$ such that $\mathcal{C}' = \mathcal{C}_{\geq 0}$.*

(2) *Let $\{X_\alpha\}$ be a small collection of objects of $\mathcal{C}$, and let $\mathcal{C}'$ be the smallest full subcategory of $\mathcal{C}$ which contains each X_α and is closed under extensions and small colimits. Then $\mathcal{C}'$ is presentable.*

Proof. We will give the proof of (1) and defer the (somewhat technical) proof of (2) until the end of this section. Fix $X \in \mathcal{C}$, and let $\mathcal{C}'_{/X}$ denote the fiber product $\mathcal{C}_{/X} \times_{\mathcal{C}} \mathcal{C}'$. Using Proposition T.5.5.3.12, we deduce that $\mathcal{C}'_{/X}$ is presentable, so that it admits a final object $f : Y \to X$. It follows that composition with f induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(Z, Y) \to \mathrm{Map}_{\mathcal{C}}(Z, X)$$

for each $Z \in \mathcal{C}'$. Proposition T.5.2.6.7 implies that $\mathcal{C}'$ is a colocalization of $\mathcal{C}$. Since $\mathcal{C}'$ is stable under extensions, Proposition 6.15 implies the existence of a (uniquely determined) t-structure such that $\mathcal{C}' = \mathcal{C}_{\geq 0}$. □

Definition 16.2. Let $\mathcal{C}$ be a presentable stable ∞-category. We will say that a t-structure on $\mathcal{C}$ is *accessible* the subcategory $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ is presentable.

Proposition 16.1 can be summarized as follows: any small collection of objects $\{X_\alpha\}$ of a presentable stable ∞-category $\mathcal{C}$ determines an accessible t-structure on $\mathcal{C}$, which is minimal among t-structures such that each X_α belongs to $\mathcal{C}_{\geq 0}$.

Definition 16.2 has a number of reformulations:

Proposition 16.3. *Let $\mathcal{C}$ be a presentable stable ∞-category equipped with a t-structure. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}_{\geq 0}$ is presentable (equivalently: the t-structure on $\mathcal{C}$ is accessible).*

(2) *The ∞-category $\mathcal{C}_{\geq 0}$ is accessible.*

(3) *The ∞-category $\mathcal{C}_{\leq 0}$ is presentable.*

(4) *The ∞-category $\mathcal{C}_{\leq 0}$ is accessible.*

(5) *The truncation functor $\tau_{\leq 0} : \mathcal{C} \to \mathcal{C}$ is accessible.*

(6) *The truncation functors $\tau_{\geq 0} : \mathcal{C} \to \mathcal{C}$ is accessible.*

Proof. We observe that $\mathcal{C}_{\geq 0}$ is stable under all colimits which exist in $\mathcal{C}$, and that $\mathcal{C}_{\leq 0}$ is a localization of $\mathcal{C}$. It follows that $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ admit small colimits, so that (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4). We have a distinguished triangle of functors

$$\tau_{\geq 0} \xrightarrow{\alpha} \mathrm{id}_{\mathcal{C}} \xrightarrow{\beta} \tau_{\leq -1} \to \tau_{\geq 0}[1]$$

in the homotopy category $\mathrm{hFun}(\mathcal{C}, \mathcal{C})$. The collection of accessible functors from $\mathcal{C}$ to itself is stable under shifts and under small colimits. Since $\tau_{\leq 0} \simeq \mathrm{coker}(\alpha)[1]$ and $\tau_{\geq 0} \simeq \mathrm{coker}(\beta)[-1]$, we conclude that (5) $\Leftrightarrow$ (6). The equivalence (1) $\Leftrightarrow$ (5) follows from Proposition T.5.5.1.2. We will complete the proof by showing that (1) $\Leftrightarrow$ (3).

Suppose first that (1) is satisfied. Then $\mathcal{C}_{\geq 1} = \mathcal{C}_{\geq 0}[1]$ is generated under colimits by a set of objects $\{X_\alpha\}$. Let S be the collection of all morphisms f in $\mathcal{C}$ such that $\tau_{\leq 0}(f)$ is an equivalence. Using Proposition 6.15, we conclude that S is generated by $\{0 \to X_\alpha\}$ as a weakly saturated class of morphisms, and therefore also as a strongly saturated class of morphisms (Definition T.5.5.4.5). We now apply Proposition T.5.5.4.18 to conclude that $\mathcal{C}_{\leq 0} = S^{-1}\,\mathcal{C}$ is presentable; this proves (3).

We now complete the proof by showing that (3) $\Rightarrow$ (1). If $\mathcal{C}_{\leq -1} = \mathcal{C}_{\leq 0}[-1]$ is presentable, then Proposition T.5.5.4.19 implies that S is of small generation (as a strongly saturated class of morphisms). Proposition 6.15 implies that S is generated (as a strongly saturated class) by the morphisms $\{0 \to X_\alpha\}_{\alpha \in A}$, where X_α ranges over the collection of all objects of $\mathcal{C}_{\geq 0}$. It follows that there is a small subcollection $A_0 \subseteq A$ such that S is generated by the morphisms $\{0 \to X_\alpha\}_{\alpha \in A_0}$. Let $\mathcal{D}$ be the smallest full subcategory of $\mathcal{C}$ which contains the objects $\{X_\alpha\}_{\alpha \in A_0}$ and is closed under colimits and extensions. Since $\mathcal{C}_{\geq 0}$ is closed under colimits and extensions, we have $\mathcal{D} \subseteq \mathcal{C}_{\geq 0}$. Consequently, $\mathcal{C}_{\leq -1}$ can be characterized as full subcategory of $\mathcal{C}$ spanned by those objects $Y \in \mathcal{C}$ such that $\mathrm{Ext}^k_{\mathcal{C}}(X, Y)$ for all $k \leq 0$ and $X \in \mathcal{D}$. Propositions 16.1 implies that $\mathcal{D}$ is the collection of nonnegative objects for some accessible t-structure on $\mathcal{C}$. Since the negative objects of this new t-structure coincide with the negative objects of the original t-structure, we conclude that $\mathcal{D} = \mathcal{C}_{\geq 0}$, which proves (1). □

The following result provides a good source of examples of accessible t-structures:

Proposition 16.4. *Let $\mathcal{C}$ be a presentable ∞-category, and let $\mathrm{Stab}(\mathcal{C})_{\leq -1}$ be the full subcategory of $\mathrm{Stab}(\mathcal{C})$ spanned by those objects X such that $\Omega^\infty(X)$ is a final object of $\mathcal{C}$. Then $\mathrm{Stab}(\mathcal{C})_{\leq -1}$ determines an accessible t-structure on $\mathrm{Stab}(\mathcal{C})$.*

Proof. Choose a small collection of objects $\{C_\alpha\}$ which generate $\mathcal{C}$ under colimits. We observe that an object $X \in \mathrm{Stab}(\mathcal{C})$ belongs to $\mathrm{Stab}(\mathcal{C})_{\leq -1}$ if and only if each of the spaces

$$\mathrm{Map}_{\mathcal{C}}(C_\alpha, \Omega^\infty(X)) \simeq \mathrm{Map}_{\mathrm{Stab}(\mathcal{C})}(\Sigma^\infty(C_\alpha), X)$$

is contractible. Let $\mathrm{Stab}(\mathcal{C})_{\geq 0}$ be the smallest full subcategory of $\mathrm{Stab}(\mathcal{C})$ which is stable under colimits and extensions, and contains each $\Sigma^\infty(C_\alpha)$. Proposition 16.1 implies that $\mathrm{Stab}(\mathcal{C})_{\geq 0}$ is the collection of nonnegative objects of the desired t-structure on $\mathrm{Stab}(\mathcal{C})$. □

Remark 16.5. The proof of Proposition 16.4 gives another characterization of the t-structure on $\mathrm{Stab}(\mathcal{C})$: the full subcategory $\mathrm{Stab}(\mathcal{C})_{\geq 0}$ is generated, under extensions and colimits, by the essential image of the functor $\Sigma^\infty : \mathcal{C} \to \mathrm{Stab}(\mathcal{C})$.

We conclude this section by completing the proof of Proposition 16.1.

Proof of part (2) *of Proposition 16.1.* Choose a regular cardinal κ such that every object of X_α is κ-compact, and let $\mathcal{C}^\kappa$ denote the full subcategory of $\mathcal{C}$ spanned by the κ-compact objects. Let $\mathcal{C}'^\kappa = \mathcal{C}' \cap \mathcal{C}^\kappa$, and let $\mathcal{C}''$ be the smallest full subcategory of $\mathcal{C}'$ which contains $\mathcal{C}'^\kappa$ and is closed under small colimits. The ∞-category $\mathcal{C}''$ is κ-accessible, and therefore presentable. To complete the proof, we will show that $\mathcal{C}' \subseteq \mathcal{C}''$. For this, it will suffice to show that $\mathcal{C}''$ is stable under extensions.

Let $\mathcal{D}$ be the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by those morphisms $f : X \to Y$ where $Y \in \mathcal{C}''$, $X \in \mathcal{C}''[-1]$. We wish to prove that the cokernel functor $\mathrm{coker} : \mathcal{D} \to \mathcal{C}$ factors through $\mathcal{C}''$. Let $\mathcal{D}^\kappa$ be the full subcategory of $\mathcal{D}$ spanned by those morphisms $f : X \to Y$ where both X and Y are κ-compact objects of $\mathcal{C}$. By construction, $\mathrm{coker} \,|\, \mathcal{D}^\kappa$ factors through $\mathcal{C}''$. Since $\mathrm{coker} : \mathcal{D} \to \mathcal{C}$ preserves small colimits, it will suffice to show that $\mathcal{D}$ is generated (under small colimits) by $\mathcal{D}^\kappa$.

Fix an object $f : X \to Y$ in $\mathcal{D}$. To complete the proof, it will suffice to show that the canonical map $(\mathcal{D}^\kappa_{/f})^\triangleright \to \mathcal{D}$ is a colimit diagram. Since $\mathcal{D}$ is stable under colimits in $\mathrm{Fun}(\Delta^1, \mathcal{C})$ and colimits in $\mathrm{Fun}(\Delta^1, \mathcal{C})$ are computed pointwise (Proposition T.5.1.2.2), it will suffice to show that composition with the evaluation maps give colimit diagrams $(\mathcal{D}^\kappa_{/f})^\triangleright \to \mathcal{C}$. Lemma T.5.3.5.8 implies that the maps $(\mathcal{C}'^\kappa[-1])^\triangleright_{/X} \to \mathcal{C}$, $(\mathcal{C}'^\kappa)^\triangleright_{/Y} \to \mathcal{C}$ are colimit diagrams. It will therefore suffice to show that the evaluation maps

$$(\mathcal{C}'^\kappa[-1])_{/X} \overset{\theta}{\leftarrow} (\mathcal{D}^\kappa_{/f}) \overset{\theta'}{\to} (\mathcal{C}'^\kappa)_{/Y}$$

are cofinal.

We first show that θ is cofinal. According to Theorem T.4.1.3.1, it will suffice to show that for every morphism $\alpha : X' \to X$ in $\mathcal{C}'[-1]$, where X' is κ-compact, the ∞-category

$$\mathcal{E}_\theta : \mathcal{D}^\kappa_{/f} \times_{\mathcal{C}'^\kappa[-1]_{/X}} (\mathcal{C}'^\kappa[-1]_{/X})_{X'/}$$

is weakly contractible. For this, it is sufficient to show that $\mathcal{E}_\theta$ is filtered (Lemma T.5.3.1.18).

We will show that $\mathcal{E}_\theta$ is κ-filtered. Let K be a κ-small simplicial set, and $p : K \to \mathcal{E}_\theta$ a diagram; we will extend p to a diagram $\overline{p} : K^\triangleright \to \mathcal{E}_\theta$. We can identify p with two pieces of data:

- (*i*) A map $p' : K^\triangleleft \to \mathcal{C}'^\kappa[-1]_{/X}$.
- (*ii*) A map $p'' : (K \star \{\infty\}) \times \Delta^1 \to \mathcal{C}$, with the properties that $p''|(K \star \{\infty\}) \times \{0\}$ can be identified with p', $p''|\{\infty\} \times \Delta^1$ can be identified with f, and $p''|K \times \{1\}$ factors through $\mathcal{C}'^\kappa$.

Let $\overline{p}' : (K^\triangleleft)^\triangleright \to \mathcal{C}'^\kappa[-1]_{/X}$ be a colimit of p'. To complete the proof that $\mathcal{E}_\theta$ is κ-filtered, it will suffice to show that we can find a compatible extension $\overline{p}'' : (K^\triangleright \star \{\infty\}) \times \Delta^1 \to \mathcal{C}$ with the appropriate properties. Let L denote the full simplicial subset of $(K^\triangleright \star \{\infty\}) \times \Delta^1)$ spanned by every vertex except $(v, 1)$, where v denotes the cone point of $K^\triangleright$. We first choose a map $q : L \to \mathcal{C}$ compatible with p'' and $\overline{p}'$. This is equivalent to solving the lifting problem

$$\begin{array}{ccc} & & \mathcal{C}_{/f} \\ & \nearrow & \downarrow \\ K^\triangleright & \longrightarrow & \mathcal{C}_{/X}, \end{array}$$

which is possible since the vertical arrow is a trivial fibration. Let $L' = L \cap (K^\triangleright \times \Delta^1)$. Then q determines a map $q_0 : L' \to \mathcal{C}_{/Y}$. Finding the desired extension $\overline{p}''$ is equivalent to finding a map $\overline{q}_0 : L'^\triangleright \to \mathcal{C}_{/Y}$, which carries the cone point into $\mathcal{C}'^\kappa$.

Let $g : Z \to Y$ be a colimit of q_0 (in the ∞-category $\mathcal{C}_{/Y}$). We observe that Z is a κ-small colimit of κ-compact objects of $\mathcal{C}$, and therefore κ-compact. Since $Y \in \mathcal{C}''$, Y can be written as the colimit of a κ-filtered diagram $\{Y_\alpha\}$, taking values in $\mathcal{C}'^\kappa$. Since Z is κ-compact, the map g factors through some Y_α; it follows that there exists an extension $\overline{q}_0$ as above, which carries the cone point to Y_α. This completes the proof that $\mathcal{E}_\theta$ is κ-filtered, and also the proof that θ is cofinal.

The proof that θ' is cofinal is similar but slightly easier: it suffices to show that for every map $Y' \to Y$ in $\mathcal{C}'$, where Y' is κ-compact, the fiber product

$$\mathcal{E}_{\theta'} = \mathcal{D}^{\kappa}_{/f} \times_{\mathcal{C}'^{\kappa}_{/Y}} (\mathcal{C}'^{\kappa}_{/Y})_{Y'/}$$

is filtered. For this, we can either argue as above, or simply observe that $\mathcal{E}_{\theta'}$ admits κ-small colimits. □

A Calculation of Geometric Realizations

Let $\mathcal{C}$ be a small category which admits finite products, and let $\mathbf{A}$ be the full subcategory of $\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}$ spanned by the product preserving functors. Propositions 13.1 and 13.2 imply that $\mathbf{A}$ has the structure of a simplicial model category, whose underlying ∞-category is equivalent to $\mathcal{P}_{\Sigma}(\mathrm{N}(\mathcal{C})^{op})$. Let let $\mathcal{A}$ denote the full subcategory of $\mathcal{S}\mathrm{et}^{\mathcal{C}}$ spanned by the product preserving functors. Then $\mathbf{A}$ can be identified with the category of simplicial objects of $\mathcal{A}$. In particular, we can identify $\mathcal{A}$ with a full subcategory of $\mathbf{A}$, so that every object $\mathcal{F} \in \mathbf{A}$ determines a simplicial object

$$\psi : \mathbf{\Delta}^{op} \to \mathcal{A} \subseteq \mathbf{A}.$$

Our goal in this section is to show that $\mathcal{F}$ can be identified with a homotopy colimit (in $\mathbf{A}$) of the diagram ψ. More generally, we have the following result:

Proposition A.1. *Let $\mathcal{C}$ be a category which admits finite products, and $\mathbf{A} \subseteq \mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{C}}$, $\mathcal{A} \subseteq \mathcal{S}\mathrm{et}^{\mathcal{C}}$ the full subcategories spanned by the product-preserving functors. Let $\mathcal{F} : \mathbf{\Delta}^{op} \to \mathbf{A}$ be a simplicial object of $\mathbf{A}$, which we can identify with a* bisimplicial *object $F : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathcal{A}$. Composition with the diagonal*

$$\mathbf{\Delta}^{op} \to \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \xrightarrow{F} \mathcal{A}$$

gives a simplicial object of $\mathcal{A}$, which we can identify with an object $|\mathcal{F}| \in \mathbf{A}$. Then the homotopy colimit of $\mathcal{F}$ is canonically isomorphic to $|\mathcal{F}|$ in the homotopy category $\mathrm{h}\mathbf{A}$.

The first step in the proof of Proposition A.1 is to replace $\mathcal{F}$ by an equivalent simplicial object which is better suited to the computation of homotopy colimits.

Notation A.2. Let $\mathcal{D}$ be a category which admits finite products and let $\mathbf{A}$ be the category of product preserving functors from $\mathcal{D}$ to $\mathcal{S}\mathrm{et}_{\Delta}$. Let $\mathcal{F} : \mathbf{\Delta}^{op} \to \mathbf{A}$ be a simplicial object of $\mathbf{A}$. We define a new simplicial object $\widetilde{\mathcal{F}} : \mathbf{\Delta}^{op} \to \mathbf{A}$ as follows. For each $D \in \mathcal{D}$, let

$$\mathrm{Map}_{\mathcal{S}\mathrm{et}_{\Delta}}(\Delta^m, \widetilde{\mathcal{F}}([n])(D)) = \mathrm{Map}_{\mathcal{S}\mathrm{et}_{\Delta}}(\Delta^m, \mathcal{F}([m] \star [n])(D)).$$

where $\star : \mathbf{\Delta} \times \mathbf{\Delta} \to \mathbf{\Delta}$ is the operation of concatenating linearly ordered sets (so that $[m] \star [n] \simeq [m+n+1]$).

We observe that this definition makes sense even for $n = -1$, so that $\widetilde{\mathcal{F}}$ underlies an *augmented* simplicial object $\widetilde{\mathcal{F}}^{+} : \mathbf{\Delta}_{+}^{op} \to \mathbf{A}$, with $\widetilde{\mathcal{F}}^{+}(\emptyset) \simeq |\mathcal{F}|$. Moreover, the inclusions $[n] \hookrightarrow [m] \star [n]$ induce a natural transformation $\alpha : \widetilde{\mathcal{F}} \to \mathcal{F}$ of simplicial objects of $\mathbf{A}$.

Lemma A.3. *Let $X : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathcal{S}\mathrm{et}$ be a bisimplicial set, and let $Y, Z : \mathbf{\Delta}^{op} \to \mathcal{S}\mathrm{et}$ be defined by the equations*

$$Y([n]) = X([n] \star [0], [n])$$

$$Z([n]) = X([0], [n]).$$

Let $\theta : Y \to Z$ be the map induced by the inclusions $[0] \hookrightarrow [n] \star [0]$. Then θ is a weak homotopy equivalence of simplicial sets.

Proof. Let $\psi : Z \to Y$ be the map induced by the projections $[n] \star [0] \to [0]$ in $\mathbf{\Delta}$. Then $\theta \circ \psi = \mathrm{id}_Z$, and $\psi \circ \theta$ is simplicially homotopic to the identity. □

Lemma A.4. *Let $\mathcal{F} : \mathbf{\Delta}^{op} \to \mathbf{A}$ be as in Notation A.2. Then the natural transformation $\alpha : \widetilde{\mathcal{F}} \to \mathcal{F}$ is a weak equivalence of simplicial objects of $\mathbf{A}$.*

Proof. It will suffice to prove that for each $n \geq 0$ and each $C \in \mathcal{C}$, the associated map

$$\widetilde{\mathcal{F}}([n])(C) \to \mathcal{F}([n])(C)$$

is a weak homotopy equivalence of simplicial sets. Shifting if necessary, we may reduce to the case $n = 0$. We now conclude by applying Lemma A.3. □

Lemma A.5. *Let $\beta : \mathcal{F} \to \mathcal{F}'$ be a natural transformation between simplicial objects of $\mathcal{S}\mathrm{et}_\Delta$, and let $\widetilde{\beta} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}'$ be the induced map (see Notation A.2). Then:*

(1) *If β is a strong cofibration (in $\mathcal{S}\mathrm{et}_\Delta^{\mathbf{\Delta}^{op}}$), then $\widetilde{\beta}$ is also a strong cofibration.*

(2) *If β is a weak equivalence, then $\widetilde{\beta}$ is a weak equivalence.*

(3) *If β is a weak equivalence, then $\widetilde{\beta}$ induces a weak equivalence $|\,\mathcal{F}\,| \to |\,\mathcal{F}'\,|$ on diagonals.*

Proof. To prove (1), it suffices to choose β to be a generator for the class of strong cofibrations in $\mathcal{S}\mathrm{et}_\Delta^{\mathbf{\Delta}^{op}}$. In this case, the desired result follows from an explicit computation.

To prove (2), we consider the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{F}} & \xrightarrow{\widetilde{\beta}} & \widetilde{\mathcal{F}}' \\ \downarrow & & \downarrow \\ \mathcal{F} & \xrightarrow{\beta} & \mathcal{F}'. \end{array}$$

The vertical arrows are weak equivalences by Lemma A.4. If β is a weak equivalence, it follows that $\widetilde{\beta}$ is a weak equivalence as well.

Assertion (3) is a standard result in the theory of bisimplicial sets; see for example [6], p. 199. □

Lemma A.6. *Let $\mathcal{F}$ be a simplicial object of $\mathcal{S}\mathrm{et}_\Delta$, and let $\widetilde{\mathcal{F}}^+ : \mathbf{\Delta}_+^{op} \to \mathcal{S}\mathrm{et}_\Delta$ be defined as in Notation A.2. Then $\widetilde{\mathcal{F}}^+$ is a homotopy colimit diagram.*

Proof. Choose a weak equivalence $\mathcal{G} \to \mathcal{F}$, where $\mathcal{G}$ is a strongly cofibrant diagram $\mathbf{\Delta}^{op} \to \mathcal{S}\mathrm{et}_\Delta$. In view of parts (2) and (3) of Lemma A.5, it will suffice to show that $\widetilde{\mathcal{G}}^+$ is a homotopy colimit diagram. To prove this, we observe that $\widetilde{\mathcal{G}}$ is a strongly cofibrant diagram (part (1) of Lemma A.5), and that $\widetilde{\mathcal{G}}^+$ is a colimit diagram in $\mathcal{S}\mathrm{et}_\Delta$ (this follows either by direct observation, or by applying Lemma T.6.1.3.17 degreewise, in the ordinary category $\mathcal{S}\mathrm{et}$). □

Proof of Proposition A.1. In view of Lemma A.4, it will suffice to show that the $\widetilde{\mathcal{F}}^+ : \mathbf{\Delta}_+^{op} \to \mathbf{A}$ is a homotopy colimit diagram. Using Proposition 13.2, we may reduce to the problem of showing that the diagram $\widetilde{\mathcal{F}}^+ : \mathbf{\Delta}_+^{op} \to \mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}}$ is a homotopy colimit. Since strongly cofibrant diagrams in $\mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}}$ remain strongly cofibrant after evaluation at any object $C \in \mathcal{C}$, it will suffice to show that each of the compositions

$$\mathbf{\Delta}_+^{op} \xrightarrow{\widetilde{F}^+} \mathbf{A} \subseteq \mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}} \to \mathcal{S}\mathrm{et}_\Delta$$

is a homotopy colimit diagram. This follows immediately from Lemma A.6. □

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