

The Frobenius (Lecture 14)

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Our goal in this lecture is to study some of the basic features of the category \mathcal{U} of unstable modules over the Steenrod algebra \mathcal{A} . We begin with a few general remarks.

For every commutative algebra R over the field \mathbf{F}_2 , there is a canonical ring homomorphism $F : R \rightarrow R$, called the *Frobenius morphism*, given by $F(x) = x^2$. The Frobenius map is functorial with respect to all homomorphisms between commutative \mathbf{F}_2 -algebras: in other words, every map $f : R \rightarrow R'$ fits into a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ \downarrow F & & \downarrow F \\ R & \xrightarrow{f} & R'. \end{array}$$

In particular, if R is a commutative Hopf algebra over \mathbf{F}_2 with comultiplication $\Delta : R \rightarrow R \otimes R$, then we have a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\Delta} & R \otimes R \\ \downarrow F & & \downarrow F \otimes F \\ R & \xrightarrow{\Delta} & R \otimes R. \end{array}$$

In other words, the Frobenius map F is a homomorphism of Hopf algebras.

We apply this remark in the case where R is the dual Steenrod algebra $\mathcal{A}^\vee \simeq \mathbf{F}_2[b_1, b_3, b_7, \dots]$. We have a map of Hopf algebras

$$F : \mathcal{A}^\vee \rightarrow \mathcal{A}^\vee.$$

Passing to the graded dual, we obtain another map of Hopf algebras $V : \mathcal{A} \rightarrow \mathcal{A}$, called the *Verschiebung*.

Let us compute the map V . Since V is a map of algebras, it will suffice to compute $V(\text{Sq}^n)$ for each $n \geq 0$. Let $\langle \cdot, \cdot \rangle : \mathcal{A} \otimes \mathcal{A}^\vee \rightarrow \mathbf{F}_2$ denote the pairing between the Steenrod algebra and its dual. By definition, we have

$$\langle V(\text{Sq}^n), x \rangle = \langle \text{Sq}^n, x^2 \rangle.$$

Since the algebra structure on \mathcal{A}^\vee is dual to the comultiplication $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, we get

$$\langle \text{Sq}^n, x^2 \rangle = \langle \Delta \text{Sq}^n, x \otimes x \rangle = \sum_{n=i+j} \langle \text{Sq}^i, x \rangle \langle \text{Sq}^j, x \rangle.$$

We note that the terms in this sum for which $i \neq j$ cancel in pairs. Moreover, if n is even, the term with $i = j = \frac{n}{2}$ coincides with

$$\langle \text{Sq}^i, x \rangle \langle \text{Sq}^j, x \rangle = \langle \text{Sq}^{\frac{n}{2}}, x \rangle.$$

We can summarize this calculation as follows:

Proposition 1. *The Verschiebung map $V : \mathcal{A} \rightarrow \mathcal{A}$ is given by the formula*

$$V(\mathrm{Sq}^n) = \begin{cases} \mathrm{Sq}^{\frac{n}{2}} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Remark 2. We could instead regard V as being *defined* by the formula of Proposition 1. Then we would need to check that V is well-defined, which is an exercise in manipulating the Adem relations.

Let M be a module over the Steenrod algebra \mathcal{A} , so that we have a ring homomorphism $\mathcal{A} \rightarrow \mathrm{End}(M)$. Composing with the Verschiebung map $V : \mathcal{A} \rightarrow \mathcal{A}$, we get a new homomorphism $\mathcal{A} \rightarrow \mathrm{End}(M)$, which gives a new \mathcal{A} -module structure on M . We will denote this new \mathcal{A} -module by ΦM . More concretely:

- (1) The elements of ΦM can be identified with the elements of M . When it is important to distinguish between the M and ΦM , we let $\Phi(x)$ denote the element of ΦM corresponding to $x \in M$.
- (2) The Steenrod algebra acts on ΦM by the formula

$$\mathrm{Sq}^n \Phi(x) = \begin{cases} \Phi(\mathrm{Sq}^{\frac{n}{2}} x) & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

The map V does not preserve the grading on the Steenrod algebra \mathcal{A} : we have instead $\deg V(a) = \frac{\deg(a)}{2}$ if a is homogeneous of even degree (and $V(a)$ vanishes if a has odd degree). If M is a graded \mathcal{A} -module, then ΦM again has the structure of a graded \mathcal{A} -module via the following convention:

- (3) For each $n \geq 0$, we let

$$(\Phi M)^n = \begin{cases} M^{\frac{n}{2}} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Note that if M is an unstable \mathcal{A} -module, then ΦM is again unstable: if x is a nonzero element of $(\Phi M)^n$, then $n = 2k$ is even and $x = \Phi(x_0)$ for some $x_0 \in M^k$. If $m > n$, then $\mathrm{Sq}^m(x)$ vanishes by definition if m is odd, and is equal to $\Phi(\mathrm{Sq}^{\frac{m}{2}}(x_0))$ if m is even; this will also vanish since $\frac{m}{2} > k$ and M is assumed to be unstable.

Proposition 3. *Let M be an unstable module over the Steenrod algebra \mathcal{A} . Then there is a canonical homomorphism of \mathcal{A} -modules*

$$f : \Phi M \rightarrow M$$

defined by the formula $f(\Phi(x)) = \Phi(\mathrm{Sq}^{\deg x}(x))$ when x is homogeneous.

Remark 4. If M is an unstable algebra over \mathcal{A} , we can rewrite the definition of f as $f(\Phi(x)) = \Phi(x^2)$. In other words, we can think of f as a kind of Frobenius map.

Proof. We must show that f is compatible with the action of the Steenrod algebra: in other words, we must show that for every homogeneous element x , we have

$$f(\mathrm{Sq}^n \Phi(x)) = \mathrm{Sq}^n \mathrm{Sq}^{\deg x}(x).$$

There are three cases to consider.

If $n > 2 \deg(x)$, then both sides vanish in view of our assumption that M is unstable. If $n = 2 \deg(x)$, then we have

$$f(\mathrm{Sq}^n \Phi(x)) = f(\Phi(\mathrm{Sq}^{\deg x}(x))) = \mathrm{Sq}^{2 \deg x} \mathrm{Sq}^{\deg x} x = \mathrm{Sq}^n \mathrm{Sq}^{\deg x} x.$$

If $n < 2 \deg(x)$, then we can rewrite $\mathrm{Sq}^n \mathrm{Sq}^{\deg x}(x)$ using the Adem relations. We get

$$\mathrm{Sq}^n \mathrm{Sq}^{\deg x}(x) = \sum_k (2k - n, \deg(x) - k - 1) \mathrm{Sq}^{\deg(x)+k} \mathrm{Sq}^{n-k}(x).$$

Terms with $2k > n$ vanish since $\deg(x) + k > \deg(\mathrm{Sq}^{n-k}(x)) = n - k + \deg(x)$. Terms with $2k < n$ vanish since $2k - n < 0$. We therefore have

$$\mathrm{Sq}^n \mathrm{Sq}^{\deg x}(x) = \begin{cases} 0 & n \text{ odd} \\ \mathrm{Sq}^{\deg x + \frac{n}{2}} \mathrm{Sq}^{\frac{n}{2}} x & n \text{ even.} \end{cases}$$

On the other hand, we have

$$\mathrm{Sq}^n(\Phi(x)) = \begin{cases} 0 & n \text{ odd} \\ \Phi(\mathrm{Sq}^{\frac{n}{2}}) & n \text{ even.} \end{cases}$$

and in the latter case $\deg(\mathrm{Sq}^{\frac{n}{2}}(x)) = \deg(x) + \frac{n}{2}$, so the desired equality holds. \square

Let us study the behavior of the homomorphism f in the case where $M = F(n)$ is the free unstable \mathcal{A} -module on one generator ν_n . In this case, M has a basis $\{\mathrm{Sq}^I \nu_n\}$, where I ranges over admissible positive sequences (i_1, \dots, i_k) of excess $\leq n$. We observe that $f(\Phi(\mathrm{Sq}^I \nu_n)) = \mathrm{Sq}^{I'} \nu_n$, where I' is the sequence $(i_0, i_1, i_2, \dots, i_k)$ with

$$i_0 = \deg \mathrm{Sq}^I \nu_n = i_1 + i_2 + \dots + i_k + n.$$

In particular, the excess $i_0 - i_1 - \dots - i_k$ of I' is precisely n . Conversely, if I' is an admissible sequence of excess n , then $I' = (\deg \mathrm{Sq}^I \nu_n, i_1, i_2, \dots, i_k)$, where $I = (i_1, i_2, \dots, i_k)$. In other words:

Proposition 5. *The map $f : \Phi F(n) \rightarrow F(n)$ is injective, and its image is spanned by expressions $\{\mathrm{Sq}^I \nu_n\}$ where I is positive, admissible, and has excess exactly n .*

The cokernel of the map $f : \Phi F(n) \rightarrow F(n)$ has a basis given by the images of the expressions $\{\mathrm{Sq}^I \nu_n\}$, where I ranges over admissible positive sequences of excess $< n$. Up to a change of grading, this is identical to the structure of the free unstable module $F(n-1)$. In order to describe the situation more systematically, we introduce the following definition:

Definition 6. Let M be an unstable module over the Steenrod algebra \mathcal{A} . We define a new unstable \mathcal{A} -module ΣM as follows:

- (1) As a vector space, $\Sigma M \simeq M$, and this isomorphism is compatible with the action of the Steenrod algebra.
- (2) The grading on ΣM is defined by the formula $(\Sigma M)^n \simeq M^{n-1}$.

In other words, ΣM is the module $M \otimes \mathbf{F}_2[-1]$, where $\mathbf{F}_2[-1]$ denotes a single copy of \mathbf{F}_2 in degree 1 (with its unique \mathcal{A} -module structure).

Warning 7. The notation introduced in Definition 6 is incompatible with our notation for suspensions of complexes used in previous lectures: if V is a complex with a good symmetric multiplication, we have an isomorphism of \mathcal{A} -modules $H^*(\Omega V) = \Sigma H^*(V)$.

If M is an unstable \mathcal{A} -module, then ΣM is again unstable. However, Σ does not define an equivalence from the category \mathcal{U} to itself, because the obvious “inverse” construction does not preserve instability. For each unstable \mathcal{A} -module M , let $\overline{\Omega}M$ denote the \mathcal{A} -module M , with the grading $(\overline{\Omega}M)^n \simeq M^{n+1}$. Then $\overline{\Omega}M$ is not necessarily unstable: an element $x \in (\overline{\Omega}M)^n$ can be identified with an element $x \in M^{n+1}$, so that x need not be annihilated by Sq^{n+1} . However, we can correct this deficiency by passing to a quotient: let ΩM denote the quotient of $\overline{\Sigma}M$ by the submodule generated by $\mathrm{Sq}^k x$ for $k > n$, $x \in (\overline{\Sigma}M)^n$. (In fact, it suffices to take $k = n + 1$ here). Then the construction $M \mapsto \Omega M$ defines a functor from the category of unstable \mathcal{A} -modules to itself, and this construction is left adjoint to the functor Σ .

We observe that, for every unstable \mathcal{A} -module M , we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(\Omega F(n), M) \simeq \mathrm{Hom}_{\mathcal{A}}(F(n), \Sigma M) \simeq (\Sigma M)^n \simeq M^{n-1}.$$

Consequently, we can identify $\Omega F(n)$ with $F(n-1)$ as an unstable \mathcal{A} -module. The adjoint of this identification is a map $F(n) \rightarrow \Sigma F(n-1)$. We can restate Proposition 5 as follows:

Proposition 8. *For each $n > 0$, we have a short exact sequence*

$$0 \rightarrow \Phi F(n) \xrightarrow{f} F(n) \xrightarrow{u} \Sigma \Omega F(n) \rightarrow 0$$

where u is the unit map for the adjunction between Ω and Σ and f is the map of Proposition 3.

Proposition 8 admits a generalization where we replace $F(n)$ by an arbitrary unstable \mathcal{A} -module M . We observe that the functors $M \mapsto \Phi M$ and $M \rightarrow \Sigma M$ are obviously exact. However, the functor $M \mapsto \Omega M$ is only right exact. We can therefore define left-derived functors $L^i \Omega M$ to be the homologies of the complex

$$\dots \rightarrow \Omega P_2 \rightarrow \Omega P_1 \rightarrow \Omega P_0 \rightarrow 0,$$

where $\dots \rightarrow P_1 \rightarrow P_0 \rightarrow M$ is a resolution of M by free unstable \mathcal{A} -modules. A standard argument in homological algebra shows that this definition is independent of the choice of resolution, up to canonical isomorphism.

Theorem 9. *For every unstable \mathcal{A} -module M , there is a canonical exact sequence*

$$0 \rightarrow \Sigma L^1 \Omega M \rightarrow \Phi M \xrightarrow{f} M \xrightarrow{u} \Sigma \Omega M \rightarrow 0$$

where u is the unit map for the adjunction between Σ and Ω , and f is the map described in Proposition 3. Moreover, the derived functors $L^i \Omega$ vanish for $i > 1$.

Proof. Choose a free resolution P_\bullet of M . Using Proposition 8, we get a short exact sequence of complexes

$$0 \rightarrow \Phi P_\bullet \rightarrow P_\bullet \rightarrow \Sigma \Omega P_\bullet \rightarrow 0.$$

The desired result now follows from the associated long exact sequence, since the complexes ΦP_\bullet and P_\bullet are exact in degrees > 0 . \square