Removal of Singularities for Stein Manifolds

Undergraduate Honors Thesis in Mathematics

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Abstract

We adapt the technique of removal of singularities to the holomorphic setting and prove a general flexibility result for holomorphic vector bundles over Stein manifolds. If $D : V \to W$ is an elliptic differential operator between holomorphic vector bundles over a Stein Manifold, then a q-tuple $(\theta_1, \ldots, \theta_q)$ of holomorphic sections generating W may be deformed to an exact holomorphic q-tuple $(D\phi_1, \ldots, D\phi_q)$ generating W. We also prove a parametric version of this theorem with holomorphic dependence on a Stein parameter X and obtain a 1-parametric h-principle. The parametric h-principle works relative to closed complex analytic subsets A of X. As corollaries we will obtain h-principles for holomorphic immersions and free maps of Stein Manifolds.

Contents

1 Introduction

Problems in smooth geometry are often subject to a partial differential inequality or relation that satisfies an h-principle. For any differential relation, there is a notion of a formal solution where the derivatives are replaced with algebraic relations. In situations where formal solutions can be deformed into actual solutions, we say that a problem satisfies an h-principle. More precisely, we can study the inclusion of the space of genuine solutions into the space of formal solutions and ask what properties this map has, e.g. if it induces an isomorphism on π_0 or if it is a homotopy equivalence.

For example, an immersion is a smooth map f with injective df. We can consider a formal immersion to be a pair (f, g) where f is a smooth map and g is a tangent bundle monomorphism covering f, not necessarily equal to df. Under some mild conditions it turns out that formal immersions can be deformed into actual immersions, which is to say the problem of finding immersions satisfies an h-principle.

There are several powerful methods, introduced by Gromov and Eliashberg, for proving h-principles. The one we will focus on in this thesis is known as the technique of removal of singularities (introduced in [3]). It can be used to prove the h-principle for immersions, among others. The goal of this thesis is to adapt this technique to a holomorphic setting and use it to prove a variety of h-principles there.

In holomorphic geometry there is much more rigidity than seen in the smooth case. Let M be a compact complex manifold. Then all holomorphic maps $M \to \mathbb{C}^q$ are constants by the maximum principle. In particular, M admits no immersions to complex Euclidean space. Even so, it is natural to ask if there are any examples of complex manifolds which admit holomorphic flexibility properties similar to those seen in smooth geometry. It turns out the answer is yes, there is an important class of complex manifolds known as Stein manifolds for which much of the theory carries over, which we will discuss in section 3.

The main result of this thesis is the following

Theorem 1.1. *Let M be a Stein manifold, and let V and W be holomorphic vector bundles over M. Let $D : \mathcal{A}(V) \to \mathcal{A}(W)$ be an elliptic differential operator of order s from the holomorphic sections of V to the holomorphic sections of W. Let $\theta_1, \ldots, \theta_q$ be holomorphic sections of W that span the fibers at every point so that q is greater than the rank of W. Then the q-tuple $(\theta_1, \ldots, \theta_q)$ is homotopic through q-tuples generating W to a q-tuple of exact holomorphic sections $(D\phi_1, \ldots, D\phi_q)$ with the same spanning property.*

We will also prove the following parametric version of the theorem in section 5.2.

Theorem 1.2. *Let $\operatorname{Sec}(V_q(W))$ denote the holomorphic sections of q-tuples spanning W. Let X be a Stein manifold, and let $\Theta(x) = (\theta_1, \ldots, \theta_q)(x)$ be a q-tuple of sections spanning W with holomorphic dependence on a parameter $x \in X$. Then there is a homotopy $H : X \times I \to \operatorname{Sec}(V_q(W))$ such that $H(x, 0) = \Theta(x)$, each q-tuple $H(x, 1)$ is exact, and each family $H(\cdot, t)$ has a holomorphic dependence on X.*

Furthermore, if $A \subset X$ is a closed complex analytic subset of X such that Θ is exact on A, then H may be chosen to be fixed on A, These theorems are very general, but give many geometrically interesting examples of h-principles as corollaries. Let $V = M \times \mathbb{C}$, where sections are just holomorphic functions, and let $W = T^*M$ be the complex cotangent bundle and D be the exterior derivative d. Theorem 1.1 then states that if $q > \dim(M)$, a q-tuple of holomorphic 1-forms generating the cotangent bundle can be deformed into a q-tuple of exact holomorphic 1-forms $(df_1, \ldots, df_q)$. By taking the map $(f_1, \ldots, f_q)$, we obtain an immersion $M \to \mathbb{C}^q$. The parametric version of this theorem gives a holomorphic analog of the classical Smale-Hirsch theorem.

Another example is the h-principle for holomorphic free maps. A map $f : M \to \mathbb{C}^q$ is said to be free if all of the first and second complex derivatives of f are linearly independent. Consider the bundle $W = J^2(M, \mathbb{C})/\mathbb{C}$, where we take second order jets and forget the constant term. Given a map $g : M \to \mathbb{C}$, there is an operator D that takes $j^2 g$ and forgets the value of g. Writing $f = (f_1, \ldots, f_q)$, a free map is exactly one for which the sections $Df_1, \ldots, Df_q$ generate W. Thus the main theorem applies, and gives an h-principle for holomorphic free maps of Stein manifolds.

A smooth analog of Theorem 1.1 was stated and proven in [3]. A similar result to Theorem 1.1 was stated in [4] with only a brief sketch of the proof and no technical details, we give the first proof here as well as the first statement and proof of the parametric h-principle in this setting. We will present necessary background material as well as tools needed in the proof in sections 2 and 3 of this thesis. In section 4, we discuss the holomorphic jet transversality theorem, and we adapt the transversality theorem to the specfic situations we need to prove the main theorem. In section 5.1 we give a detailed proof of Theorem 1.1, in 5.2 we will prove Theorem 1.2, as well as a proof of the 1-parametric h-principle, and in 5.3 we will discuss applications to holomorphic immersions and free maps of Stein manifolds into $\mathbb{C}^q$.

2 Jets and the h-principle

This section contains background material and language needed to discuss and prove h-principles. It will also include some examples and results, without proofs, that are precursors of the main results in the holomorphic setting. A good and accessible reference for the material in this section is the book [6] of Eliashberg and Mishachev. A more advanced reference is the monograph [9].

2.1 Jet Bundles and Transversality

2.1.1 Jet Bundles

We carry out the construction of jet bundles in the case of smooth fibrations. Later on we will need these in the holomorphic setting, but the construction goes through in exactly the same manner and we will not repeat it. Given a smooth map $f : \mathbb{R}^n \to \mathbb{R}^q$, its s-jet captures all derivatives up to order s. Let $f = (f_1, \ldots, f_q)$ and let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ range over all tuples such that $0 \leq |\alpha| = \sum \alpha_i \leq s$. Then $j^s f(x)$ is the tuple of all partial derivatives

$$\partial^\alpha f_i = \frac{\partial^{|\alpha|} f_i}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(x)$$

We can consider the manifold $J^s(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^{c(n,s,q)}$ to be the bundle of s-jets over $\mathbb{R}^n$, where $c(n, s, q)$ is the number of all partial derivatives up to order s. We can compute $c(n, s, q) = q\binom{s+n}{n}$ by a simple combinatorial argument. Notice that this is the dimension of the space $P^s(n, q)$ of polynomials from $\mathbb{R}^n \to \mathbb{R}^q$, and there is a parametrization

$$\mathbb{R}^n \times P^s(n, q) \cong J^s \mathbb{R}^n, \mathbb{R}^q) : (x, f) \to (x, j^s f(x))$$

In order to define the jet space of an arbitrary smooth fiber bundle, we will first want to give an invariant definition of s-jets. We say that two smooth functions f and g are s-tangent at x if $j^s f(x) = j^s g(x)$. By the chain rule in local coordinates, s-tangency is independent of the choice of coordinates. The jet space $J^s(\mathbb{R}^n, \mathbb{R}^q)$ is then the set of s-tangency classes of maps $\mathbb{R}^n \to \mathbb{R}^q$. Local coordinates on $\mathbb{R}^n$ and $\mathbb{R}^q$ give local coordinate charts on $J^s(\mathbb{R}^n, \mathbb{R}^q)$ which give it a natural topology and smooth structure so that the natural map $J^s(\mathbb{R}^n, \mathbb{R}^q) \to \mathbb{R}^n$ is a smooth fibration.

If $X \to M$ is any smooth fiber bundle, we can define the s-jet space $X^{(s)} \to M$ in the same way by defining s-tangency of local sections $M \to X$ in local coordinates. Given a section $\phi : M \to X$ we may define its s-jet extension $j^s \phi : M \to X^{(s)}$ to be the s-tangency class of ϕ. There are natural projections $X^{(s)} \to X^{(k)}$ for $k \leq s$, since s-tangency of a section implies k-tangency. These projections carry the natural structure of affine bundles. If X is a vector bundle, then the bundle $X^{(s)} \to M$ also carries a canonical vector bundle structure.

When studying sections of jet bundles, there is a distinguished class of sections that are the s-jet extensions of sections $X \to M$.

Definition 2.1. *A section $\Phi : M \to X^{(s)}$ is holonomic if there is a section $\varphi : M \to X$ such that $\Phi = j^s\varphi$.*

It may not be obvious from the definition of jet spaces, but holonomic sections of $X^{(s)}$ are quite special and are rare in the space of all sections. To see this consider the case $s = 1$. Locally a section of $X^{(1)}$ over M is given by a section ϕ of X together with a non vertical subspace of $T_{\phi(m)}X$ of dimension $\dim(M)$. This subspace is tangent to ϕ if and only if the section is holonomic, and evidently most sections are not holonomic.

2.1.2 Holonomic Splitting and Transversality

Despite the rarity of holonomic sections, locally $X^{(s)}$ may be foliated by holonomic sections. The key to this is the polynomial parametrization of $J^s(\mathbb{R}^n, \mathbb{R}^q)$. If X has fiber dimension q over a manifold of dimension M, then taking local trivializations allows us to identify small neighborhoods of $X^{(s)}$ with $J^s(\mathbb{R}^n, \mathbb{R}^q)$. This gives us the following lemma.

Lemma 2.1. *(Holonomic Splitting, see 1.7.1 of [6]). Let $f : M \to X^{(s)}$ be a holonomic sections, and $U \subset M$ be an open ball. Then there exists an embedding*

$$F : U \times \mathbb{R}^{c(n,q,s)} \to X^{(s)}$$

onto a neighborhood of f such that $F(m, 0) = f(m)$ and for all t the map $m \to F(m, t)$ is a holonomic section of $X^{(s)}$ over U.

This lemma is one of the main ingredients needed to prove the Thom transversality theorem, for which jet bundles are the natural setting. The other ingredient is Sard's theorem, which we recall below.

Theorem 2.1. *(Sard) Let $f : M \to N$ be a smooth map between manifolds. Then the set of critical values of f has measure zero in N.*

Definition 2.2. *A map $f : M \to N$ is said to be transversal to a submanifold W of N if for each $m \in M$ either $f(m) \notin W$ or $T_{f(m)}N$ is spanned by the image $df(T_mM)$ together with $T_{f(m)}W$. We will use $f(M) \pitchfork W$ to denote that f is transverse to W.*

By the implicit function theorem, if f is transverse to W then $f^{-1}(W)$ is a submanifold of the same codimension as W in N. Transversality is a ubiquitous phenomenon in geometry, and in many problems the key step is showing that a certain transversality condition can be achieved. The prototype for many such arguments is the Thom Transversality Theorem.

Theorem 2.2. *(Thom Transversality Theorem) Let $X \to M$ be a smooth fiber bundle, and let $\Sigma \subset X^{(s)}$ be a stratified subset. Then the set of smooth sections φ of X for which $j^s\varphi$ is transversal to Σ is residual in the space of all smooth sections. Here we use residual in the Baire sense.*

Proof. (Sketch) Let $B \subset M$ be a small compact ball, and let $T_B(\Sigma)$ to denote the set of sections whose s-jet extensions are transverse to Σ over B. Then $T_B(\Sigma)$ is open and dense in the space of all smooth sections. Openness is easy from the compactness of B, so density is the real issue. Let f be a section of X, and let $F : B \times \mathbb{R}^{c(n,q,s)} \to X^{(s)}$ be the holonomic tubular neighborhood of f guaranteed by the holonomic splitting theorem. By using bump functions we may assume that the sections $F(\cdot, t)$ are global holonomic sections of $X^{(s)}$. Since F is a submersion, $F^{-1}(\Sigma)$ is a submanifold, and the natural projection $F^{-1}(\Sigma) \to \mathbb{R}^{c(n,q,s)}$ has regular values arbitrarily close to the origin. But t is a regular value of this map if and only if $F(\cdot, t)$ is transversal to Σ, which gives density of $T_B(\Sigma)$ by Sard's Theorem since $F(\cdot, t)$ approximates f over any compact set as $t \to 0$. M admits a countable covering B_i by small compact balls, and the intersection of $T_{B_i}(M)$ is a residual in the space of all sections. This intersection consists of sections which are transverse to Σ over every B_i, and thus transverse to Σ everywhere. □

The transversality theorem is powerful and general. It shows, for instance, that the generic map $f : M^n \to \mathbb{R}^{2n}$ is an immersion, or that morse functions are generic in the space of smooth functions as f is Morse if df is transversal to the zero section of T^*M. A holomorphic analog of the Thom transversality theorem will be essential for the proof of the main theorem, and the only step that cannot be done holomorphically is extending the local sections of the holonomic splitting to global ones. This obstruction is impossible to overcome in general, but for certain bundles over Stein Manifolds (such as holomorphic vector bundles, or complex homogenous fiber bundles) we will be able to find a workaround.

2.2 Differential Relations and the h-principle

2.2.1 Differential Relations

Definition 2.3. *A differential relation of order s is a subset $\mathcal{R}$ of the jet space $X^{(s)}$*

Most differential relations don't have any geometric meaning, but there any many that do. Any partial differential equation, for instance, corresponds to a differential relation by replacing the derivatives with the corresponding coordinates on jet space and considering the submanifold of jet space cut out by the new equation.

A more pertinent example for us is the differential relation for immersions. If M and N are two manifolds, then maps $M \to N$ are sections of the trivial fiber bundle $M \times N$ over M. The jet space $(M \times N)^{(1)} = J^1(M, N)$ fibers over $M \times N$, with the fiber consisting of linear maps $T_mM \to T_nN$. The differential relation for immersions is the subset $\mathcal{R}_{\text{imm}}$ of $J^1(M, N)$ consisting of injective linear maps. A section of $J^1(M, N)$ over M lying in $\mathcal{R}_{\text{imm}}$ is a tangent bundle monomorphism $TM \to TN$, and an immersion is a holonomic section of $J^1(M, N)$ lying in $\mathcal{R}_{\text{imm}}$.

Definition 2.4. *Let $\mathcal{R} \subset X^{(s)}$ be a differential relation. By a formal solution of $\mathcal{R}$ we mean a section of $X^{(s)}$ lying in $\mathcal{R}$. We will denote the space of formal solutions by* $\text{Sec}(\mathcal{R})$*, topologized as a subspace of the space of sections of $X^{(s)}$. By a genuine solution of $\mathcal{R}$ we* *mean a holonomic section of $X^{(s)}$ lying in $\mathcal{R}$. We will denote the space of genuine solutions by* $\mathrm{Hol}(\mathcal{R})$*, topologized in the same way.*

Given a differential relation $\mathcal{R}$, we are usually interested in finding genuine solutions, and by themselves formal solutions are not of major interest. However, the existence of formal solutions is a necessary condition for the existence of genuine solutions, and finding formal solutions to a differential relation is usually a problem of an algebraic or topological nature. For example, finding a formal solution to the immersion relation $\mathcal{R}_{\text{imm}}$ boils down to the existence of sections of a certain homogenous fiber bundle over M (we will see this in more detail in the holomorphic setting at the end of section 3.2). Quite surprisingly, it turns out that for many interesting classes of differential relations, the existence of a formal solution is also sufficient for the existence of genuine solutions.

2.2.2 The h-principle

A differential relation $\mathcal{R}$ is said to satisfy the *h*-principle, or homotopy principle, if every formal solution of $\mathcal{R}$ is homotopic to a genuine one through solutions of $\mathcal{R}$. This is often the weakest statement that one tries to prove, and there are different kinds of h-principles. A differential relation is said to satisfy the 1-parametric *h*-principle if every path between genuine solutions in $\mathrm{Sec}(\mathcal{R})$ can be deformed through paths in $\mathrm{Sec}(\mathcal{R})$ into a path in $\mathrm{Hol}(\mathcal{R})$ such that the deformation is fixed on the endpoints of the path. More generally we can ask if the inclusion $\mathrm{Hol}(\mathcal{R}) \to \mathrm{Sec}(\mathcal{R})$ is a weak homotopy equivalence. There are many other kinds of *h*-principles one can look to prove, including versions involving approximation or parametric versions where the differential relation itself depends on some parameter. We refer the reader to Chapter 6 of [6] for more details about the *h*-principle as well as many examples of h-principles and some general techniques used to prove them.

Example 2.1. *The inclusion* $\mathrm{Hol}(\mathcal{R}_{\text{imm}}) \to \mathrm{Sec}(\mathcal{R}_{\text{imm}})$ *is a weak homotopy equivalence whenever* $\dim(N) > \dim(M)$*. (Equality can hold if M is an open manifold).*

We give an application to Smale's startling eversion of spheres. Let $i : S^2 \to \mathbb{R}^3$ be the standard embedding of S^2 as the unit sphere in $\mathbb{R}^3$, and let $f = -i$. Then f and i are homotopic through immersions. To prove this we apply the h-principle for $\mathcal{R}_{\text{imm}}$ and note it suffices to find a path between these sections in $\mathcal{R}_{\text{imm}}$. It is trivial to find a path between f and i since $\mathbb{R}^3$ is contractible, so we must find a path between df and di covering a path between f and i through bundle monomorphisms. To do so, note that df and di both give elements of $\pi_2(\mathrm{SO}(3))$, and by some elementary topology we have $\pi_2(SO(3)) = 0$. Indeed, $SO(3) \cong \mathbb{R}P^3$, and $\pi_2(\mathbb{R}P^3) = \pi_2(S^3) = 0$, which give us a homotopy from df to di in $SO(3)$ and thus through bundle monomorphisms since the tangent bundle to $\mathbb{R}^3$ is trivial.

The h-principle for immersions as applied here only gives an abstract regular homotopy between f and i, a quick youtube search for sphere eversion gives some stunning concrete examples of such a regular homotopy.

Strictly speaking, the statement of Theorem 1.1 does not fit into the framework of the *h*-principle as we have laid it out above, although the proof will reduce to finding holonomic solutions of a certain differential relation. That being said, the h-principle is more of a philosophy then a certain collections of theorems. We can sometimes, surprisingly, reduce hard geometric problems to homotopy theoretic problems which may be easy or hard, but in most cases easier than the original geometric problem. Theorem 1.1 shows, in essence, that finding holomorphic q-tuples of the form $D\Phi$ generating the fibers of W is equivalent to finding any holomorphic q-tuple with the same property. This theorem, when combined with Oka-Grauert Theory, reduces the original analytic problem to a purely topological one and gives an h-principle.

2.2.3 The Removal of Singularities

Philosophy aside, we now discuss some of the techniques used to prove h-principles. In [9], Gromov discusses several broad methods used to prove h-principles. We will focus on one of these techniques, presented in Chapter 2.1 of [9], known as the removal of singularities. We will adapt this method of proof for the proof of the main theorem. The original source for this idea is the paper [3], which proves the smooth precursor of Theorem 1.1.

The technique of the removal of singularities considers differential relations $\mathcal{R} = X^{(s)}/\Sigma$, which are the complements of a closed stratified singularity set Σ of positive codimension. The goal is to construct holonomic sections $j^s\phi$ so that $j^s\phi^{-1}(\Sigma)$ is empty, which can be done whenever Σ can in some sense be made transversal to certain subsets of the base manifold M.

More concretely, we can explain the basic idea behind the removal of singularities by discussing some ideas behind a proof of the h-principle for immersions $M \to \mathbb{R}^q$ for $q > \dim(M)$. A formal immersion into $\mathbb{R}^q$ is equivalent to the data of q 1-forms which generate the cotangent bundle T^*M. An immersion is equivalent to the data of exact 1-forms $df_1, \ldots, df_q$ with the same property. The idea is to construct the functions f_i one by one. We give the construction of df_1 avoiding many technical details, and leave out the induction. We will carry this out in detail in the holomorphic setting later on, and the details can be seen in the proof of 2.1.C of [9]. To replace θ_1 with df_1, we note that generically θ_1 is only needed to span T^*M on some positive codimension subset A of M. In order to replace θ_1 with df_1, we thus only need to control the value of df_1 along A. In particular, we need df_1 to miss the span of $\theta_2, \ldots, \theta_q$ along A, which we denote by Σ. We can consider a vector field V along A, defined by $\theta_1(V) = 1$, $\theta_i(V) = 0$ for $i > 1$ and extending V to all of M. Constructing f_1 is then equivalent to solving the equation $Vf_1 \neq 0$ along A, which turns out to be possible after obtaining certain transversality conditions of V to A. If V is tangent up to finite order to A then this is possible.

The basic strategy of the proof of Theorem 1.1 is similar to the above argument. We will proceed inductively and attempt to resolve some singularity set. Obtaining the relevant conditions will be more difficult and require new ideas and techniques. We will need to find versions of the transversality conditions that work holomorphically, obtain them without the ability to localize constructions using bump functions, and exploit such conditions to solve equations like $Vf_1 \neq 0$ in the setting of Stein manifolds.

3 Geometry of Several Complex Variables

This section contains background material from complex geometry needed to understand and prove the main results. Much of this material is fairly standard, and references include the books [12] and [10] of Hormander and Gunning-Rossi. The Oka-Grauert theory is a little more specialized, and the monograph [8] is a good reference. We assume familiarity with complex analysis as well as some differential geometry and topology.

3.1 Basics of Several Complex Variables

3.1.1 Holomorphic Functions

The theory of several complex variables begins with the local theory in $\mathbb{C}^n$. In single variable complex analysis one considers holomorphic functions on $\mathbb{C}$, which are C^1 functions $f : \mathbb{C} \to \mathbb{C}$ such that df commutes with multiplication by i. Equivalently, f is holomorphic if it satisfies the Cauchy-Riemann equations

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 2\frac{\partial f}{\partial \bar{z}} = 0$$

We can generalize this definition to $\mathbb{C}^n$ as follows

Definition 3.1. *A C^1 function $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic if f satisfies the Cauchy-Riemann equation with respect to each coordinate, i.e*

$$\frac{\partial f}{\partial \bar{z}_i} = 0, i = 1, \ldots, n$$

Being holomorphic is a local property, and hence we may talk about holomorphic functions on any open set U of $\mathbb{C}^n$. A function $U \to \mathbb{C}^m$ is holomorphic if each of the m coordinate functions is holomorphic. We will use $\mathcal{O}(U)$ to denote the ring of holomorphic functions on U.

Much of the theory of single-variable complex analysis generalizes to the case of several complex variables. Let f be a holomorphic function defined in a neighborhood of 0 and consider the polydisk $B(r_1, \ldots, r_n)$ which is the product of disks of radius r_i about the origin in each factor of $\mathbb{C}$. We can define $\partial_0 B(r_1, \ldots, r_n)$ to be the product of the boundaries of these disks (this is a proper subset of the boundary of the polydisc). Then applying the Cauchy Integral Formula inductively gives

$$f(0) = (2\pi)^{-n} \int_{\partial_0 B(r_1,\ldots,r_n)} \frac{f(w_1, \ldots, w_n) dw_1 \cdots dw_n}{\prod_{i=1}^n w_i}$$

Just as in the single-variable case, this implies that any holomorphic function is automatically C^∞ and in fact analytic. Similarly, uniform convergence of holomorphic functions implies that the limit (a priori only continuous) is holomorphic and that convergence occurs uniformly in all derivatives. The maximum principle also holds for the same reason.

There are, however, several new phenomena that occur in several complex variables that don't occur in the single variable case. The polydisk $B(r_1, \ldots, r_n)$ is not biholomorphic to the unit ball in $\mathbb{C}^n$ for $n \geq 2$, which is in stark contrast with the Riemann Mapping Theorem. Holomorphic functions of several complex variables also do not have isolated singularities. Any holomorphic function defined on the unit ball minus the origin can be extended over the origin, a special case of Hartog's theorem. Single variable holomorphic functions can have isolated poles and even essential singularities, and more generally any open set $D \subset \mathbb{C}$ has holomorphic functions which cannot be extended to any larger open set. It is natural to ask about the domains of holomorphic functions in $\mathbb{C}^n$, and this line of investigation proves extremely fruitful.

3.1.2 Domains of Holomorphy

Definition 3.2. *An open subset $D \subset \mathbb{C}^n$ is a called a domain of holomorphy if there does not exist a pair of open sets U, V where the following is true.*

(i) $U \subset (V \cap D)$

(ii) V *is connected and is not a subset of* D

(iii) For every holomorphic function f on D there is a holomorphic g on V such that $f|_U = g|_U$. Note that since V is connected such a g is uniquely determined by its value on U.

This definition is roughly capturing that it is impossible to extend all of the holomorphic functions on D past any part of its boundary. It turns out this is equivalent to having just one holomorphic function which may not be extended beyond any part of D.

Theorem 3.1. *D is a domain of holomorphy if and only if there exists a holomorphic function f on D such that there does not exist U and V satisfying (i) and (ii) of 3.2 and a holomorphic function g on V such that $g|_U = f|_U$.*

Proof. The if direction is automatic, the only if is given by Theorem 2.5.5 of [12]. □

We will now give several characterizations of domains of holomorphy in $\mathbb{C}^n$, first in terms of holomorphic convexity, and then in terms of plurisubharmonic functions and pseudoconvexity. Both of these are essential characterizations that will later generalize to Stein manifolds.

Definition 3.3. *Let D be an open set in $\mathbb{C}^n$, and $K \subset D$ be a compact subset. We say that K is holomorphically convex if for all $f \in \mathcal{O}(D)$ there is a point $p \in D$ such that $f(p) \geq \sup_{z \in K} f(z)$. We may define the holomorphically convex hull of K to be*

$$\hat{K}_{\mathcal{O}(D)} = \{w \in D : f(w) \leq \sup_{z \in K} f(z) \text{ for all } f \in \mathcal{O}(D)\}$$

We say that a domain D is holomorphically convex if $\hat{K}_{\mathcal{O}(D)}$ is compact for any compact $K \subset D$.

A second characterization of Domains of holomorphy is given by the following theorem

Theorem 3.2. *D is a domain of holomorphy if and only if it is holomorphically convex.*

Proof. See Theorem 2.5.5 of [12] □

This gives us several easy corollaries, and a way to prove that certain domains are domains of holomorphy. For example, any convex open set is holomorphically convex, and thus convex open subsets of $\mathbb{C}^n$ are domains of holomorphy.

3.1.3 Pseudoconvexity

We will give one more characterization of domains of holomorphy in $\mathbb{C}^n$, but this first requires a discussison of plurisubharmonic functions and pseudoconvexity. Part of the reason this characterization is important (and in fact one way to prove this characterization) is that the $\bar{\partial}$ operator has a rich analytic theory for pseudoconvex domains which carries over to Stein manifolds. This allows one to establish the powerful tools discussed in sections 3.2 and 3.3. We will not discuss this theory as we only need some of the resulting applications for the proof of the main theorem, but we refer the reader to Chapters 4 and 5 of [12] for a more thorough discussion.

Definition 3.4. *An upper-semi continuous function $f : D \to \mathbb{R}$ is plurisubharmonic if for every holomorphic function $g : \mathbb{D} \to D, f \circ g$ is a subharmonic function on $\mathbb{D}$. Here $\mathbb{D} \subset \mathbb{C}$ denote the unit disk.*

Example 3.1.

(i) $\log |f|$ *for a holomorphic function f is an example of a plurisubharmonic function*

(ii) $|f|^\alpha$ *for f holomorphic and $\alpha > 0$ is another example of a plurisubharmonic function.*

Plurisubharmonic functions satisfy the maximum principle, and plurisubharmonicity is a local property. Moreover, plurisubharmonic functions are preserved by holomorphic maps, i.e. if g is holomorphic then $f \circ g$ remains plurisubharmonic. This will allow us to define plurisubharmonic functions on complex manifolds.

There are various other characterizations of plurisubharmonic functions, but we give one in terms of the Levi form. Here we take f to be any C^2 function from $\mathbb{C}^n \to \mathbb{R}$. Associated to the complex Hessian of f is a Hermitian quadratic form called the Levi form defined by

$$\mathcal{L}_{f,z}(v) = \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}(z) v_j \bar{v}_k,$$

We are taking $v \in \mathbb{C}^n$ to be a tangent vector at z. A function f is plurisubharmonic if and only if the Levi form is nonnegative definite at every point. We say that a C^2 function f is strictly plurisubharmonic if the Levi form is positive definite at every point. The Levi form can be defined invariantly, and thus be defined on complex manifolds, which will allow us to define strongly plurisubharmonic functions on complex manifolds.

Example 3.2. *If $(f_1, \dots, f_m)$ are holomorphic functions, then $\sum |f_i|^2$ is a plurisubharmonic function, and it is strongly plurisubharmonic whenever the collection df_i span the cotangent space. If M is a complex manifold and $(f_1, \dots, f_m) : M \to \mathbb{C}^m$ is a holomorphic immersion, then $g = \sum |f_i|^2$ is a strongly plurisubharmonic function. If this map is an embedding than g is a strongly plurisubharmonic exhaustion function.*

Definition 3.5. *Given a domain $D \subset \mathbb{C}^n$ and a compact $K \subset D$, we may define its plurisubharmonic convex hull by*

$$\hat{K}_{\mathrm{psh}(D)} = \{w \in D : f(w) \leq \sup_K f \text{ for all plurisubharmonic } f\}$$

D is said to be pseudoconvex if $\hat{K}_{\mathrm{psh}(D)}$ is compact whenever K is compact.

There is much more to be said about pseudoconvexity and plurisubharmonicity, but we will end with the following major theorem.

Theorem 3.3. *Let $D \subset \mathbb{C}^n$ be an open set, then the following are equivalent*

(i) D is a domain of holomorphy

(ii) D is holomorphically convex

(iii) D is pseudoconvex

(iv) D admits a plurisubharmonic exhaustion function.

Proof. See Theorems 2.6.5, 2.6.7 and 4.2.8 of [12]. □

3.2 Stein Manifolds

3.2.1 Complex Manifolds

Having outlined parts of the local theory of several complex variables for domains in $\mathbb{C}^n$, we now generalize to the case of complex manifolds.

Definition 3.6. *Let M be a smooth manifold of dimension $2n$. An atlas (U_α, ϕ_α) is holomorphic if the transititon maps are holomorphic between open subsets of $\mathbb{R}^{2n} \cong \mathbb{C}^n$. A complex manifold of dimension n is a smooth manifold of dimension $2n$ equipped with a maximal holomorphic atlas.*

A function $f : M \to N$ between complex manifolds is holomorphic if it is holomorphic in local charts. We may also define holomorphic fiber bundles and holomorphic vector bundles in the same way as in smooth topology, and leave out the details for which we refer the reader to [12] or chapter 1 of [8]. Likewise the tangent and cotangent bundles of a complex manifold become holomorphic vector bundles, and we can define the holomorphic jet bundles of holomorphic fiber bundles in the same way as in section 2. Complex manifolds of dimension 1 will be called Riemann Surfaces.

Example 3.3. *We give some examples of complex manifolds.*

(i) $\mathbb{C}^n$ *and open subsets of* $\mathbb{C}^n$.

(ii) $\mathbb{C}$, $\mathbb{C}P^1$ *and* $\mathbb{D}$ *(the unit disk) are the unique simply connected Riemann surfaces. Note that* $\mathbb{D}$ *and* $\mathbb{C}$ *are diffeomorphic but not biholomorphic.*

(iii) A complex manifold G *that has holomorphic group operations is a complex Lie group. Prominent examples include* $\mathrm{GL}_n(\mathbb{C})$ *and* $\mathrm{SL}_n(\mathbb{C})$.

(iv) A complex manifold X *that admits a transitive holomorphic action by a complex Lie group is a homogenous complex manifold. Of particular interest to us will be the complex Stiefel manifold* $V_{k,n}(\mathbb{C})$ *with* $k \leq n$. *The Stiefel manifold* $V_{k,n}$ *consist of all complex* $k \times n$ *matrices of maximal rank, which is an open subset of* $\mathbb{C}^{kn}$. $V_{k,n}$ *admits a transitive* $\mathrm{GL}_n(\mathbb{C})$ *action given by right multiplication, and can also be described as all* n*-tuples of vectors which span* $\mathbb{C}^k$

During the proof of the main theorem, we will need to work with subsets of complex manifolds more general than complex submanifolds.

Definition 3.7. *A subset* Σ *of a complex manifold* M *is a complex analytic subset if every point* $p \in \Sigma$ *has a neighborhood* $U \subset M$, *such that there are finitely many holomorphic functions* $f_1, \ldots, f_m$ *defined on* U *such that*

$$\Sigma \cap U = \bigcap_{i=1}^{m} \{f_i = 0\}$$

All the notions from section 3.1 generalize immediately to complex manifolds. We may define holomorphically convex sets, the holomorphically convex hull of a set and plurisubharmonic functions on complex manifolds. With this in hand, we are now prepared to discuss Stein manifolds.

3.2.2 Stein Manifolds

Definition 3.8. *A Stein Manifold* M *is a complex manifold satisfying the following additional assumptions*

(i) Holomorphic functions separate points, i.e. for distinct $x, y \in M$ *there is a global holomorphic function* f *such that* $f(x) \neq f(y)$.

(ii) Every neighborhood has a coordinate neighborhood given by holomorphic functions $f_1, \ldots, f_n$ *which extend to global holomorphic functions on* M.

(iii) M *is holomorphically convex, i.e. if* K *is compact, the holomorphic convex hull* $\hat{K}_{\mathcal{O}}$ *is also compact.*

Example 3.4.

(i) Domains of holomorphy in $\mathbb{C}^n$ are Stein.

(ii) Open Riemann surfaces are Stein.

(iii) The product of two Stein manifolds is Stein.

Stein manifolds are a generalization of domains of holomorphy in $\mathbb{C}^n$, and have many holomorphic functions as is clear from the definition. Another way in which Stein manifolds have many holomorphic functions is given more quantitatively by the following theorem.

Theorem 3.4. *Let K be a holomorphically convex compact subset of a Stein manifold and f a holomorphic function defined in a neighborhood of K. Then there are global holomorphic functions g which approximate f uniformly on K.*

Proof. See Chapter 5 of [12]. □

Theorems like this, which will generalize to coherent analytic sheaves in section 3.3, are the main tools for working with Stein manifolds and are extremely powerful. We give a characterization similar to that of Theorem 3.3.

Theorem 3.5. *The following are equivalent*

(i) M is Stein

(ii) M can be properly holomorphically embedded in some $\mathbb{C}^q$.

(iii) M admits an exhausting strongly plurisubharmonic function

Proof. See Chapter 5 of [12]

The directions (b) implies (c) and (b) implies (a) are both relatively easy. That (a) implies (b) is a fairly direct construction using properties of analytic polyhedra to construct proper holomorphic maps and then use the abundance of holomorphic maps on a Stein manifold to create an embedding. That (c) implies (a) is the last remaining direction, and is a deep theorem with a hard proof. □

Corollary 3.1. *Stein Manifolds of complex dimension n, and hence real dimension $2n$, have the homotopy type of an n-dimensional CW-complex.*

Proof. We may assume that M has a proper plurisubharmonic Morse function ϕ. Critical points of plurisubharmonic Morse functions must have Morse Index less or equal to n (in order for the Levi form to be positive definite), and hence standard Morse theory gives the result. □

We will end our brief discussion of Stein manifolds with a discussion of Oka-Grauert theory, following Chapter 5 of [8]. When combined with Corollary 3.1 this will give some concrete applications of the Main theorem to holomorphic immersions of Stein manifolds.

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3.2.3 Oka-Grauert Theory

Oka-Grauert Theory is in some sense the story of holomorphic flexibility properties of Stein manifolds and has two components, a cohomological theory and a homotopy theoretic one. Part of the cohomological story will be discussed in section 3.3 with Cartan's Theorem B and has been formulated as the heurestic that analytic problems on Stein manifolds which can be cohomologically formulated have only topological obstructions. An example is the second Cousin problem, which looks to find meromorphic functions with prescribed zeros and poles. On Stein manifolds, if this problem is solvable for continuous functions it is also solvable for holomorphic functions. This is to say that imposing the holomorphicity condition adds no extra constraint.

We will focus on the homotopy theoretic aspects of Oka-Grauert theory. The idea is to study complex manifolds Y for which all continuous maps from a Stein manifold M are homotopic to holomorphic ones. A stronger formulation is to ask for the inclusion of holomorphic maps $\mathcal{O}(M, Y)$ into the space of continuous maps to be a weak homotopy equivalence. These manifolds are known as Oka manifolds, and the contain complex homogenous manifolds as an important subclass. The main theorem that we will use is as follows.

Theorem 3.6. *Let M be a Stein manifold and let $\pi : X \to M$ be a holomorphic fiber bundle with complex homogenous fibers and structure group a complex Lie group acting transitively on the fibers. Then the inclusion of holomorphic sections of X into the space of continuous sections of X is a weak homotopy equivalence.*

Proof. See Theorem 5.3.2 of [8]. □

For the purpose of applying Oka-Grauert, we note that Theorem 3.6 implies that under the conditions of the theorem, any continuous section is homotopic to a holomorphic one. Let W be a holomorphic vector bundle over a Stein manifold of rank m. Let $q \geq m$. Then associated to W is the Stiefel bundle $V_q(W)$, which is a complex homogenous fiber bundle with structure group $\mathrm{GL}_q(\mathbb{C})$. This is the bundle of q-tuples of vectors spanning W. One construction is given by taking the vector bundle $\mathrm{Hom}(W, \mathbb{C}^q)$ of vector bundle maps from W to $M \times \mathbb{C}^q$, and then taking the sub-bundle given by taking in each fiber the injective maps.

In Theorem 1.1 we start with a tuple $(\theta_1, \ldots, \theta_q)$ of global holomorphic sections of W that span the fiber of W at every point. Such a tuple is exactly given by a holomorphic section of $V_q(W)$. To find such a holomorphic section, Theorem 3.6 tells us it suffices to find a continuous section.

Lemma 3.1. *$V_{q,m}(\mathbb{C})$ is $(2q-2m)$-connected, i.e. $\pi_i(V_{q,m}(\mathbb{C}) = 0$ for $i \leq 2q-2m$.*

Proof. $V_{q,m}(\mathbb{C})$ can be presented as $\mathbb{C}^{qm}/\Sigma$ where Σ is a closed complex analytic subset of complex codimension $q-m+1$, or real codimension $2q-2m+2$. Σ consists of those $q \times m$ matrices of less than maximal rank. By the jet transversality theorem, if $i \leq 2q-2m$, any homotopy between maps $S^i \to \mathbb{C}^{nq}$ can be made to miss Σ, which shows that $\pi_i(\mathbb{C}^{qm}/\Sigma) = 0$ since $\mathbb{C}^{qm}$ is contractible. □

If M is a Stein manifold of complex dimension n, then corollary 3.1 gives M the structure of a CW-complex with cells of degree less than or equal to n. By inductively building over the skeleton of M, we may construct a continuous section of $V_q(W)$ whenever $2q-2m \geq n$. The way this works is that every time we add a cell to M, we may trivialize the bundle $V_q(W)$ over the cell and extend the section from the boundary of the cell since the obstruction to doing so is in $\pi_i(V_{q,n}(\mathbb{C}))$, which vanishes.

Taken together, we have the following corollary.

Corollary 3.2. *Let W be a holomorphic vector bundle of rank m over a Stein manifold M of complex dimension n. Then if $2q-2m \geq n$, we may find a tuple $(\theta_1, \ldots, \theta_q)$ of holomorphic sections of W that span the fiber of W at every point of M.*

3.3 Cartan's Theorems

3.3.1 Coherent Analytic Sheaves

In the theory of Stein manifolds, coherent analytic sheaves play a central role. In some ways they are a natural generalization of holomorphic vector bundles. The sheaf of sections of any holomorphic vector bundle gives an example of a coherent analytic sheaf, and the category of coherent analytic sheaves is an enlargement of the category of holomorphic vector bundles that makes it into an abelian category. For more details and a more thorough presentation of the material in this section, we refer the reader to Chapter 7 of [12], which also contains a brief presentation of the basic notion of a sheaf. We will use $\mathcal{O}_X$ to denote the structure sheaf of holomorphic functions on a complex manifold X.

Definition 3.9. *A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is locally finitely generated if every point p has a neighborhood U_p, and sections $f_1, \ldots, f_q$ over U_p such that the stalk $\mathcal{F}_x$ is generated by the germs of the f_i over $\mathcal{O}_{X,x}$ at every point $x \in U_p$.*

Given an analytic sheaf $\mathcal{F}$ (by which we mean a sheaf of $\mathcal{O}_X$-modules), an open set U and sections $f_1, \ldots, f_q \in \Gamma(U, \mathcal{F})$. We may define a sheaf of relations $\mathcal{K}(f_1, \ldots, f_q) \subset \mathcal{O}^q$ over U by q-tuples $(a_1, \ldots, a_q)$ of holomorphic functions over open subsets V of U such that $\sum a_i f_i = 0$ on V.

Definition 3.10. *An analytic sheaf $\mathcal{F}$ on X is called coherent if $\mathcal{F}$ is locally finitely generated and every sheaf of relations is locally finitely generated.*

Example 3.5.

(i) The structure sheaf $\mathcal{O}_X$ of a complex manifold is coherent analytic. This is a hard theorem due to Oka (See Theorem 6.4.1 of [12]). More generally for all n, $\mathcal{O}_X^n$ is coherent analytic and any locally finitely generated subsheaf of $\mathcal{O}_X^n$ is coherent.

(ii) Since locally the sheaf of holomorphic sections of a holomorphic vector bundle looks like $\mathcal{O}_X^n$, all of these sheaves are coherent.

(iii) *Let Σ be a closed analytic subset, then the ideal sheaf I_Σ of holomorphic functions vanishing on Σ is coherent analytic.*

(iv) *More generally, we can consider the sheaf of sections of a holomorphic vector bundle vanishing to a certain order along a complex analytic set. This locally looks like a locally finitely generated subsheaf of $\mathcal{O}_X^n$ and is therefore coherent.*

The theory of coherent analytic sheaves on Stein manifolds is especially nice. They admit many global sections in a precise way, given by the following useful and powerful theorems.

Theorem 3.7. *(Cartan's Theorem A) Let M be a Stein manifold and $\mathcal{F}$ a coherent analytic sheaf on M. Then at every point x in M the stalk $\mathcal{F}_x$ is generated as an $\mathcal{O}_{X,x}$-module by germs of global sections of $\mathcal{F}$.*

Proof. See Theorem 7.2.8 of [12] □

Theorem 3.8. *Let M be a Stein manifold and $\mathcal{F}$ be a coherent analytic sheaf on M. Let $f, f_1, \dots, f_q$ be global sections of M such that at every point $p \in M$, the germ f_p is in the $\mathcal{O}_{X,p}$ submodule generated by the $f_{i,p}$. Then there are global holomorphic functions $a_1, \dots, a_q$ such that*

$$f = \sum a_i f_i$$

Proof. See Theorem 7.2.9 of [12]. □

Theorem 3.9. *Let $\mathcal{F}$ be a coherent analytic sheaf over a Stein manifold M and let K be a holomorphically convex compact subset of M. Let f be a section of $\mathcal{F}$ in a neighborhood of K. Then there are global sections of $\mathcal{F}$ which uniformly approximate f on K.*

Proof. See Theorem 7.2.7 of [12] □

The above theorems are the main tools, along with Cartan's Theorem B, that we will use for working with sections of coherent analytic sheaves to prove Theorem 1.1.

3.3.2 Cartan's Theorem B

The last major theorem from the theory of coherent analytic sheaves over a Stein manifold that we will discuss is Cartan's Theorem B. This first requires the development of the theory of sheaf cohomology. We refer the reader to sections 7.3 and 7.4 of [12] for more details. For the subsequent discussion we will use X to denote an arbitrary topological space, and $\mathcal{F}$ a sheaf of abelian groups on X.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of X. For any tuple $a = (a_1, \dots, a_n) \in A^n$, we denote $U^a = \cap_{i=1}^n U_{a_i}$. An $(n-1)$-cochain is an assignment f giving to each $a \in A^n$ a section $f(a)$ of $\mathcal{F}$ over U^a that is alternating. By alternating we mean that if a' is obtained from a by swapping two elements, then $f(a') = -f(a)$. We denote the set of all n-cochains by $C^n(\mathcal{U}, \mathcal{F})$. $C^n(\mathcal{U}, \mathcal{F})$ naturally inherits a group structure from $\mathcal{F}$, as we can always add sections over open sets.

There is a coboundary operator $\delta : C^n(\mathcal{U}, \mathcal{F}) \to C^{n+1}(\mathcal{U}, \mathcal{F})$ as follows. Let $a' = (a_1, \ldots, a_{n+2}) \in A^{n+2}$. Denote by a'_i the corresponding $n+1$ tuple with a_i-removed. Then we may define δ by

$$\delta f(a') = \sum_{i=1}^{n+2} (-1)^i f(a'_i)$$

Implicit in the definition of δ are the natural restriction maps from $U^{a'_i}$ to $U^{a'}$. Readjusting so we have $a' \in A^{n+3}$, and using a'_{ij} to denote the corresponding $n+1$-tuple with distinct indices i and j deleted, we can calculate

$$\delta^2 f(a') = \sum_{i=1}^{n+3} (-1)^i \delta f(a'_i) = \sum_{i=1}^{n+3} \left(\sum_{j=1}^{i-1} (-1)^{i+j} f(a'_{i,j}) + \sum_{j=i+1}^{n+3} (-1)^{i+j-1} f(a'_{i,j}) \right)$$

If $i \neq j$, this sum hits each $f(a'_{i,j})$ twice with opposite signs. If $j < i$, deleting i and then j carries a sign of $(-1)^{i+j}$, whereas deleting j and then i carries a sign of $(-1)^{i+j-1}$. Thus $\delta^2 = 0$, and we can define cohomology of the resulting complex. Let $Z^n(\mathcal{U}, \mathcal{F}) = \ker(\delta) \subset C^n$ be the group of n-cocycles, and $B^n(\mathcal{U}, \mathcal{F}) = \mathrm{Im}(\delta) \subset C^n$ be the group of n-coboundaries. Since $\delta^2 = 0$, we have that $B^n \subset Z^n$ and thus may define the cohomology group

$$H^n(\mathcal{U}, \mathcal{F}) = Z^n(\mathcal{U}, \mathcal{F}) / B^n(\mathcal{U}, \mathcal{F})$$

Notice that H^0 is the same as the group of global sections of $\mathcal{F}$. A 0-cochain is a choice $f_\alpha \in \Gamma(U_\alpha, \mathcal{F})$ for each $\alpha \in A$. A cochain is a 0-cocycle if $f_\alpha - f_\beta = 0$ on $U_\alpha \cap U_\beta$, which is precisely the data of a global section of a sheaf $\mathcal{F}$ since the U_α cover X.

If $\mathcal{V}$ is a refinement of the cover $\mathcal{U}$, the restriction of cochains from U to V commutes with the coboundary operators and thus induces a map $H^n(\mathcal{U}, \mathcal{F}) \to H^n(\mathcal{V}, \mathcal{F})$. Taking the colimit over covers gives rise to the cohomology groups $H^n(X, \mathcal{F})$. Homomorphisms between sheaves of abelian groups induce maps on cohomology in the obvious way (by mapping cocycles to cocycles).

For paracompact spaces, short exact sequences of sheaves induce long exact sequences on cohomology, more precisely we have the following theorem.

Theorem 3.10. *If X is paracompact, and $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves of abelian groups, then there is an exact sequence*

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F}) \to \cdots$$

The maps between cohomology groups of the same degree are the obvious ones, and one can define a map $H^i(X, \mathcal{H}) \to H^{i+1}(X, \mathcal{F})$ in a natural way using the snake lemma.

Proof. See Theorem 7.3.7 of [12]. ☐

The main goal of developing the cohomology of sheaves is for the following powerful theorem.

Theorem 3.11. *(Cartan's Theorem B). Let $\mathcal{F}$ be a coherent analytic sheaf on a Stein Manifold M, then for $n > 0$ we have that $H^n(M, \mathcal{F}) = 0$.*

Proof. See Theorem 7.4.3 of [12] ☐

Corollary 3.3. *We can extend sections of holomorphic vector bundles defined on closed complex analytic subsets of Stein Manifold.*

Proof. Let $\mathcal{A}_V$ be the sheaf of sections of a holomorphic vector bundle, and let $\mathcal{A}_{V,\Sigma}$ be the push forward of the sheaf of sections of $V|_\Sigma$, where Σ is a closed complex analytic subset. Let $I_{V,\Sigma}$ be the sheaf of sections of V vanishing on Σ. Then there is an exact sequence of coherent analytic sheaves

$$0 \to I_{V,\Sigma} \to \mathcal{A}_V \to \mathcal{A}_{V,\Sigma} \to 0$$

By the long exact sequence on cohomology and Cartan's Theorem B, there is a surjection $H^0(\mathcal{A}_V) \to H^0(\mathcal{A}_{V,\Sigma}) \to 0$. This says that every global section of the latter is the restriction of a global section of the former, i.e. that every section defined on Σ extends to a global section.

☐

4 Holomorphic Transversality

Perhaps unsurprisingly, a version of jet transversality hold for Stein manifolds and certain bundles over them. We give a proof here of the basic jet transversality theorem following Forsternic's proof in [7], before adapting it to several cases that we will use to prove the main result. The main result we will use prove Theorem 1.1 is Theorem 4.4

Lemma 4.1. *Let X and Y be complex manifolds, then with the topology induced by uniform convergence on compact sets, $\mathcal{O}(X,Y)$ is a Baire Space.*

Proof. We first recall that every manifold embeds as a closed subset of $\mathbb{R}^n$. The restriction of any complete metric is complete, so every manifold can be a made into a complete metric space. Now exhaust X by an increasing sequence $K_1 \subset K_2 \subset \cdots$ of compact sets exhausting X, and let $d_{K_i}(f,g)$ by the sup metric on K_i where Y is equipped with a complete metric. We can consider the metric on $\mathcal{O}(X,Y)$ given by

$$d(f,g) = \sum_{i=1}^{\infty} 2^{-i} \frac{d_{K_i}(f,g)}{1 + d_{K_i}(f,g)}$$

A sequence is Cauchy in this metric if and only if it is Cauchy in each of the d_{K_i}, which since Y is a complete metric space implies uniform convergence over each K_i. Hence $\mathcal{O}(X,Y)$ is a complete metric space and thus a Baire space. □

Definition 4.1. *Let X and Y be complex manifolds. Holomorphic maps satisfy the condition* Ell_1 *if for every holomorphic map $f : X \to Y$ there is a holomorphic map $F : X \times \mathbb{C}^N \to Y$ such that $F(x,0) = f(x)$ and $F(x,\cdot) : \mathbb{C}^N \to Y$ has rank equal to* $\dim(Y)$ *at* 0.

This condition may seem obscure, but in fact is satisfied by many manifolds we care about in a natural way. Complex Lie groups and more generally complex homogeneous manifolds satisfy the condition Ell_1 for maps from any complex manifold by using the exponential map.

Theorem 4.1. *Let X be a Stein Manifold and Y be a complex manifold such that maps from X to Y satisfy the condition* Ell_1. *Then holomorphic maps from X to Y satisfy jet transversality with respect to stratified complex submanifolds $\Sigma \subset J^k(X,Y)$, i.e. the space of of maps with k-jets transverse to Σ is residual in the space of all holomorphic maps from X to Y.*

The basic idea is to 'perturb' maps from $X \to Y$ by polynomials in order to get sufficient variation of the k-jets. Once this happens we may use some standard applications of Sard's theorem to obtain transversality. In this case the key property of Stein manifolds that we need is that they embed in $\mathbb{C}^q$.

Proof. Let $K \subset X$ be a compact set. Let $T_K(X, Y)$ denote the space of holomorphic maps $X \to Y$ whose k-jet is transverse to Σ over K. Note that $T_K(X, Y)$ is open in $\mathcal{H}(X, Y)$ since K is compact, we will show that it is dense.

We first analyze the basic case to illustrate some of the main ideas, i.e. transversality for 0-jets when $\Sigma \subset Y$. I claim that $T_K(X, Y)$ is dense in $\mathcal{H}(X, Y)$. Let $f \in \mathcal{H}(X, Y)$ be any holomorphic map. Then there is a map $F : X \times \mathbb{C}^N \to Y$ satisfying the condition Ell_1. The condition Ell_1 implies that there is a small open U' around the origin in $\mathbb{C}^N$ and U around K so that F is a submersion on $U \times U'$ (obtaining U' is where we crucially use compactness of K). Since F is a submersion, $F^{-1}(\Sigma)$ is a submanifold of $U \times U'$. Let t be a regular value of the projection $F^{-1}(\Sigma) \to U'$. By Sard's Theorem we may take t arbitrarily close to 0. Then the map $X \to Y$ given by $x \to F(x, t)$ is holomorphic and transverse to B over K. As $t \to 0$, the sections $F(\cdot, t)$ approximate f over any compact set, which gives density of $T_K(X, Y)$ in $\mathcal{H}(X, Y)$. Let K_i be a compact exhaustion of X, then the intersection $T_{K_i}(X, Y)$ is precisely the set $T(X, Y)$ of holomorphic maps transverse to Σ, but this is a countable intersection of open-dense subsets, hence is residual since $\mathcal{H}(X, Y)$ is Baire.

We now handle the case $k > 0$. Take $f : X \to Y$ and again let $F : X \times \mathbb{C}^N \to Y$ be the map guaranteed by the condition Ell_1. Since X is stein, we may embed X in euclidean space $\mathbb{C}^q$ and let P_k denote the finite dimensional vector space of polynomials $\mathbb{C}^q \to \mathbb{C}^N$ with degree less than or equal to k. There is a map $X \times P_k \to Y$ given by $(x, p) \to F(x, p(x))$.

Lemma 4.2. *The map $X \times P_k \to Y$ induces a map to $J^k(X, Y)$ by taking k-jets. The resulting map $X \times P_k \to J^k(X, Y)$ is a submersion in a neighborhood of* 0.

Proof. See Lemma 4.5 of [7].

In the proof of the smooth jet transversality theorem, we needed to use the fact that we could extend the sections of a local holonomic splitting to global sections using bump functions. In the holomorphic case we cannot do this, but this gives us an analogous result that works the same way. □

This allows us to finish the argument. By compactness of K there are open neighborhoods $K \subset U$ and $U' \subset \{0\}$ of P_k such that this map to J^k is a submersion on $U \times U'$. By precisely the same argument as in the $k = 0$ case, we can pull back a submanifold B of J^k and find polynomials arbitrarily close to 0 giving transversality over K by Sard's theorem. We note that the sections $X \times P_k \to J^k(X, Y)$ given by fixing some polynomial p are all holonomic by construction. Taking another countable intersection over compact sets covering X gives the result. □

The main reason for handling the case basic case separately is because the argument is simpler, and we note that it holds even when X is not Stein. We perturbed our maps by polynomials to get at the higher jets, which fails when X is not Stein. With this basic transversality theorem in hand, we can now easily obtain transversality in a few other interesting case.

Theorem 4.2. *Let $\pi : Z \to X$ be a holomorphic fiber bundle, such that for any section $s : X \to Z$ there is a map $F : X \times \mathbb{C}^N \to Z$ satisfying the following properties*

(i) $F(x, 0) = s(x)$.

(ii) For each $t \in \mathbb{C}^N$, we have that $\pi(F(m, t)) = m$, i.e. that each $F(\cdot, t)$ is a holomorphic section of Z.

(iii) For each fixed m, the map $F(m, \cdot) \to Z_m$ is a submersion at the origin of $\mathbb{C}^N$.

Then sections of $X \to Z$ satisfy jet transversality.

Proof. The proof is the same as the proof of Theorem 4.1 up to small changes in notation. □

We recall the following corollary of Oka-Grauert Theory.

Lemma 4.3. *Let E be a holomorphic vector bundle over a Stein Manifold M, there are finitely many sections $s_1, \ldots, s_q$ of E such that at every point $s_1(p), \ldots, s_q(p)$ span the fiber E_p.*

Given $s : M \to E$, we obtain a map F given by $M \times \mathbb{C}^q \to E$, by $F(m, t_1, \ldots, t_q) = s + \sum t_j s_j$. This gives jet transversality for any holomorphic vector bundle after applying Theorem 4.2.

There are many more specific holomorphic jet transversality theorems that one can prove, following the general principal that if we can perturb maps by polynomials, then we can obtain jet transversality. For instance we can specify the vanishing locus of section.

Theorem 4.3. *Let Σ be a proper complex analytic subset of a Stein Manifold. Then functions vanishing on Σ satisfy jet-transversality theorem over X/Σ. More precisely let B be a stratified complex submanifold of $J^k(X, Y)$ lying over X/Σ, then the set of functions vanishing on Σ whose k-jet extension is transverse to B is residual in the set of all functions vanishing on Σ.*

Proof. We first prove a preliminary lemma.

Lemma 4.4. *Σ is cut out by finitely many global holomorphic functions.*

Proof. By coherence of the sheaf I_Σ, we know that Σ is locally cut out by finitely many global sections of I_Σ by Cartan's Theorem A. This means we have a (possibly infinite family) of global holomorphic functions $\{f_\alpha\}$ which cut out Σ. Proposition 5.7 of [1] then gives us finitely many global holomorphic functions cutting out Σ. □

Σ is the zero locus of finitely many holomorphic maps $f_1, \ldots, f_n$ and let f be any function vanishing on Σ. There is a map $F : X \times \mathbb{C}^n \to \mathbb{C}$ defined by $F(x, t_1, \ldots, t_n) = f(x) + \sum t_i f_i(x)$. Then for fixed $(t_1, \ldots, t_n)$, the map $F(\cdot, t_1, \ldots, t_n)$ is a holomorphic function vanishing on Σ, and since the f_i cut out X/Σ, for each $x \in \Sigma$, $F(x, \cdot)$ is a submersion $\mathbb{C}^n \to \mathbb{C}$ at the origin. With this setup in hand, the proof of the jet transversality theorem works verbatim over X/Σ, as all conditions are satisfied. □

Applying the same argument for holomorphic vector bundles where we may specify both a locus and order of vanishing gives rise to the following theorem, which we will use repeatedly to obtain necessary transversality conditions in the proof of the main theorem.

Theorem 4.4. *Let V be a holomorphic vector bundle over a Stein Manifold M, and let Σ be a closed complex analytic subset of M, and $s \geq 0$ a non-negative integer. Let B be a stratified complex submanifold of $J^k(V)$ defined over M/Σ. Then generic sections of V vanishing to order s along Σ have k-jet extension transverse to B.*

Proof. There exist finitely many sections $v_1, \ldots, v_q$ of V which span the fiber at every point of M/Σ and vanish to order s along Σ. (For instance by multiplying sections generating V by functions which vanish to order s along Σ). Let v vanish to order s along Σ. Then there is a map $F : M \times \mathbb{C}^q \to V$ defined by $F(x, t_1, \ldots, t_q) = v(x) + \sum t_i v_i(x)$. This map satisfies all the properties of Theorem 4.2, and additionally for any fixed $t \in \mathbb{C}^q$, the section $F(\cdot, t)$ of V vanishes to order s along Σ. Thus the argument of theorem 4.1 applies, all that changes is some small notation to denote that we are working in the space of sections vanishing along Σ to order s. $\square$

5 Main Results

We first recall Theorem 1.1

Theorem 5.1. *Let M be a Stein-manifold, and let V and W be holomorphic vector bundles over M. Let $D : \mathcal{A}(V) \to \mathcal{A}(W)$ be an elliptic differential operator of order s from the holomorphic sections of V to the holomorphic sections of W. Let $\theta_1, \dots, \theta_q$ be holomorphic sections of W that span the fibers at every point so that q is greater than the rank of W, then the q-tuple $(\theta_1, \dots, \theta_q)$ is homotopic through q-tuples generating W to a q-tuple of exact holomorphic sections $D\phi_1, \dots, D\phi_q$ with the same spanning property.*

We mean D is elliptic if we can write $D = \Delta \circ j^s$, where $\Delta : J^sV \to W$ is a fiberwise surjective linear homomorphism of vector bundles. Note that such a Δ always exists and is unique. We will call a tuple $(D\phi_1, \dots, D\phi_q)$ an exact q-tuple (inspired by the case where D is the exterior derivative d).

5.1 Proof of Main Theorem

The strategy of the proof is to replace the θ_i one by one with a $D\phi_i$ satisfying the same result. The homotopy in this state will be given by $t\theta_i + (1-t)D\phi_i$. This homotopy doesn't a priori preserve the spanning property, but our construction of the ϕ_i will make this clear. To obtain the section ϕ_i, we will have to perturb the q-tuple $(\theta_1, \dots, \theta_q)$ to obtain some transversality condition. The perturbation will have to be carried out carefully to not spoil the previous construction of the $D\phi_1$ through $D\phi_{i-1}$.

Proof. We will present the construction of ϕ_i assuming we have constructed $\phi_1, \dots, \phi_{i-1}$. We are in the situation where we have a tuple $(D\phi_1, \dots, D\phi_{i-1}, \theta_i, \dots, \theta_q)$ of sections which generate the bundle W. Let $\Sigma \subset M$ be the subset where θ_i is necessary. Σ is the subset where $(D\phi_1, \dots, D\phi_{i-1}, \theta_{i+1}, \dots, \theta_q)$ generate a linear subspace of dimension $\operatorname{rank}(W) - 1$.

Lemma 5.1. *The q-tuple $(D\phi_1, \dots, D\phi_{i-1}, \theta_i, \theta_{i+1}, \dots, \theta_q)$ is homotopic through holomorphic q-tuples generating W to one where we may assume that the first $i-1$ sections remain exact, while Σ is a proper closed complex analytic subset of M.*

Proof. That Σ is a closed, complex analytic subset is clear. It is locally cut out by an intersection of vanishing determinants of holomorphic matrices, so it is complex analytic. That Σ is closed follows from the fact that having rank less than or equal to $\operatorname{rank}(W) - 1$ is a closed condition, but it is impossible to have rank less than $\operatorname{rank}(W) - 1$ if adding θ_i causes the rank to reach $\operatorname{rank}(W)$. The key point is thus to ensure that Σ is a proper subset of M, i.e. that there is at least one point in M where θ_i is not needed to generate the fiber.

If Σ is all of M then $(D\phi_1, \dots, D\phi_{i-1}, \theta_{i+1}, \dots, \theta_q)$ generate a linear subspace of dimension $\operatorname{rank}(W) - 1$ everywhere. Here is where the $q > \operatorname{rank}(W)$ condition comes to play. Without loss of generality there is a point where $D\phi_1$ is not needed (since not all of the forms can be needed everywhere). Let Σ_1 be the complex analytic subset where $D\phi_1$ is needed, which has codimension at least 1. Let α be a section of V vanishing to high order along Σ_1, so that $D\alpha = 0$ along Σ_1. Perturbing $D\phi_1$ by $D\alpha$ preserves our desired properties. Away from Σ_1, since $D\phi_1$ is not needed, and θ_i is needed, we must have that $D\phi_1$ is in the span of $(D\phi_2, \ldots, D\phi_{i-1}, \theta_{i+1}, \ldots, \theta_q)$. By Theorem 4.4 we may ensure that $D\alpha$ misses the span in at least one point (since D is elliptic), and therefore that θ_i is not needed at that point and hence Σ is a proper subset of M.

□

The proof of the above lemma is fairly easy, but allows us to demonstrate a technique by which we can obtain necessary transversality conditions without breaking our delicate induction. Namely we can perturb our sections by terms of the form $D\alpha$ for carefully chosen α.

We may now assume that Σ is a proper, closed complex analytic subset of M. Recall that we have written $D = \Delta \circ j^s$. Let Ω denote $\Delta^{-1}(\langle D\phi_1, \ldots, D\phi_{i-1}, \theta_{i+1}, \ldots, \theta_q\rangle)$, the pullback of the span in J^sV. Because Δ is a fiberwise surjective, Ω is a codimension 1 subspace of J^sV along Σ. By Cartan's Theorem B, we may pullback the section θ_i to a section θ' of J^sV. Indeed, let K be the kernel of the map Δ. The bundle K is also a holomorphic vector bundle, and we get a short exact sequence

$$0 \to K \to J^sV \to W \to 0$$

of associated sheaves of sections. These sheaves are coherent analytic, and thus $H^1(K) = 0$. Taking the long exact sequence on cohomology thus means that the map on global sections $H^0(M, J^sV) \to H^0(M, W)$ induced by Δ is surjective, which gives the result.

The condition that that this q-tuple spans along W precisely means that θ' misses Ω along Σ. We may thus define a section L of the holomorphic vector bundle $(J^sV)^*$ by $L(\Omega) = 0$ along Σ and $L(\theta') = 1$ along Σ. This uniquely defines L on Σ, which we may extend over all of M by Corollary 3.3 of Cartan's Theorem B. Note that by definition L is independent of the choice of pullback θ'.

In order to define our desired section ϕ_i of V, it suffices to find some section ϕ_i of V such that $L(j^s\phi_i) \neq 0$ along Σ. We will go further and show that under appropriate transversality conditions on L, we can always solve the equation $L(j^s\phi)|_\Sigma = g$ for any holomorphic function g on Σ.

Definition 5.1. *The definition of transversality of L to Σ is that for holomorphic sections ϕ vanishing to order $s-1$ along Σ, the collection $L(j^s\phi)$ generate of ideal of functions A, where $A|_\Sigma = \mathcal{O}|_\Sigma$. We have no control over the sheaf A away from Σ.*

The definition of semi-transversality of L to Σ is that Σ admit a stratification $\Sigma_0 \subset \cdots \subset \Sigma_k = \Sigma$, so that L is transverse to Σ_i/Σ_{i-1}, and for holomorphic sections ϕ vanishing to order $s-1$ along Σ_i, the $Lj^s\phi$ generate an ideal of functions A, where $A|_{\Sigma_i}$ contains the ideal sheaf $I_{\Sigma_{i-1}}$.

Semi-transvesality means L is transverse to Σ locally except at some smaller complex analytic set Σ', and L is transverse to Σ' except at some smaller analytic set. Repeating this procedure will reach the empty set in finitely many steps. In coordinates, transversality means we have control of the top derivatives of L in some transverse direction, we give a precise definition below. This geometric interpretation will be used to obtain the semi-transversality condition, as it is generic by the Jet transversality theorem. It is not a priori obvious that this coordinate definition of semi-transversality implies the analytic definition given above, we will prove this below in Lemma 5.3.

Definition 5.2. *L is geometrically transverse to Σ, if there locally exists a holomorphic coordinate neighborhood U with coordinates $(z_1, \ldots, z_n)$ such that $\Sigma \cap U$ is contained in the hypersurface $z_1 = 0$ and L splits as $L_1 + L_2$ in the following way:*

(i) There is locally a spitting $V = V_1 \oplus V_2$ of holomorphic bundles over U where V_1 is a trivial line bundle

(ii) $L_i : V_i \to U \times \mathbb{C}$ are differential operators of order s such that L_1 has a leading term of $\frac{\partial^s}{\partial z_i^s}$.

L is geometrically semi-transverse to Σ if we may decompose $\Sigma = \Sigma_k \supset \ldots \supset \Sigma_0$ so that L is geometrically transverse to each Σ_i/Σ_{i-1} and if v is a section of V vanishing to order $(s-1)$ along Σ_i then $L(v)$ vanishes along Σ_{i-1}.

This definition of semi-transversality is a holomorphic analog of the definition given in [3] to prove the removal of singularities, and indeed a section is semi-transverse holomorphically if it is semi-transverse smoothly.

Lemma 5.2. *Suppose that L is semi-transverse to Σ, then for any holomorphic function g on Σ, we may find a holomorphic section α of V so that $L(j^s\alpha)|_\Sigma = g$.*

Kolaric gives some examples in [13] of situations where $L\phi = g$ is not solvable in the case that L is holomorphic tangent vector field. Another way to think about semi-transversality is that it is a generalization of the notion of tangency to finite order for vector fields.

Proof. In the proof, we will write $L_s = L \circ j^s$.

We first attempt to solve the equation $L_s\alpha = g$ along Σ_0. Extend g arbitrarily to all of M, which is possible by Cartan's Theorem B. Let $f_1^0, \ldots, f_m^0$ be global holomorphic functions cutting out Σ_0. We may use the transversality condition of L along Σ_0 to find sections $\alpha_1, \ldots, \alpha_n$ vanishing to order $(s-1)$ along Σ_0 so that $L_s\alpha_1, \ldots, L_s\alpha_k$ together with $f_1^0, \ldots, f_m^0$ have no common vanishing locus. Note: the transversality condition only tells us that the collection of such α is infinite, using Proposition 5.7 of [1] allows us to reduce this to a finite collection by talking convergent infinite linear combinations. Then by Theorem 3.8, we may find global holomorphic functions a_i and b_j such that

$$\sum_i a_i L_s \alpha_i + \sum_j b_j f_j^0 = g$$

Since the α_i vanish to order $s-1$ along Σ_0, we have that $a_i L_s\alpha_i - L_s(a_i\alpha_i) = 0$ along Σ_0. Moreover since the f_j^0 vanish along Σ_0, we obtain that

$$L_s(\sum a_i\alpha_i)|_{\Sigma_0} = g|_{\Sigma_0}$$

We now extend to solving the equation $L_s(\alpha)|_{\Sigma_1} = g$ for any holomorphic function g on Σ_1. Extend g to all of M. By our previous argument, we may find some section α' so that $L_s\alpha' - g$ vanishes along Σ_0. Let $f_1^1, \ldots, f_m^1$ be global holomorphic functions cutting out Σ_1. By finding an infinite family and reducing to a finite family using 5.7 of [1], the semi-transversality condition implies that we may find sections $\alpha_1, \ldots, \alpha_k$ which vanish to order $s-1$ along Σ_1 such that $L_s\alpha_1, \ldots, L_s\alpha_k, f_1^1, \ldots, f_m^1$ cut out I_{Σ_0}. $L_s\alpha' - g$ is a global section of I_{Σ_0}, and therefore there exists global holomorphic functions a_i and b_j such that

$$\sum a_i L_s\alpha_i + \sum b_j f_j^i = L_s\alpha' - g$$

Proceeding in precisely the same way we find that

$$L_s(\sum a_i\alpha_i)|_{\Sigma_1} = L_s\alpha' - g$$

and therefore that $L_s(\sum a_i\alpha_i + \alpha')|_{\Sigma_1} = g$.

We finish by proceeding inductively, solving on each Σ_i in precisely the same manner (the only thing that changes is notation). This finishes the proof.

□

We have now proven that, assuming L is semi-transverse to Σ, we can find ϕ so that $Lj^s\phi = 1$ along Σ, and thus we can set $\phi_i = \phi$ and replace θ_i with $D\phi_i$ using the homotopy $(1-t)\theta_i + tD\phi_i$. The reason that this works is because we have that $L((1-t)\theta' + tj^s\phi) = (1-t) + t = 1$ and therefore that we can replace θ_i with $\Delta((1-t)\theta' + tj^s\phi) = (1-t)\theta_i + tD\phi_i$ without ruining our desired spanning property. Inducting in the obvious way proves the main theorem. It therefore remains to show that we can obtain our desired semi-transversality condition.

Lemma 5.3. *Geometric semi-transversality to an analytic set Σ' implies semi-transversality.*

Proof. Suppose that L is geometrically transverse to Σ_1 except at some subset Σ_0 (where Σ_0 is possibly empty) and that L is geometrically transverse to Σ_0. It suffices to show that this implies L being semitransverse to Σ_1 up to Σ_0.

By Cauchy-Kovalevskaya in local coordinates about each point in Σ_1, there are local sections $\{\phi_i\}$ of V vanishing to order $s-1$ along Σ_1 such that the $L\phi$ cut out I_{Σ_0} locally. To obtain analytic transversality, we need to find global sections with the same property. We apply Theorem 3.9 to the sheaf of sections of V vanishing to order $(s-1)$ along Σ_1. This allows us to uniformly approximate the ϕ_i by global sections ψ_i^n in sufficiently small holomorphically convex balls about each point in Σ_1. The failure of transversality at Σ_0 means the $L\psi_i^n$ still vanish at Σ_0, but the Cauchy integral formula in local coordinates means that the $L\psi_i^n$ uniformly approximate the $L\phi_i$. For each i we may thus take n large enough so that $L\psi_i^n$ locally cut out I_{Σ_0}, and this give us the desired result.

For general Σ' (with perhaps many layers as in the definition of semi-transversality), we can induct layer by layer and obtain the desired result. □

Note that geometric transversality only depends on $L|_\Sigma$, whereas the definition of analytic transversality a priori requires the knowledge of L in a neighborhood of Σ. We now show that we can obtain semi-transversality of L to Σ, generalizing an argument given in [13] which establishes the relevant transversality condition in the case of holomorphic immersions.

Lemma 5.4. *Let Σ_1 be the subset where $D\phi_1$ is needed to span. There exists β vanishing to high order along Σ_1 so that the new tuple $(D\phi_1 + D\beta, \ldots, D\phi_{i-1}, \theta_i, \ldots, \theta_q)$ has L is geometrically semi-transverse to Σ on M/Σ_1. This new tuple satisfies all of our desired properties, and if Σ' is the subset of non-semi-transversality, Σ' has been reduced to $\Sigma' \cap \Sigma_1$ as nothing has changed along Σ_1.*

Proof. Let $\mathcal{A}^{s+1}(V, \Sigma_1)$ denote the space of holomorphic sections of V vanishing to order $s+1$ (or higher) along Σ_1. Let $\Sigma(\beta)$ by the subset where θ_i is needed (noting the dependence on β). Let p be a point of M/Σ_1. If $D\phi_2, \ldots, D\phi_{i-1}, \theta_{i+1}, \ldots, \theta_q$ suffices to span W at p, then they suffice in a neighborhood U_p of p, and $\Sigma(\beta) \cap U_p$ is empty and transversality holds trivially. Thus we assume that θ_i is needed at p together with $n-1$ of the other sections which we label $\alpha_1, \ldots, \alpha_{k-1}$ for $k = \mathrm{rank}(W)$. These sections suffice to span W in a neighborhood U_p of p over which $W = U_p \times \mathbb{C}^k$. Let W_α be the subset of W spanned by the α_i. We define L_α by $L_\alpha(\Delta^{-1}(W_\alpha)) = 0$ and $L_\alpha(\theta') = 1$ over U_p. Note that on $\Sigma \cap U_p$, L_α necessarily agrees with L. This means that on $\Sigma(\beta)$ the resulting L is ultimately independent of β. Choose holomorphic coordinates on U_p so that L_α has a component with leading term $\partial^s/\partial z_1^s$.

Fixing β, we define a holomorphic function over U_p by

$$f_\beta = \det(\alpha_1, \ldots, \alpha_{k-1}, D\phi_1 + D\beta)$$

Whenever $f_\beta \neq 0$, we may replace θ_i with $D\phi_1 + D\beta$ and it will not be needed to span W, thus $\Sigma(\beta) \cap U_p \subseteq \{f_\beta = 0\} = Z_\beta$. From the definition of semi-transversality, we note that if L_α is semi-transverse to Z_β, then it is semi-transverse to $\Sigma(\beta)$, so we will show that for an open and dense subsets of $\mathcal{A}^{s+1}(V, \Sigma_1)$ we can make L_α semi-transverse to Z_β. But note that by our coordinate description of L, the non-transversality locus of L_α to Z_β is contained in $Z_\beta \cap \{\frac{\partial}{\partial z_1} f_\beta = 0\}$, and so on. Thus if for some large m we have

$$\emptyset = \bigcap_{j=0}^{m} \{\frac{\partial^j}{\partial z_1^j} f_\beta = 0\}$$

then we will obtain that L_α is semi-transverse to Z_β. Indeed, each of the equations $f_\beta = 0$ and $\frac{\partial^j}{\partial z_1^j} f_\beta = 0$ cuts out transversely intersecting hyperplanes in $J^{m+s}(V)$ as all of these equations are linear in the $(m+s)$-jet of β. Taking $m = \dim(M) + 1$, the jet-transversality theorem 4.4 then guarantees an open dense subset A_{U_p} of $\mathcal{A}^{s+1}(V, \Sigma_1)$ for which L is transverse to $\Sigma(\beta)$ over U_p. Now we may cover M/Σ_1 by neighborhoods of the form U_p and take a countable subcover $\{U_j\}$. Intersecting the A_{U_j} gives a residual set in $\mathcal{A}^{s+1}(V, \Sigma_1)$ of β so that L is transverse to $\Sigma(\beta)$, and thus proves the lemma.

□

Lemma 5.5. *The holomorphic q-tuple $(D\phi_1, \ldots, D\phi_{i-1}, \theta_i, \ldots, \theta_q)$ generating W is homotopic to a holomorphic q-tuple $(D\phi'_1, \ldots, D\phi'_{i-1}, \theta_i, \ldots, \theta'_q)$ generating W through holomorphic q-tuples generating W to one where L is semi-transverse to Σ.*

Proof. Using notation as in the proof of Lemma 5.4, we may use the homotopy deforming by $D\phi_1 + tD\beta$ to obtain semi-transverality on M/Σ_1 and $\Sigma'(\beta) \subseteq \Sigma' \cap \Sigma_1$ where Σ' denotes the non-semitransversality locus. Notice that we use nothing special about $D\phi_1$ and equally could have used any of the other sections.

We will reduce Σ' to empty in the following way. Pick a point p on an irreducible component of Σ'. Then there is a section of our q-tuple, without loss of generality $D\phi_1$, which is not needed to span W at that point. Modifying $D\phi_1$ by $D\beta$ as in Lemma 5.4 then drops the dimension of that irreducible component of Σ' since p does not lie on Σ_1 and does not create more irreducible components or raises their dimension, and does so through a homotopy of holomorphic q-tuples with our desired properties. Repeating in this fashion to the countably many irreducible components of Σ' drops all of the dimensions to at most $\dim(M) - 2$. Inducting on the dimensions of the irreducible components of Σ' then gives the result

□

By our above remarks, Lemma 5.5 completes the proof of the Main theorem.

□

5.2 Parametric h-principle

We give a few general corollaries of the main theorem, generalizing the parametric h-principle for immersions proven in [13].

Theorem 5.2. *(Parametric h-principle) All notation is as the statement of the main theorem and we again take q greater than the rank of W. Let $\mathrm{Sec}(V_q(W))$ denote the holomorphic sections of q-tuples spanning W. Let X be a Stein manifold, and let $\Theta(x) = (\theta_1, \ldots, \theta_q)(x)$ be a q-tuple of sections spanning W with holomorphic dependence on a parameter $x \in X$. Then there is a homotopy $H : X \times I \to \mathrm{Sec}(V_q(W))$ such that $H(x, 0) = \Theta(x)$, each q-tuple $H(x, 1)$ is exact, and each family $H(\cdot, t)$ has a holomorphic dependence on X.*

Furthermore, if $A \subset X$ is a closed complex analytic subset of X such that Θ is exact on A, then H may be chosen to be fixed on A,

Proof. Let $\pi : M \times X \to M$ be the standard projection. Then Θ is equivalent to a holomorphic spanning q-tuple of the pullback bundle π^*W. We may also pullback D and V to $M \times X$. An exact holomorphic q-tuple spanning W with dependence on X is simply an exact q-tuple of π^*W with respect to the operator π^*D. The differential operator π^*D

remains elliptic, and hence applying Theorem 1.1 to Θ and π^*D over $M \times X$ (since the product of Stein manifolds is Stein) gives the basic parametric h-principle.

To prove the relative version of the parametric h-principle, we must check that all steps of the proof of Theorem 1.1 can be carried out without changes on $M \times A$. Obtaining semi-transversality is no issue, since we may additionally ask for all perturbations in the proof of the main theorem to vanish on the complex analytic subset $M \times A$ of $M \times X$ and we only need to obtain semi-transversality away from $M \times A$. The difficult step is in solving the differential equation as in Lemma 5.2. We now use notation as in the proof of Lemma 5.2. Suppose that along A, $\Theta = (\pi^*D\psi_1, \ldots, \pi^*D\psi_q)$. Note that for a section α of π^*V, $\pi^*D(\alpha)|_{M\times A}$ depends only on $\alpha|_{M\times A}$, and hence if α agrees with ψ_i on $M \times A$ then $D\alpha$ will agree with $D\psi_i$ on $M \times A$. We also note that from the definition of $L \in J^sV^*$ and Cartan's Theorem B, we may extend L from Σ to all of $M \times X$ so that L is tangent to all slices $M \times \{x\}$. Therefore $L_s(\phi)$ vanishes along $M \times A$ if ϕ does.

We now have to show that when we solve the equation $L_s\alpha|_\Sigma = 1$ in order to replace θ_i with $D\alpha$, we can choose α so that it is equal to ψ_i on a $M \times A$.

Lemma 5.6. *Suppose L is semi-transverse to Σ away from $M \times A$, then for any holomorphic function g which vanishes on $\Sigma \cap (M \times A)$, there is a section α of π^*V vanishing on $M \times A$ so that $L(j^s\alpha)|_\Sigma = g$*

Proof. The proof is identical to the proof of Lemma 5.2, merely requiring that all objects in the proof vanish on $M \times A$, which is easily obtained since $M \times A$ is closed and complex analytic. $\square$

By the definitions of L and ψ_i, we have that $1 - L\psi_i$ vanishes on $\Sigma \cap (M \times A)$, hence we may choose α so that $L\alpha|_\Sigma = 1 - \psi_i$, and replacing θ_i with $D(\psi_i + \alpha)$ gives the result, as α can be taken to vanish on $M \times A$. This step of the proof thus can be fixed on A, and hence the parametric h-principle relative to A holds.

$\square$

By Oka-Grauert theory, any continuous family of such q-tuples parametrized by a Stein manifold X is homotopic to a holomorphic family. Indeed, consider the holomorphic Steifel bundle $V_q(\pi^*W)$ over $M \times X$, all continuous sections are homotopic to holomorphic ones by Theorem 3.6. We also note that every real analytic manifold is a deformation retract of a Stein manifold. If N is real analytic, then arbitrarily small neighborhoods of N in its complexification $N_\mathbb{C}$ are Stein (see chapter 2 of [2]). If S^k is sphere, let $S^k_\mathbb{C}$ denote such a Stein manifold. Then any continuous family of q-tuples parametrized by S^k may be extended to $S^k_\mathbb{C}$ and deformed into a holomorphic family, which may then be deformed to a family of exact q-tuples. If $\mathrm{Exa}(D, q, W)$ denotes the space of exact holomorphic spanning q-tuples, and $\mathrm{Sec}(V_q(W))$ denote the space of holomorphic spanning q-tuples, this tells us that the map $\pi_i(\mathrm{Exa}(D, q, W)) \to \pi_i(\mathrm{Sec}(V_q(W)))$ is surjective. One consequence of the following corollary is that this map in injective on π_0 and thus is an isomorphism on π_0.

Corollary 5.1. *(1-parametric h-principle) Let $D\Phi = (D\phi_1, \ldots, D\phi_q)$ and $D\Psi = (D\psi_1, \ldots, D\psi_q)$ be exact holomorphic spanning q-tuples homotopic through holomorphic spanning q-tuples*

$\Theta(t) = (\theta_1(t), \ldots, \theta_q(t))$. i.e we have $t \in [0,1]$, $\Theta(0) = D\Phi$ and $\Theta(1) = D\Psi$. Then $D\Phi$ and $D\Psi$ are homotopic through exact holomorphic q-tuples which span W. Precisely, there is a homotopy $G : [0,1] \times [0,1] \to \mathrm{Sec}(V_q(W))$, such that $G(0,t) = \Theta(t)$, $G(t,0) = D\Phi$, $G(t,1) = D\Psi$, and $G(1,t)$ is exact.

Proof. The homotopy $\Theta(t)$ may be extended from $[0,1]$ to a $\mathbb{C}$-parameter family of spanning q-tuples. By 5.4.9 of [8], without loss of generality we may assume Θ depends holomorphically on the $\mathbb{C}$-parameter, and that $\Theta(0) = D\Phi$ and $\Theta(1) = D\Psi$. We may apply the relative parametric h-principle in Theorem 5.2 to the complex analytic subset $A = \{0\} \cup \{1\}$ and obtain a family H of holomorphic spanning q-tuples parametrized by $[0,1] \times \mathbb{C}$ so that the tuples $H(1,z)$ are all exact and $H(0,z) = \Theta(z)$, and that $H(t,0) = D\Phi$ and $H(t,1) = D\Psi$. We may define $G(s,t) = H(s,t)$ for $s,t \in [0,1]$ and this gives the desired result. □

5.3 Applications

As we mentioned in the introduction, the main theorem has some concrete applications to geometrically interesting h-principles, and we give an account of some of these in this section.

5.3.1 Holomorphic Immersions

We consider the case where $W = T^*M$ is the holomorphic cotangent bundle, and the operator D is the exterior derivative d. d is elliptic as the symbol Δ is the natural projection $J^1(M,\mathbb{C}) \to T^*M$. As mentioned in the introduction, a map $f = (f_1, \ldots, f_q)$ from M to $\mathbb{C}^q$ is an immersion if and only the df_i span T^*M at every point i.e. if the pullback $T^*\mathbb{C}^q \to T^*M$ is surjective. Given any injective monomorphism of holomorphic bundles $TM \to T\mathbb{C}^q$, pulling back the forms dz_i on $T^*\mathbb{C}^q$ gives rise to a q-tuple $(\theta_1, \ldots, \theta_q)$. Likewise given any q-tuple we can define an injective bundle monomorphism $F : TM \to T\mathbb{C}^q$ by mapping M to a point and requiring that $dz_i(F(v)) = \theta_i(v)$. Such a q-tuple can thus be considered a formal holomorphic immersion, and the main theorem states that any formal holomorphic immersion is homotopic through formal holomorphic immersions to a genuine holomorphic immersion.

Theorem 5.3. *(h-principle for holomorphic immersions). Let $q > \dim(M)$, then if $(\theta_1, \ldots, \theta_q)$ are holomorphic 1-forms generating the cotangent bundle, they may be deformed through q-tuples of holomorphic 1-forms generating the cotangent bundle to such a q-tuple $(df_1, \ldots, df_q)$ of exact holomorphic 1-forms. The map $(f_1, \ldots, f_q) : M \to \mathbb{C}^q$ is an immersion.*

Of course the rank of T^*M is n, and thus by Corollary 3.2, if $2q - 2n \geq n$ we can always find such a q-tuple, giving the following corollary.

Corollary 5.2. *Every Stein manifold M admits a holomorphic immersion into $\mathbb{C}^q$ for $q = \lceil \frac{3n}{2} \rceil$ and n is the complex dimension of M.*

5.3.2 Free maps

Lastly, we discuss an h-principle for holomorphic free maps.

Definition 5.3. *A holomorphic map $f : M \to \mathbb{C}^q$ is said to be free around each point $p \in M$ if the collection of vectors*

$$\{\frac{\partial f_i}{\partial z_j}\} \cup \{\frac{\partial f_i}{\partial z_j \partial z_k}\}$$

is linearly independent for local holomorphic coordinates $(z_1, \ldots, z_n)$ around p. This notion does not depend on the choice of local coordinates.

Free maps are higher order analogs of immersions. Given a map $f : M \to \mathbb{C}^q$, one can consider the first osculating space which is given by $Df(T_pM)$, and the second osculating space which also includes the second order derivatives. Free maps are those maps whose second osculating space is of maximal possisble dimension, just as immersions are those for which the first osculating space is of maximal dimension. Constructing free maps can be very difficult. In the smooth case it is not known whether there exists a free map $T^2 \to \mathbb{R}^5$ despite no obvious obstructions. Despite this, the smooth removal of singularities gives an h-principle for free maps $T^2 \to \mathbb{R}^6$ (which exist since the bundle $J^2(T^2, \mathbb{R})$ is trivial).

To see why Theorem 1.1 implies the h-principle for free maps, consider the vector bundle given by $J^2(M, \mathbb{C})/\mathbb{C}$, where we ignore the constant term of jet space (given by the value of a holomorphic function f). A free map is then a function $(f_1, \ldots, f_q)$ such that the $j^2 f_i$ span the space $J^2(M, \mathbb{C})/\mathbb{C}$. This is proved by working in local coordinates. If the $j^2 f_i$ span the bundle, then the matrix whose rows are $j^2 f_i$ will have maximal rank. The columns of the matrix are the first and second derivatives of f, which must also have maximal rank and thus be linearly independent by dimension reasons.

There is a second order operator $D : M \times \mathbb{C} \to J^2(M, \mathbb{C})/\mathbb{C}$ which take the second jet of a holomorphic functions and forgets the value of the function. D is clearly elliptic as the associated homomorphism Δ is simply projection from $J^2(M, \mathbb{C})$. Thus Theorem 1.1 applies and we obtain

Theorem 5.4. *(h - principle for holomorphic free maps). If $q \geq \frac{1}{2}n(n+3) + 1$, then holomorphic free maps $f : M^n \to \mathbb{C}^q$ satisfy the h-principle.*

Applying Corollary 3.2, noting that $J^2(M, \mathbb{C})/\mathbb{C}$ has fiber dimension $\frac{n(n+3)}{2}$, we obtain

Corollary 5.3. *If M^n is a Stein manifold, then M admits a holomorphic free map to $\mathbb{C}^q$ for $q \geq \frac{n(n+3)}{2} + \lceil \frac{n}{2} \rceil$*

More generally, one can consider k-free holomorphic maps, those for which the derivatives up to order k are linearly independent. Proceeding in the same way gives an h-principle for k-free maps of Stein manifolds.

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