Some Problems on Theta Functions

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1. The delta distribution $\delta(x)$ is a periodic "generalized function" defined by the property that
$$\frac{1}{2\pi}\int_0^1 \delta(x)f(x)dx = f(0)$$
Using this definition, compute the fourier series of $\delta(x)$. When does this series converge? When it converges, what value does it converge to?

2. Solve the heat equation
$$\frac{\partial f}{\partial t} = \frac{1}{4\pi}\frac{\partial^2 f}{\partial x^2}$$
on the interval $[0, 1]$, with periodic boundary conditions, and initial conditions given by the delta distribution. This is called the "fundamental solution" of the heat equation. Show that the fourier series defining $f(x, t)$ converges when t is positive and diverges when t is negative.

3. Define the Riemann theta function by $\Theta(z) = \Theta(z; \tau) = f(z, -i\tau)$ where f is the fundamental solution to the heat equation. In other words, Θ is what you get by evaluating the fundamental solution at imaginary times. For the same reason that the fundamental solution is only defined for positive times, the Θ function is only defined when the "auxiliary parameter" τ has positive imaginary part. Much like the exponential function, the Θ function has a hidden symmetry, which only emerges when one considers Θ as a function of a complex variable. In particular, show that it satisfies the following identities:
$$\Theta(z + 1) = \Theta(z)$$
$$\Theta(z + \tau) = e^{-2\pi i(z+\tau/2)}\Theta(z)$$
$$\Theta(-z) = \Theta(z)$$

4. Show that $\Theta(\frac{1+\tau}{2}) = 0$. Find all the other zeroes of $\Theta(z)$.

5. Fix a complex number τ. Define a function
$$g(z) = \left(\frac{\Theta(z - \frac{1}{2})}{\Theta(z)}\right)^2$$
Note that g is meromorphic, since it has a pole when $z = \frac{1+\tau}{2}$. Show that it is doubly periodic. In other words, show that it satisfies the following two identities:
$$g(z + 1) = g(z)$$
$$g(z + \tau) = g(z)$$
Can you use this trick to construct other doubly periodic functions? Try to determine precisely when the trick works.

6. Show that any holomorphic, doubly periodic function is constant.

7. Let $g'(z)$ be the derivative of the function $g(z)$ defined above. Show that $g'(z)$ is a doubly periodic function. Can you find a formula for $g'(z)$ along the lines of the formula defining $g(z)$? In other words, can you express it as a ratio of translates of the Θ function?