Syllabus

18.218: Topics in Combinatorics

Algebraic Methods in Extremal Combinatorics

Spring 2022

Course Description:
This graduate topics class covers different algebraic methods used in extremal combinatorics. These methods include linear algebra methods, spectral graph theory methods, and various polynomial methods (for example, the Combinatorial Nullstellensatz and the slice rank polynomial method). Some of the results discussed in the class are several decades old while others are very recent.

Instructor: Lisa Sauermann
Email: `lsauerma@mit.edu`

Time: Tuesday, Thursday, 9:35am – 10:55am

Location: Room 4-237

Office hours:
Monday 1:05 – 1:55pm in room 2-171 (starting from February 7)

Homework: There will be five problem sets (roughly every other week). The problem sets will be due at 8pm on February 24, March 10, March 31, April 14 and May 5 (each problem set will be posted two weeks before the due date). Late submissions will only be accepted in exceptional circumstances, with an accommodation letter from Student Support Services or GradSupport (see below).

Homework submission: The problem sets will be posted on Canvas and homework solutions will need to be submitted via Gradescope. Please start each problem on a new page (and don't forget to specify which page is for which problem after uploading your solution).

Collaboration on homework problems: Students are permitted to work together on homework problems, as long as everyone is actively involved in the discussion about the problem (for example, it is not allowed to ask another student who has already solved the problem for their solution). Every student must write down their solutions individually. If a student worked together with others when solving a problem, this must be indicated at the start of the solution.

Grading: Grades will be based entirely on the homework assignments.

Schedule:
A tentative schedule for the class is as follows (this schedule is subject to change):

- Week 1 (Feb 1, Feb 3): Linear independence and rank methods (Odd-Town theorem, Even-Town theorem, Fisher's inequality, Bollobás' Theorem via exterior algebras). [M, Miniatures 3,4,33]
- Week 2 (Feb 8, Feb 10): More linear algebra methods (equiangular lines, point-sets with only two distances, another proof of Bollobás' Theorem). [M, Miniatures 9,15]
- Week 3 (Feb 15, Feb 17): More linear algebra methods (counterexample to Borsuk's conjecture). [M, Miniatures 17,18]
- Week 4 (only Feb 24): Polynomial methods relying on the number of zeros of a one-variable polynomial (monomial counting, Finite Field Nikodym Problem) [G, Chapter 2]
- Week 5 (Mar 1, Mar 3): More polynomial methods relying on the number of zeros of a one-variable polynomial (the Finite Field Kakeya Problem, the Joints Problem). [M, Miniature 25], [G, Chapter 2]
- Week 6 (Mar 8, Mar 10): Spectral Methods (the Friendship Theorem, expanders, Huang's proof of the Sensitivity Conjecture). [14, pp. 55–57], [3, Section 9.2], [12]
- Week 7 (Mar 15, Mar 17): The Combinatorial Nullstellensatz (and different applications, including the Cauchy–Devenport Theorem and the Chevalley–Warning Theorem). [1]
- Week 8 (Mar 29, Mar 31): More applications of the Combinatorial Nullstellensatz (the Erdős–Ginzburg–Ziv theorem and Reiher's proof of Kemnitz' conjecture). [1, 15]
- Week 9 (Apr 5, Apr 7): Alon and Dubiner's result proving linearity of the Erdős–Ginzburg–Ziv constant $\mathfrak{s}(\mathbb{F}_p^d)$ for fixed d. [2]
- Week 10 (Apr 12, Apr 14): The slice rank polyomial method (and different applications, including Ellenberg and Gijswijt's bound for the cap-set problem, the multi-colored sum-free problem, and bounds for sunflower-free sets). [8, 9, 13, 17]
- Week 11 (Apr 19, Apr 21): More applications of the slice rank polynomial method and its consequences (the arithmetic triangle removal lemma, and upper bounds for the Erdős–Ginzburg–Ziv constant $\mathfrak{s}(\mathbb{F}_p^d)$ for fixed p). [10, 11, 16]
- Week 12: No class, no office hours
- Week 13 (May 3, May 5): Lower bounds for extremal numbers of bipartite graphs: explicit constructions coming from algebraic geometry, and randomized algebraic constructions of Bukh and Conlon. [4, 5, 6, 7]
- Week 14 (only May 10): Continuation of randomized algebraic constructions of Bukh and Conlon. [4, 5, 6, 7]

References:
There are no required materials for this course. The course does not follow any particular textbook, but for the first five weeks, we will cover some parts of the following two books:

[G] L. Guth, *Polynomial methods in combinatorics*, University Lecture Series, 64. American Mathematical Society, ISBN: 978-1-4704-2890-7.

[M] J. Matoušek, *Thirty-three Miniatures: Mathematical and Algorithmic Applications of Linear Algebra*, Student Mathematical Library, 53, American Mathematical Society, ISBN: 978-0-8218-4977-4.
Freely accessible at `http://kam.mff.cuni.cz/~matousek/stml-53-matousek-1.pdf`

Here is the list of references for the remaining weeks of the course:

[1] N. Alon, *Combinatorial Nullstellensatz*, Combin. Probab. Comput. **8** (1999), 7–29.

[2] N. Alon and M. Dubiner, *A lattice point problem and additive number theory*, Combinatorica **15** (1995), 301–309.

[3] N. Alon and J. H. Spencer, *The probabilistic method*, Fourth edition, Wiley Series in Discrete Mathematics and Optimization, ISBN: 978-1-119-06195-3.

[4] B. Bukh, *Random algebraic construction of extremal graphs*, Bull. London Math. Soc. **47** (2015), 939–945.

[5] B. Bukh, *Extremal graphs without exponentially-small bicliques*, preprint, 2021, arXiv:2107.04167.

[6] B. Bukh and D. Conlon, *Rational exponents in extremal graph theory*, J. Eur. Math. Soc. **20** (2018), 1747–1757.

[7] D. Conlon, *Graphs with few paths of prescribed length between any two vertices*, Bull. Lond. Math. Soc. **51** (2019), 1015–1021.

[8] E. Croot, V. Lev, and P. Pach, *Progression-free sets in $\mathbb{Z}_4^n$ are exponentially small*, Ann. of Math. (2) **185** (2017), 331–337.

[9] J. Ellenberg and D. Gijswijt, *On large subsets of $\mathbb{F}_q^n$ with no three-term arithmetic progression*, Ann. of Math. (2) **185** (2017), 339–343.

[10] J. Fox and L. M. Lovász, *A tight bound for Green's arithmetic triangle removal lemma in vector spaces*, Adv. Math. **321** (2017), 287–297.

[11] J. Fox and L. Sauermann, *Erdős-Ginzburg-Ziv constants by avoiding three-term arithmetic progressions*, Electron. J. Combin. **25** (2018), Paper 2.14, 9 pp.

[12] H. Huang, *Induced subgraphs of hypercubes and a proof of the Sensitivity Conjecture*, Ann. of Math. (2) **190** (2019), 949–955.

[13] E. Naslund and W. Sawin, *Upper bounds for sunflower-free sets*, Forum Math. Sigma **5** (2017), e15, 10pp.

[14] B. Nica, *A brief introduction to Spectral Graph Theory*, expanded lecture notes, 2016, arXiv:1609.08072.

[15] C. Reiher, *On Kemnitz' conjecture concerning lattice-points in the plane*, Ramanujan J. **13** (2007), 333–337.

[16] L. Sauermann, *On the size of subsets of $\mathbb{F}_p^n$ without p distinct elements summing to zero*, Israel J. Math. **243** (2021), 63–79.

[17] T. Tao, *A symmetric formulation of the Croot–Lev–Pach–Ellenberg–Gijswijt capset bound*, blog post, 2016, `http://terrytao.wordpress.com/2016/05/18/a`.

Policies for students needing to isolate due to Covid:
If you have to miss class because of being required to isolate due to Covid, you can contact me to get access to the material covered in the classes you missed (of course, you are also welcome to simply ask another student for their notes from class instead). Students in isolation are expected to submit their homework as usual (homework submission is online via Gradescope). If you are unable to submit your homework on time because of feeling sick, please contact Student Support Services or GradSupport (see below) for an accommodation letter.

Student Support Services (S^3) and GradSupport:
Personal and medical issues can make it hard to focus on academics. If you find that something is getting in the way of your ability to attend class, complete work, or take an exam, you should contact a dean in Student Support Services (S^3) if you are an undergraduate student or in GradSupport if you are a graduate student. The deans will provide you with support and help with determining next steps. You can e-mail `s3-support@mit.edu` or `gradsupport@mit.edu` (or you can visit the websites `https://studentlife.mit.edu/s3` and `https://oge.mit.edu/development/gradsupport/` for more ways to connect with S^3 and GradSupport).

Disability and Access Services (DAS):
MIT is committed to the principle of equal access and we want all of our students to feel welcome here. Students who need disability accommodations are encouraged to speak with Kathleen Monagle, Associate Dean, prior to or early in the semester so that accommodation requests can be evaluated and addressed in a timely fashion. Even if you are not planning to use accommodations, it is recommended that you meet with DAS staff to familiarize yourself with the services and resources of the office. You may also consult with Disability and Access Services in 5-104 or at 617-253-1674. If you have already been approved for accommodations, please contact me early in the semester so that we can work together to get your accommodation logistics in place.

Mental Health Resources:
Your mental health is very important. If you feel in distress, please consider reaching out to Student Support Services or GradSupport (see above). In emergency situations, you can reach MIT Medical Student Mental Health and Counseling at 617-253-2916 (days) or 617-253-4481 (nights and weekends).

Important Dates for Spring 2022:
First Day of Classes ..... January 31
Add Date ..... March 4
Drop Date ..... April 19
Spring Break ..... March 21–25
Last Day of Classes ..... May 10