Oddtown, Eventown, and Fisher's inequality

Applications of Linear Algebra

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MIT

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Can form $2^{n/2}$ different clubs!

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Theorem

Let $\mathcal{F} \subset 2^{[n]}$ be a collection of sets such that each $A \in \mathcal{F}$ has $|A|$ odd, and each $|A \cap B|$ is even for $A, B \in \mathcal{F}$. Then $|\mathcal{F}| \leq n$.

Proof

Notation

For each $A \in \mathcal{F}$, take $1_A \in \mathbb{F}_2^n$, its *incidence vector*, where $\mathbb{F}_2$ is the field with 2 elements.

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Since $k \geq 1$, we also have

$$\sum_{A \in \mathcal{F}} \alpha_A = 0.$$

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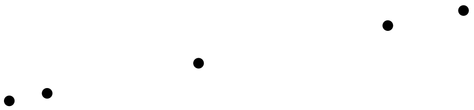

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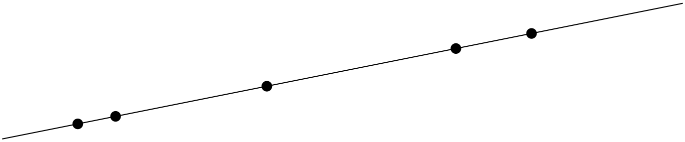

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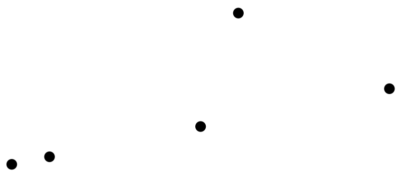

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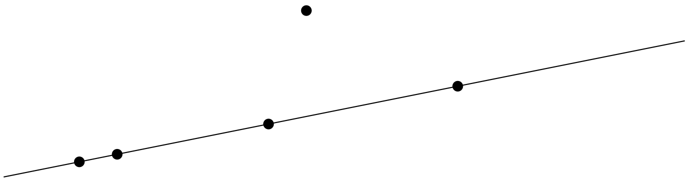

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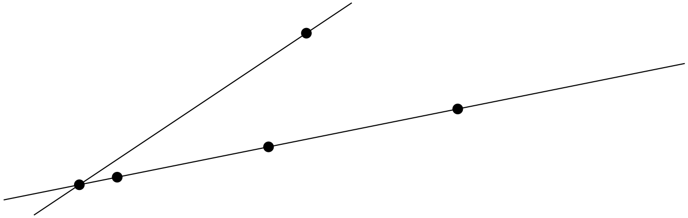

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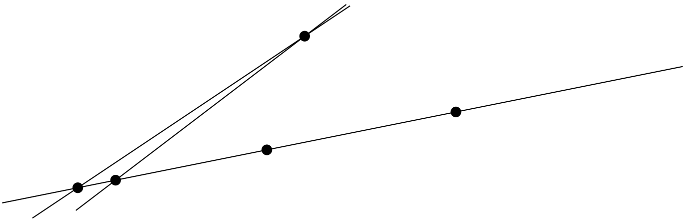

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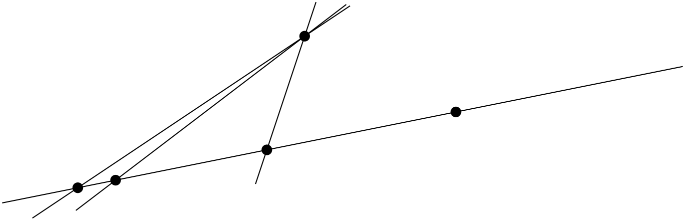

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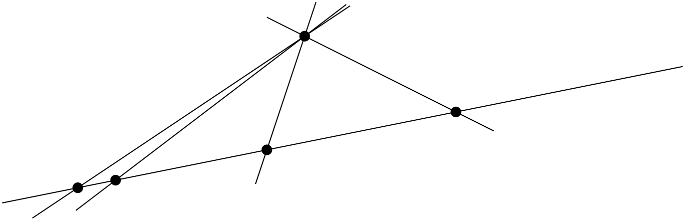

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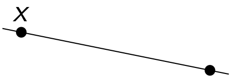

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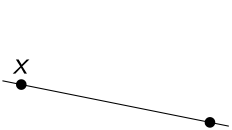

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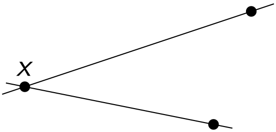

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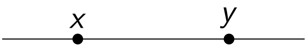
Each set A_x has $|A_x| \geq 2$.

Claim 2

For $x \neq y \in P$, we have $|A_x \cap A_y| = 1$. In particular, the sets A_x for distinct points $x \in P$ are distinct.



Fisher's inequality application

Theorem

Suppose P is a set of k points in the plane, not all on a single line. Then pairs of points from P define at least k lines.

Proof: Let L be the set of lines, and for each point $x \in P$, let A_x be the set of lines in L that contain x.

Claim 1

Each set A_x has $|A_x| \geq 2$.

Claim 2

For $x \neq y \in P$, we have $|A_x \cap A_y| = 1$. In particular, the sets A_x for distinct points $x \in P$ are distinct.

Fisher's inequality now implies that $k = |P| \leq |L|$.