Oddtown, Eventown, and Fisher's inequality

Applications of Linear Algebra

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Eventown

In a town, residents are allowed to form clubs. In order to restrict the number of clubs, the following rules are implemented:

- Every club must have an even number of members.
- Two clubs must not have the exact same set of members.
- Any two clubs must share an even number of members.

If there are n residents, how many clubs are possible?
$2^{n/2}$!

Odd-even town

What if each club is odd, and intersection of any two is even?
Then there can be at most n.
We show that the vectors 1_A for $A \in \mathcal{F}$ are linearly independent over $\mathbb{F}_2$.
Suppose $\sum \alpha_A 1_A = 0$. We then have, for any $\widetilde{A}$,

$$0 = \langle \sum \alpha_A 1_A, 1_{\widetilde{A}} \rangle = \sum \alpha_A \langle 1_A, 1_{\widetilde{A}} \rangle = \alpha_{\widetilde{A}}.$$

Fisher's inequality

What if the rules say that any pair of clubs must have the same intersection size?
Fisher's inequality: There are at most n clubs.
Take 1_A but this time over $\mathbb{R}$.
Will show that the vectors 1_A are linearly independent.

$$0 = \|\sum_{A\in\mathcal{F}} \alpha_A 1_A\|^2 = \langle \sum_{A\in\mathcal{F}} \alpha_A 1_A, \sum_{B\in\mathcal{F}} \alpha_B 1_B \rangle$$

$$= \sum_{A\in\mathcal{F}} \alpha_A^2 |A| + \sum_{A\neq B\in\mathcal{F}} \alpha_A \alpha_B k = k \left(\sum_{A\in\mathcal{F}} \alpha_A \right)^2 + \sum_{A\in\mathcal{F}} (|A| - k)\alpha_A^2.$$

Fisher's inequality application

Theorem: If we have a set P of n points in the plane, not all on a line, there are at least n lines between them.

Proof: For each point $x \in P$, let A_x be the set of lines that contain x.

Then each $|A_x| \geq 2$, for $x \neq y$, we have $|A_x \cap A_y| = 1$.

By Fisher's inequality, we are done!