A Fast New Algorithm for Weak Graph Regularity

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Joint work with Jacob Fox and Yufei Zhao

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Szemerédi's regularity lemma

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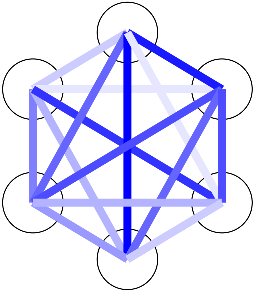

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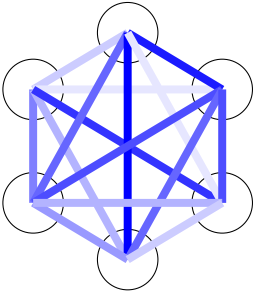

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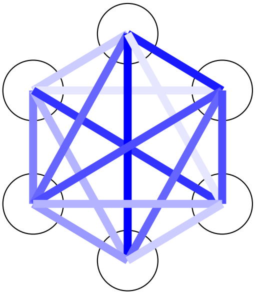

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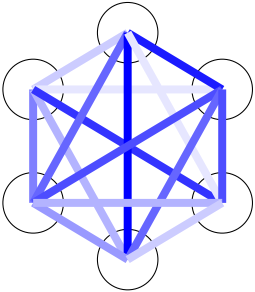

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How can we find such a partition algorithmicaly?

Regularity between sets

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Definition

Given a graph G and two sets of vertices X and Y, we say the pair (X, Y) is ϵ-*regular* if for any $X' \subset X$ with $|X'| \geq \epsilon|X|$, $Y' \subset Y$ with $|Y'| \geq \epsilon|Y|$, we have

$$\Big| d(X', Y') - d(X, Y) \Big| \leq \epsilon.$$

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Roughly says graph between X and Y is "random-like".

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Szemerédi's regularity lemma

For every $\epsilon > 0$, there is an $M(\epsilon)$ such that for any graph $G = (V, E)$, there is an equitable, ϵ-regular partition of the vertices into at most $M(\epsilon)$ parts.

Drawback of regularity lemma

The standard proof gives $M(\epsilon) \leq T(\epsilon^{-5})$, where T is the tower function, i.e. $T(5) = 2^{2^{2^{2^{2}}}} = 2^{65536}$.

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Frieze-Kannan regularity lemma: weaker regularity property, better bounds.

Frieze-Kannan (weak) regularity lemma

Definition

Given a partition $\mathcal{P} = \{V_1, V_2, ..., V_k\}$ of the set of vertices V, it is *Frieze-Kannan ϵ-regular* (FK-ϵ-regular) if for any pair of sets $S, T \subseteq V$, we have

$$\left| e(S, T) - \sum_{i,j=1}^{k} d(V_i, V_j)|S \cap V_i||T \cap V_j| \right| \leq \epsilon |V|^2$$

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Frieze-Kannan regularity lemma

Let $\epsilon > 0$. Every graph has a Frieze-Kannan ϵ-regular partition with at most $2^{2/\epsilon^2}$ parts.

Cut distance

Definition

Given two (weighted) graphs G_1 and G_2 on the same vertex set V, we define their *cut distance*

$$d_{\square}(G_1, G_2) = \frac{1}{|V|^2} \max_{S,T \subseteq V} |e_{G_1}(S, T) - e_{G_2}(S, T)|.$$

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Given two (weighted) bipartite graphs G_1 and G_2 between the same vertex sets V and W, we define their *cut distance*

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Given a partition $\mathcal{P}$, let $G_{\mathcal{P}}$ be the weighted graph obtained by taking, between each $X, Y \in \mathcal{P}$, weighted edges with weight $d(X, Y)$.

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Counting lemma

Given two graphs G_1 and G_2 on the same vertex set, for any graph H on k vertices, we have

$$|\hom(H, G_1) - \hom(H, G_2)| \leq e(H) d_\square(G_1, G_2) n^k.$$

Algorithmic regularity

Alon-Duke-Lefmann-Rödl-Yuster (1994)

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Folklore/Tao blog post (2010)

Randomized algorithm in time $O_\epsilon(1)$, ϵ-regular partition.

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There is an $\epsilon^{-O(1)}n^2$-time algorithm which, given $\epsilon > 0$, an n-vertex graph G, outputs $r \leq \epsilon^{-O(1)}$, subsets $S_1, S_2, \ldots, S_r, T_1, T_2, \ldots, T_r \subset V(G)$ and numbers $c_1, c_2, \ldots, c_r = \pm\epsilon^8/300$ such that

$$d_\square(G, c_1 K_{S_1,T_1} + c_2 K_{S_2,T_2} + \ldots + c_r K_{S_r,T_r}) \leq \epsilon.$$

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- that each singular value of A is at most ϵn, i.e. its operator norm $\|A\| \leq \epsilon n$, or
- sets $S, T \subseteq [n]$ such that

$$\left| \sum_{i \in S, k \in T} a_{i,k} \right| \geq \frac{\epsilon^8}{100} n^2.$$

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Idea of testing along expanders first appeared in Kohayakawa-Rödl-Thoma paper.

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Sequence of iterative steps, $A = A_0, A_1, \cdots, A_s$.

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Set $A_{l+1} = A_l - t 1_S 1_T^{\mathbf{T}}$, $t = c\epsilon^8$. L^2-norm of each row and column cannot increase, and $\|A\|_{Fr}^2$ decreases by $\gtrsim \epsilon^{16} n^2$.

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Must end after $O(\epsilon^{-16})$ steps.

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Algorithmic problem

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How fast can we approximate the count within an additive $\epsilon n^{|V(H)|}$?

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Can be done in time $\epsilon^{-O_H(1)} n^2$.