Stationary curves under the Möbius-Plateau energy

Max Lipton and Gokul Nair

August 28, 2022

Abstract

Plateau problems with elastic boundary energy have been of recent theoretical and applied interest. However, avoidance of self-intersection on the boundary during the gradient descent requires strong assumptions on the initial conditions. To make progress towards an alternative to these strong assumptions, we introduce the novel Möbius-Plateau energy, where we instead use the repulsive Möbius energy first introduced by O'Hara, and later expounded upon by Freedman, He, and Wang. This new energy also has myriad scientific utility and presents interesting theoretical phenomena.

We first prove the existence of minimizing curves in the case of immersed discs bounded by knots, placing our work on solid footing. We then investigate the Möbius-Plateau energy of helicoidal strips, which are classified as screws or ribbons based on the orientations of the boundary helices. Through direct methods of the variational equation, we show that stable helicoidal screws are plentiful, whilst stable helicoidal ribbons impose strong constraints on their parameters: they must have high coiling, thin width in comparison to the coiling, and remain close to axis.

1 Problem Formulation

Denote the unit disk, by $D \subset \mathbb{R}^2$ and let $\gamma : \partial D \to \mathbb{R}^3$ be an simple closed curve. We define the *Möbius energy* [1, 2] as

$$E_M[\gamma] = \iint_{\partial D \times \partial D} \left[\frac{1}{|\gamma(y) - \gamma(x)|^2} - \frac{1}{D(\gamma(y), \gamma(x))^2} \right] |\dot{\gamma}(x)| \, |\dot{\gamma}(y)| \, dydx. \tag{1}$$

The Möbius energy blows up in self intersections, so the gradient descent will not induce self-intersections, an issue which can occur in Euler-Plateau problems requiring strong assumptions to avoid [3]. Furthermore, minimizers of the Möbius energy enjoy nice regularity properties. Freedman, He, and Wang proved the minimizers are $C^{1,1}$ [2], a result later strengthened to full C^∞ regularity by He [4].

For a fixed curve γ, we define the following class of surfaces that span the curve,

$$\mathcal{D}_\gamma := \{u : D \to \mathbb{R}^3 | \; u \in W^{1,2}(D) \cap C^0(\overline{D}), \; u : \partial D \to \gamma \text{ is monotone and onto}\}$$

The area of an immersion, $u \in \mathcal{D}_\gamma$ can be computed as

$$\text{Area}(u) = \int_D (|u_x|^2 \, |u_y|^2 - \langle u_x, u_y \rangle^2)^{\frac{1}{2}} dxdy \tag{2}$$

where u_x and u_y are the derivatives of u in x and y. We then have the following definition of the *Plateau energy*,

$$E_P[\gamma] = \inf_{u \in \mathcal{D}_\gamma} \text{Area}(u). \tag{3}$$

The problem of interest can be stated as follows,

Problem 1. *Find a closed curve γ that minimizes the Möbius-Plateau energy,*

$$E[\gamma] = E_M[\gamma] + E_P[\gamma]. \tag{4}$$

2 Energy Minimization

We define the following admissible class of closed curves for some irreducible knot-type K,

$$\mathcal{A} = \{\gamma \in C^{0,1}(\partial D, \mathbb{R}^3) |\ \gamma(\partial D) \text{ is simple closed, } \gamma \text{ is of knot type K, } \text{Length}(\gamma) = 2\pi,\ \gamma(0) = 0\}$$

Here, $C^{0,1}(\partial D, \mathbb{R}^3)$ denotes Lipschitz maps from $\mathbb{S}^1$ to $\mathbb{R}^3$. The constraint $\gamma(0) = 0$ and fixed length together ensure that the images of the curves in $\mathcal{A}$ are contained in some ball of fixed radius.

We also state some key results that are useful in the proof of the existence theorem,

Lemma 2 ([2] Lemma 1.2)**.** *Let $\gamma : X \to \mathbb{R}^3$ be a rectifiable curve in $\mathbb{R}^3$ parametrized by arc length. If $E_M[\gamma]$ is finite, then γ is C bi-Lipschitz with constant C depending only on $E_M[\gamma]$. Furthermore, $C \to 1$ as $E_M[\gamma] \to 0$.*

As noted in [2], subarcs of a curve with finite Möbius energy can be made to have arbitrarily small energy given that their length is sufficiently small, yielding the next useful corollary.

Lemma 3 ([2] Corollary 1.3)**.** *With γ as above such that $E_M[\gamma]$ is finite, we have that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, \gamma) > 0$ such that γ is a $(1+\varepsilon)$ bi-Lipschitz embedding on subarcs of γ with length less than δ.*

The property in this lemma is a weakened version of an isometric embedding. In proving the existence of minimizers for the Möbius-Plateau energy, we will need to ensure that arc length is preserved in the limit of a minimizing sequence. Were we to have the luxury of our curves being isometric embeddings of fixed length, the desired property would trivially follow. In general, total arc length is not preserved in sequences of curves, even in a uniformly converging limit. In particular, the length could shorten. However, it will turn out that this weaker property will be sufficient to prove that total length is preserved in uniform limits.

Lemma 4 ([2] Lemma 4.2)**.** *Let $\gamma_i : \mathbb{R}/L_i\mathbb{Z} \to \mathbb{R}^3$ be a sequence of rectifiable simple closed curves of uniformly bounded energy, E_M. Assume that the curves are all parametrized by arc-length and that $L = \lim_{i\to\infty} L_i > 0$ exists. If $\gamma_i(0)$ is a bounded sequence of points, then there is a subsequence γ_{i_k} of γ_i which converges locally uniformly to a rectifiable simple curve $\gamma : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$. Moreover, E_M is sequentially lower semicontinuous under this convergence, i.e.,*

$$E_M[\gamma] \leq \liminf_{i_k \to \infty} E_M[\gamma_{i_k}]$$

Theorem 5 ([2] Theorem 4.3)**.** *Let K be an irreducible knot. There exists a simple loop $\gamma_K : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$ with knot-type K such that $E_M[\gamma_K] \leq E_M[\gamma]$ for any other simple closed loop $\gamma : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$ of the same knot-type.*

Remark. We can make $E_M[\gamma]$ arbitrarily large by repeatedly introducing Reidemeister Type-I moves with strands sufficiently close to each other. Then by the continuous dependence of the energy integrand on γ, we can construct a sequence γ_i of knots with type K uniformly converging to a limiting knot γ_∞, with $E_M[\gamma_i] \to E_M[\gamma_\infty] = C$, with C being any fixed value such that $C \geq \inf\{E_M[\gamma] : \gamma \text{ has knot type } K\}$. The construction given in the proof of 5 still applies even when the limiting Möbius energy has changed, so when K is an irreducible knot type, we can assume the limit also has type K.

We begin the proof of existence by considering a minimizing sequence of E, $\{\gamma_k\} \subset \mathcal{A}$ i.e.,

$$\lim_{k\to\infty} E[\gamma_k] = \inf_{\gamma\in\mathcal{A}} E[\gamma].$$

This gives us a uniform bound on $E_M[\gamma_k]$, therefore, from Lemma 4 we have a subsequence (for which we will use the same labelling) that converges uniformly to a simple closed $C^{0,1}$ curve $\gamma_\infty : \partial D \to \mathbb{R}^3$. Furthermore, by applying the remark after Theorem 5 we can conclude that γ_∞ has the same knot-type K, even though the Möbius energy of the limit is not necessarily the infimum. It is also straightforward to conclude that $\gamma_\infty(0) = 0$ and apply Fatou's Lemma to see

$$\text{Length}(\gamma_\infty) = \int_0^{2\pi} |\dot\gamma_\infty(s)|\, ds \leq \liminf_{k\to\infty} \int_0^{2\pi} |\dot\gamma_k(s)|\, ds = \liminf_{k\to\infty}\ \text{Length}(\gamma_k) = 2\pi.$$

However, we must still show $\text{Length}(\gamma_\infty) \geq 2\pi$. Let $\varepsilon > 0$. Applying Lemma 3 we have that each γ_i is a $(1+\varepsilon)$ bi-Lipschitz embedding on subarcs of length $\leq \delta = \delta(i,\varepsilon)$. As the sequence converges locally uniformly, γ_∞ is also a $(1+\varepsilon)$ bi-Lipschitz embedding, and we can assume this δ is independent of i and also works for γ_∞. Now take a partition $0 = s_0 < s_1 < \cdots < s_N = 2\pi$ with $|s_{i+1} - s_i| < \delta$ and N large enough so that $N\delta > 2\pi$. By the bi-Lipschitz property, we have $|\gamma_\infty(s_{i+1}) - \gamma_\infty(s_i)| \geq (1+\varepsilon)^{-1}|s_{i+1} - s_i| > (1+\varepsilon)^{-1}\delta$. Summing this inequality over i yields $\text{Length}(\gamma_\infty) \geq (1+\varepsilon)^{-1}N\delta > (1+\varepsilon)^{-1}2\pi$. As ε was arbitrary, we conclude $\text{Length}(\gamma_\infty) \geq 2\pi$, as desired, so therefore $\gamma_\infty \in \mathcal{A}$.

We would like to now show that E is weakly lower semicontinuous with respect to uniform convergence $\gamma_k \to \gamma_\infty$. We already have lower semicontinuity in E_M. So it remains to show that E_P is lower semicontinuous, i.e.

$$E_P[\gamma_\infty] \leq \liminf_k E_P[\gamma_k] \tag{5}$$

Lower Semicontinuity of E_P

We define the Dirichlet energy of a mapping as

$$\mathrm{e}(u) = \int_D |\nabla u|^2\, dxdy.$$

Lemma 6. *Let $\gamma \in \mathcal{A}$, then*

$$\inf_{u\in\mathcal{D}_\gamma} \text{Area}(u) = \inf_{u\in\mathcal{D}_\gamma} \mathrm{e}(u)$$

Proof. Refer to [5] Lemma 4.4. $\square$

First we state the following version of the classical Plateau's problem,

Theorem 7 ([5] Theorem 4.1)**.** *Given a piecewise C^1 closed Jordan curve $\Gamma \subset \mathbb{R}^3$, there exists a map $u : D \to \mathbb{R}^3$ such that*

1. *$u : \partial D \to \Gamma$ is monotone and onto.*
2. *$u \in C^0(\overline{D}) \cap W^{1,2}(D)$*
3. *The image of u minimizes area among all maps from disks with boundary Γ.*

Applying Theorem 7 to each of the simple closed curves $\gamma_k(\partial D)$ $k = 1, 2, ..., \infty$ and using Lemma 6, we get a sequence of maps $u_k \in \mathcal{D}_{\gamma_k}$ such that

$$\mathrm{e}(u_k) = \mathrm{Area}(u_k) = \inf_{u \in \mathcal{D}_{\gamma_k}} \mathrm{Area}(u) = E_P[\gamma_k].$$

We can then rewrite (5) as

$$\mathrm{e}(u_\infty) \leq \liminf_k \mathrm{e}(u_k). \tag{6}$$

Equicontinuity at the Boundary

We first prove the following simple result,

Lemma 8. *Suppose $\{\mu_k : [a, b] \to \mathbb{R}^3\}$ is a sequence of continuous curves that converges uniformly, i.e. $\mu_k \to \mu_\infty$. Then the diameters of the images converges,* $\mathrm{diam}(\mu_k([a, b])) \to \mathrm{diam}(\mu_\infty([a, b]))$.

Proof. Choose $\epsilon > 0$ and let $N > 0$ be such that $\|\mu_k - \mu_\infty\|_{C^0} < \frac{\epsilon}{2}$ for $k > N$. Let $x, y \in [a, b]$ such that $|\mu_\infty(x) - \mu_\infty(y)| = \mathrm{diam}(\mu_\infty([a, b]))$. From uniform convergence we have, for any $s, t \in [a, b]$

$$|\mu_k(s) - \mu_\infty(t)|\,, \;\; |\mu_k(s) - \mu_\infty(t)| < \frac{\epsilon}{2}.$$

We have the following lower-bound for $\mathrm{diam}(\mu_k([a, b]))$,

$$\begin{aligned}
\mathrm{diam}(\mu_\infty([a, b])) &= |\mu_\infty(x) - \mu_\infty(y)| \\
&\leq |\mu_\infty(x) - \mu_k(x)| + |\mu_k(x) - \mu_k(y)| + |\mu_k(y) - \mu_\infty(y)| \\
&\leq \mathrm{diam}(\mu_k([a, b])) + \epsilon.
\end{aligned}$$

let $s, t \in [a, b]$, then we have

$$\begin{aligned}
|\mu_k(s) - \mu_k(t)| &\leq |\mu_k(s) - \mu_\infty(s)| + |\mu_\infty(s) - \mu_\infty(t)| + |\mu_\infty(t) - \mu_k(t)| \\
&\leq \mathrm{diam}(\mu_\infty([a, b])) + \epsilon.
\end{aligned}$$

Taking a max over s, t in the above inequality yields

$$\mathrm{diam}(\mu_k([a, b])) \leq \mathrm{diam}(\mu_\infty([a, b])) + \epsilon.$$

$\square$

We also state the Courant-Lebesgue lemma which is essential for proving equicontinuity of our maps at the boundary. First, for $p \in \overline{D}$ and $\rho > 0$, we define

$$C_\rho(p) = \{q \in \overline{D} : \ |p - q| = \rho\}.$$

Lemma 9 (Courant-Lebesgue)**.** *Let $u : D \to \mathbb{R}^3$ and $u \in C^0(\overline{D}) \cap W^{1,2}(D)$ with $\mathrm{e}(u) \leq K/2$ for some $K > 0$. Then, for all $\delta < 1$, there exist $\rho \in [\delta, \sqrt{\delta}]$ such that $\forall p \in \overline{D}$,*

$$[diam(u(C_\rho(p)))]^2 \leq \frac{8\pi^2 K}{-\log \delta}.$$

Proof. Refer [5] Lemma 4.11. □

For the rest of the text, fix three distinct points $p_1, p_2, p_3 \in \partial D$. For each k, we can find a unique conformal diffeomorphism $\varphi_k : D \to D$ such that

$$u_k \circ \varphi_k(p_i) = \gamma_k(p_i) \text{ for } i = 1, 2, 3. \tag{7}$$

Proposition 1. *The maps $\tilde{u}_k := u_k \circ \varphi_k$ are equicontinuous on ∂D.*

Proof. Since $\tilde{u}_k$ correspond to a minimizing sequence (and $\mathrm{e}[\cdot]$ is conformally invariant), we can assume that $\mathrm{e}(\tilde{u}_k) \leq K/2$ for some $K > 0$. Let $\epsilon > 0$ and without loss of generality assume that

$$\epsilon < \liminf_k \left(\min_{i \neq j} |\gamma_k(p_i) - \gamma_k(p_j)| \right),$$

which exists since $\gamma_k(\partial D)$ are all simple closed curves. Since γ_∞ is simple, closed and has finite arc-length, there is some $d_0 > 0$ such that for $p, q \in \partial D$ with $0 < |\gamma_\infty(p) - \gamma_\infty(q)| < d_0$, $\gamma_\infty(\partial D) \setminus \{p, q\}$ has exactly one component with diameter, $\mathrm{diam}_\infty \leq \frac{\epsilon}{2}$ [5, Proof of Lemma 4.14]. Call the closure of this component Γ. Let $N > 0$ such that $\|\gamma_k - \gamma_\infty\|_{C^0} < \frac{\epsilon}{4}$ for $k > N$. Then, from Lemma 8, the sub-arc of $\gamma_k(\partial D)$ parametrized by the same part of ∂D that parametrizes Γ has diameter (denoted diam_k) $\frac{\epsilon}{2}$-close to the diameter of Γ, i.e. $|\mathrm{diam}_k - \mathrm{diam}_\infty| < \frac{\epsilon}{2}$. Thus, $\mathrm{diam}_k < \epsilon$. It follows that $|\gamma_k(p) - \gamma_k(q)| < \frac{\epsilon}{2} + d_0$. Henceforth, $d := \frac{\epsilon}{2} + d_0$ and $k > N$.

Take $\delta < 1$ small enough such that $\frac{8\pi^2 K}{-\log \delta} < d^2$ and not more than one of the p_i's is in $B(p, \sqrt{\delta})$. Given any $p \in \partial D$, from Lemma 9, there exists $\rho \in [\delta, \sqrt{\delta}]$ such that

$$\mathrm{diam}(\tilde{u}_k(C_\rho(p)))^2 < \frac{8\pi^2 K}{-\log \delta} < d^2,$$

i.e., $\mathrm{diam}(\tilde{u}_k(C_\rho(p))) < d$. $C_\rho(p)$ divides ∂D into two components, A_1 and A_2. Without loss of generality, let A_1 be the component that contains fewer than 2 of p_1, p_2, p_3.

Denote the images, $G_1 = u(A_1)$ and $G_2 = u(A_2)$. Due to monotonicity, G_1 will also contain fewer than 2 of $\gamma_k(p_1), \gamma_k(p_2), \gamma_k(p_3)$. Since $\mathrm{diam}(\tilde{u}_k(C_\rho(p))) < d$, from Lemma 9 $\mathrm{diam}(G_i) < \epsilon$ for at least one of $i = 1, 2$. However, from the chosen value of ϵ, this component cannot contain more than 1 of $\gamma_k(p_1), \gamma_k(p_2), \gamma_k(p_3)$. Thus, the component has to be G_1.

In conclusion, we have shown that there is $\rho > 0$ such that

$$|p - q| < \rho \implies |\gamma_k(p) - \gamma_k(q)| < \epsilon.$$

□

Using the Arzela-Ascoli theorem, we therefore have $\|u_k - u_\infty\|_{C^0(\partial D)} \to 0$.

Convergence in the Interior

Since u_k minimizes the Dirichlet energy, we can show by taking a first variation that u_k satisfies the following Euler-Lagrange equation

$$\Delta u_k = 0 \text{ in } D \tag{8}$$

where Δ is the component-wise Laplacian. Using the maximum principle [6, Theorem 8.1], we have

$$\|u_k - u_\infty\|_{C^0(D)} \leq \|u_k - u_\infty\|_{C^0(\partial D)} \to 0$$

therefore $u_k \to u_\infty$ in $C^0(\overline{D})$. We can also get stronger convergences in the interior as follows:

Consider the weak-form of (8),

$$\int_D \nabla u \cdot \nabla v dxdy = 0 \quad \forall v \in W_0^{1,2}. \tag{9}$$

Take open sets $V \subset\subset W \subset\subset D$ and let $\varphi \geq 0$ be a bump function with spt $\varphi \subset W$ and $\varphi = 1$ on V. Set $v = \varphi^2 u$ in (9),

$$\int_D \nabla u \cdot (2\varphi \nabla\varphi \otimes u + \varphi^2 \nabla u) dxdy = 0$$

Since φ is smooth, we can bound the first term,

$$\left| \int_D 2\varphi \nabla u \cdot (\nabla\varphi \otimes u) dxdy \right| \leq \tilde{C} \int_D \varphi \left|\nabla u\right| \left|u\right| dxdy.$$

Thus we have

$$\int_D \varphi^2 \left|\nabla u\right|^2 dxdy \leq \tilde{C} \int_D \varphi \left|\nabla u\right| \left|u\right| dxdy$$

Let $\epsilon > 0$ and use Cauchy's inequality [7] on the right hand side to get

$$\int_D \varphi^2 \left|\nabla u\right|^2 dxdy \leq \tilde{C}\epsilon \int_D \varphi^2 \left|\nabla u\right|^2 + \frac{\tilde{C}}{4\epsilon} \int_D \left|u\right|^2 dxdy.$$

Choose $\epsilon = \frac{1}{2\tilde{C}}$ and re-arrange the terms to obtain

$$\int_V \left|\nabla u\right|^2 dxdy \leq \int_D \varphi^2 \left|\nabla u\right|^2 dx \leq \tilde{C}^2 \int_D \left|u\right|^2 dxdy.$$

We can also re-write this (by adding the L^2-norm on both the sides) as

$$\|u\|_{W^{1,2}(V)} \leq \tilde{\tilde{C}} \|u\|_{L^2(D)} \leq C \|u\|_{C^0(\overline{D})}.$$

for $V \subset\subset D$. From linearity, we have

$$\|u_k - u_\infty\|_{W^{1,2}(V)} \leq C \|u_k - u_\infty\|_{C^0(\overline{D})} \to 0.$$

Thus, $u_k \to u_\infty$ in $W^{1,2}_{\mathrm{loc}}(D)$ and consequently $|\nabla u_k|^2 \to |\nabla u_\infty|^2$ in $L^1_{\mathrm{loc}}(D)$. By taking a sequence of enlarging disks $D_{r_i} \subset\subset D$ with $r_i \nearrow 1$, extracting subsequences $\left|\nabla u_k^i\right|^2$ that converge a.e. in D_{r_i} and appropriate diagonalization, we can construct a subsequence $|\nabla u_k|^2$ (not relabelled) that converges a.e. in D. Fatou's lemma then implies that

$$\liminf_{k\to\infty} \int_D |\nabla u_k|^2 \, dxdy \geq \int_D |\nabla u_\infty|^2 \, dxdy,$$

which is equivalent to (5).

With this, we have established the lower semicontinuity of E, and $\gamma_\infty \in \mathcal{A}$ is an energy minimizer. We can thus state the main theorem as follows,

Theorem 10. *There exists $\gamma_\infty \in \mathcal{A}$ that minimizes E.*

3 Möbius-Plateau Stationary Helix Pairs

We now consider the general Möbius-Plateau energy problem, with energy $E = E(\alpha, \beta) = \alpha E_M + \beta E_P$ for $\alpha, \beta \in \mathbb{R}$ being physical constants. In the case of helicoidal ribbons, we will instead look at the Möbius energy of links, where $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{R}^3$ are disjoint smooth curves. The Möbius energy of the link is defined as

$$E_M(\gamma_1, \gamma_2) = \int_{\mathbb{R}\times\mathbb{R}} \frac{|\dot{\gamma_1}(u)||\dot{\gamma_2}(v)| dudv}{|\gamma_1(u) - \gamma_2(v)|^2}.$$

Note that in the case of a link, we do not subtract an intrinsic distance term in the integrand, as it is not needed for the integral to converge. At a given point $\gamma_1(u)$, the L^2 gradient of the Möbiusenergy is given by the vector-valued integral

$$G_{\gamma_1,\gamma_2}(u) = 2 \int_{\mathbb{R}} \left[\frac{2P_{\dot{\gamma_1}(u)^\perp}(\gamma_2(v) - \gamma_1(u))}{|\gamma_2(v) - \gamma_1(u)|^2} - \mathbf{N}_{\gamma_1(u)} \right] \frac{|\dot{\gamma_2}(v)| dv}{|\gamma_2(v) - \gamma_1(u)|^2}.$$

The derivation of the variational equation for the Möbius energy of knots is given in [2], and He [8] showed the equation also holds in the case of links. Here, $P_{\dot{\gamma_1}(u)^\perp}$ refers to the projection onto the plane normal to the tangent vector at $\gamma_1(u)$, and $\mathbf{N}_{\gamma_1(u)}$ refers to the normal vector in the Frenet frame along γ_1. The gradient is defined similarly along γ_2 by switching u and v.

Whilst the Plateau energy will necessarily be infinite, the variational equation describing critical parametrizations remains the same, describing when the force for mutual electric repulsion between the curves cancels with the attractive force between the curve which minimizes the area of the minimal surface bounded by the curves as efficiently as possible. Let $\boldsymbol{\nu}$ be the oriented unit surface normal to Σ. Now, the direction of the L^2 gradient of the Plateau energy at $\gamma_1(u)$ (resp. $\gamma_2(v)$) is the unit conormal $\mathbf{n} = -\mathbf{T} \times \boldsymbol{\nu}$ (resp. $\mathbf{T} \times \boldsymbol{\nu}$). This unit vector is oriented so that it is pointing away from the surface at each of the boundary components, which in turn defines the orientation of $\boldsymbol{\nu}$. The derivation of the variational equation for the Plateau energy is a standard calculation which can be found in [9]. Thus, the variational equation for the Möbius-Plateau energy

$$\alpha G_{\gamma_1,\gamma_2} = -\beta \mathbf{n}.$$

A general helicoid $\Sigma = \Sigma(\omega)$ has parametrization

$$\begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix} = \begin{bmatrix} s\cos(\omega t) \\ s\sin(\omega t) \\ t \end{bmatrix}. \tag{10}$$

The two boundary curves γ_1 and γ_2 of a helicoidal ribbon are found by setting $s = A$ and $s = B$ with $A < B$. We will use u and v as the parameters of the two curves.

Definition 1. *A* ***helicoidal screw*** *is the parametrized surface in* (10) *with* $A \leq s \leq B, t \in \mathbb{R}$*, where* $A < 0 < B$*. A* ***helicoidal ribbon*** *corresponds to the case where* $0 < A < B$*.*

Hence the space of helicoidal screws and ribbons are given by the three parameters of A, B, and ω. For given α and β, we seek to find the values of A, B, and ω which satisfy the variational equation of the Möbius-Plateau energy. The variational equations for stable helix pairs differs in the case of screws and ribbons, because of the differing orientations of $\mathbf{N}$, which we will see results in two trigonometric integrals that nominally look similar, but greatly vary in how the signs of their evaluations depend on their parameters.

The helicoid and the integrand of G_{γ_1,γ_2} is invariant under "screw" transformations which rotate the xy-plane by angle ωt whilst translating in the z direction by t. Hence, to calculate the entire gradient curves along γ_1 and γ_2, it suffices to compute them at two particular points along the boundary curves and then apply screw transformations.

Observe that

$$\dot{\gamma_1}(u) = \begin{bmatrix} -\omega A\sin(\omega u) \\ \omega A\cos(\omega u) \\ 1 \end{bmatrix},$$

and that $|\dot{\gamma_1}(u)| = \sqrt{\omega^2A^2+1}$ for all u. We also have $|\dot{\gamma_2}(u)| = \sqrt{\omega^2B^2+1}$ for all v. Furthermore,

$$\ddot{\gamma_1}(u) = \begin{bmatrix} -\omega^2A\cos(\omega u) \\ -\omega^2A\sin(\omega u) \\ 0 \end{bmatrix}.$$

Next, for given surface parameters s and t, the tangent plane to Σ is given by

$$\begin{aligned} T\Sigma &= \operatorname{span}\left\{ \partial_s \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \partial_t \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\} \\ &= \operatorname{span}\left\{ \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \\ 0 \end{bmatrix}, \begin{bmatrix} -\omega s\sin(\omega t) \\ \omega s\cos(\omega t) \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

So at $u = 0$, we have the following:

$$\gamma_1(0) = \begin{bmatrix} A \\ 0 \\ 0 \end{bmatrix}, \dot{\gamma_1}(0) = \begin{bmatrix} 0 \\ \omega A \\ 1 \end{bmatrix}, \ddot{\gamma_1}(0) = \begin{bmatrix} -\omega^2A \\ 0 \\ 0 \end{bmatrix}.$$

From this, we can see that $\mathbf{N}_{\gamma_1(0)} = (\pm 1, 0, 0)$, depending on the sign of A. We can also see that $T_{\gamma_1(0)}\Sigma = \operatorname{span}\{(1,0,0),(0,\omega A,1)\}$, and thus $\boldsymbol{\nu} = \frac{1}{\sqrt{\omega^2 A^2+1}}(0,\omega A,1)$. Hence, $\mathbf{n} = (-1,0,0)$.

Next, observe

$$\begin{aligned}
P_{\dot\gamma_1(0)^\perp}\big(\gamma_2(v)-\gamma_1(0)\big) &= \big(\gamma_2(v)-\gamma_1(0)\big) - \langle \gamma_2(v)-\gamma_1(0), \dot\gamma_1(0)\rangle \frac{\dot\gamma_1(0)}{|\dot\gamma_1(0)|^2} \\
&= \begin{bmatrix} B\cos(\omega v)-A \\ B\sin(\omega v) \\ v \end{bmatrix} - \frac{\omega AB\sin(\omega v)+v}{\omega^2A^2+1}\begin{bmatrix} 0 \\ \omega A \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} B\cos(\omega v)-A \\ \frac{B\sin(\omega v)-A\omega v}{\omega^2A^2+1} \\ v-\frac{v+\omega AB\sin(\omega v)}{\omega^2A^2+1} \end{bmatrix}.
\end{aligned}$$

Finally, it is straightforward to see that $|\gamma_2(v)-\gamma_1(0)|^2 = A^2 - 2AB\cos(\omega v) + B^2 + v^2$. Now, we will make the assumption that $A < 0 < B$, which means $\mathbf{N} = (1,0,0)$. Putting all of this together, the variational equation becomes:

$$2\alpha \int_{-\infty}^{\infty} \left[\frac{2\left(B\cos(\omega v)-A\right)}{A^2-2AB\cos(\omega v)+B^2+v^2} - 1\right] \frac{\sqrt{\omega^2B^2+1}}{A^2-2AB\cos(\omega v)+B^2+v^2}\,dv = \beta \tag{11}$$

$$\int_{-\infty}^{\infty} \frac{B\sin(\omega v) - \frac{v+\omega^2A^2B\sin(\omega v)}{\omega^2A^2+1}}{(A^2-2AB\cos(\omega v)+B^2+v^2)^2}\,dv = 0$$

$$\int_{-\infty}^{\infty} \frac{v - \frac{\omega AB\sin(\omega v)+v}{\omega^2A^2+1}}{\left(A^2-2AB\cos(\omega v)+B^2+v^2\right)^2}\,dv = 0.$$

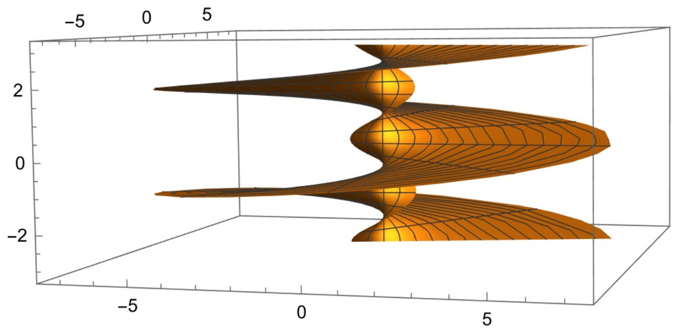
The last two integrands are odd functions in v, which means the integrals are always going to be zero, so only equation (11) relevant to us. Through identical computations by taking $v=0$, noting that in this case we have $\mathbf{N}_{\gamma_2(0)} = (-1,0,0)$ and $\mathbf{n} = (1,0,0)$, we see the first component of the variational equation at $\gamma_2(0)$ is

$$2\alpha \int_{-\infty}^{\infty} \left[\frac{2\left(A\cos(\omega u)-B\right)}{A^2-2AB\cos(\omega u)+B^2+u^2} + 1\right] \frac{\sqrt{\omega^2A^2+1}}{A^2-2AB\cos(\omega u)+B^2+u^2}\,du = -\beta. \tag{12}$$

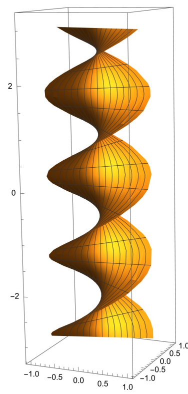
Relabelling the variable of integration and adding (11) and (12) yields

$$4\alpha \int_{-\infty}^{\infty} \frac{(A+B)(\cos(\omega v)-1)}{\left(A^2-2AB\cos(\omega v)+B^2+v^2\right)^2}\,dv = \beta\left(\frac{1}{\sqrt{\omega^2B^2+1}} - \frac{1}{\sqrt{\omega^2A^2+1}}\right). \tag{13}$$

This single equation leaves us with a two parameter family of stable helicoids controlled by the constants α and β. From direct inspection, it is clear that $A=-B$ is a solution, which we call the symmetric stable helicoidal screw. It is straightforward to numerically search for other solutions of (13). For instance, setting $\alpha=2, \beta=1, A=-1, \omega=2$, and solving for B gives solutions $B=1$ and $B \approx 6.15375$. The two helicoidal screws are pictured in 1. In the numerical searches we ran, we generally found that under most choices of parameters, the symmetric solution had a larger basin of attraction than the asymmetric solution. An analysis of the attraction properties of the solutions is yet to be done.



(a) A stable asymmetric helicoidal screw.



(b) A stable symmetric helicoidal screw.

Figure 1: Two stable helicoidal screws with identical parameters $\alpha = 2, \beta = 1, A = -1, \omega = 2$, except $B \approx 6.15375$ in the figure on the left, whilst $B = 1$ in the figure on the right.

The full description of this family, especially with respect to its singularities, is not fully known. When we tried to find solutions to the variational equation for helicoidal ribbons, derived below, we had difficulties finding solutions, and we suspected at first that no solutions existed. During our attempt to prove this lack of solutions, we had difficulty in bounding the negative contribution to the variational integral, and we later found that solutions do indeed exist, but only under comparatively strict conditions. Furthermore, the solutions we did manage to find have parameters which differ in orders of magnitude, which contributed to the difficulty in the numerical search.

Theorem 11. *If α and β have the same sign, then a stable helicoidal ribbon must have high frequency, small width, and remain close to the axis. That is, the following conditions on the parameters must hold:*

- $\omega > 1$
- $|B - A| < C\omega$, *with C being a constant proportional to $\frac{\beta}{\alpha}$*
- $|A + B| < 2$.

Proof. Without loss of generality assume α and β are positive and $0 < A < B$. We may also assume $\omega > 0$ because the cosine function is even. Were we to have $0 < A < B$, we would have that $\mathbf{N}_{\gamma_1(0)} = (-1, 0, 0)$, so the -1 term in (11) corresponding to $-\mathbf{N}$ becomes a $+1$ whilst (12) remains unchanged. If we add these two nontrivial variational equations at $\gamma_1(0)$ and $\gamma_2(0)$, we get

$$4\alpha \int_{-\infty}^{\infty} \frac{A^2 - A + B^2 - B + (A + B - 2AB)\cos(\omega v) + v^2}{\left(A^2 - 2AB\cos(\omega v) + B^2 + v^2\right)^2} dv = \beta \left(\frac{1}{\sqrt{\omega^2 B^2 + 1}} - \frac{1}{\sqrt{\omega^2 A^2 + 1}} \right). \tag{14}$$

We will show that except in a very limited circumstance, the integrand on the lefthand side of (14) is always positive. When the integrand can take negative values, we will then show that unless all of the conditions listed are met, then the integrand is positive. The righthand side of (14) is negative, as $B > A$, so we will get our desired result.

The denominator is obviously positive. As for the numerator, we break down into cases. Suppose $A + B - 2AB \leq 0$. Then

$$\begin{aligned} A^2 - A + B^2 - B + (A + B - 2AB)\cos(\omega v) + v^2 &\geq A^2 - A + B^2 + A + B - 2AB + v^2 \\ &= (B - A)^2 + v^2 > 0. \end{aligned}$$

For the remainder of the proof, assume $A + B - 2AB > 0$ and we will break down into further cases. Assume $A + B \geq 2$. Then

$$\begin{aligned} A^2 - A + B^2 - B + (A + B - 2AB)\cos(\omega v) + v^2 &\geq A^2 - A + B^2 - B - (A + B - 2AB) + v^2 \\ &= (A + B)(A + B - 2) + v^2 > 0. \end{aligned}$$

Next, assume $A + B < 2$ with $\omega \leq 1$. From the Taylor formula, $\cos(\omega v) \geq 1 - \frac{\omega^2 v^2}{2} \geq 1 - \frac{v^2}{2}$. Now see that

$$\begin{aligned} A^2 - A + B^2 - B + (A + B - 2AB)\cos(\omega v) + v^2 &\geq A^2 - A + B^2 - B + (A + B - 2AB)\left(1 - \frac{v^2}{2}\right) + v^2 \\ &= (B - A)^2 + \frac{v^2}{2}(2AB - A - B + 2) \\ &> \frac{v^2}{2}(2AB - A - B + 2) \\ &\geq \frac{v^2}{2}(2AB) \geq 0. \end{aligned}$$

So in all of the cases we have considered for A, B, and ω so far, the integrand in (14) is positive and so the equation will have no solutions.

Now, assume $A + B < 2$ with $\omega > 1$. It is only in this scenario that the numerator could take negative values and possibly lead to the integral in (14) being negative. For instance, when $A = 0.1, B = 1, \omega = 10$, the integrand takes negative values as shown in 2.

To finish the proof, we now assume $\omega < (B - A)$. As the integrand is an even function in v, it suffices to prove

$$\int_0^\infty \frac{A^2 - A + B^2 - B + (A + B - 2AB)\cos(\omega v) + v^2}{(A^2 - 2AB\cos(\omega v) + B^2 + v^2)^2} dv > 0.$$

Our method will be to overestimate the magnitude of the contribution of the negative part of the integrand whilst underestimating the contribution of the positive part of the integrand, and show that the positive contribution still outweighs the negative contribution. We will get a lower bound of the integral expressed as a rational function in ω, which we can easily verify is positive for all $\omega > 1$.

If $\cos(\omega v) \geq 0$, then

$$\begin{aligned} A^2 - A + B^2 - B + (A + B - 2AB)\cos(\omega v) + v^2 &\geq A^2 - A + B^2 - B + v^2 \\ &\geq -\frac{1}{2} + v^2. \end{aligned}$$

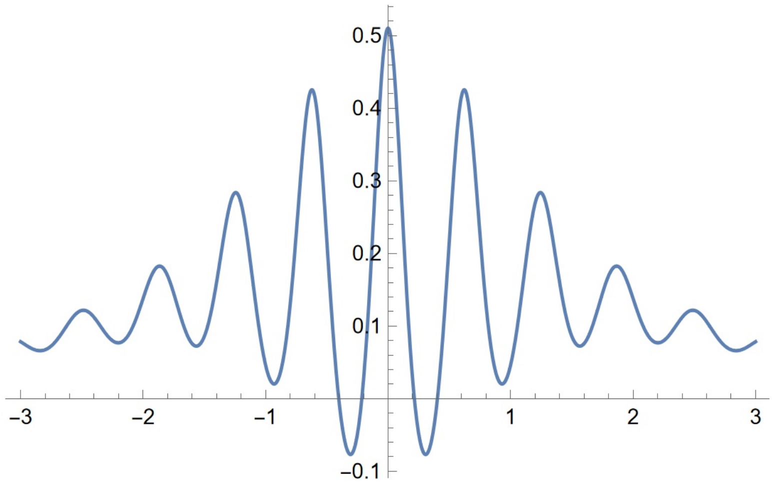


Figure 2: A graph of the integrand in (14) with $A = 0.1, B = 1$, and $\omega = 10$ as a function of v.

The last inequality follows from the fact that

$$\inf_{\substack{A+B<2 \\ 0<A<B}} A^2 - A + B^2 - B = -\frac{1}{2}. \tag{15}$$

Therefore, a necessary condition for the integrand to be negative whilst $\cos(\omega v) > 0$ is for $v \leq \frac{1}{4}$. Using a computer algebra system, one gets that the Macularin series of the integrand expanded at $v = 0$ is

$$\frac{A^2 - A + B^2 - B + (A + B - 2AB)\cos(\omega v) + v^2}{(A^2 - 2AB\cos(\omega v) + B^2 + v^2)^2} = \frac{1}{(B-A)^2} - \frac{2 + (A + B + 2AB)\omega^2}{2(B-A)^4} v^2 + O(v^4). \tag{16}$$

As the coefficient of the v^2 term is negative, we have

$$\frac{A^2 - A + B^2 - B + (A + B - 2AB)\cos(\omega v) + v^2}{(A^2 - 2AB\cos(\omega v) + B^2 + v^2)^2} \geq \frac{1}{(B-A)^2} - \frac{2 + (A + B + 2AB)\omega^2}{2(B-A)^4} v^2.$$

We will now integrate the quadratic Macluarin polynomial over $[0, \frac{1}{4}]$ to get a lower bound of the integral which includes all the points where the integrand could take negative values despite $\cos(\omega v) > 0$. Observe

$$\begin{aligned}
&\int_0^{\frac{1}{4}} \frac{A^2 - A + B^2 - B + (A + B - 2AB)\cos(\omega v) + v^2}{(A^2 - 2AB\cos(\omega v) + B^2 + v^2)^2} dv \\
&\geq \int_0^{\frac{1}{4}} \left[\frac{1}{(B-A)^2} - \frac{2 + (A + B + 2AB)\omega^2}{2(B-A)^4} v^2\right] dv \\
&= \frac{1}{4(B-A)^2} - \frac{2 + (A + B + 2AB)\omega^2}{384(B-A)^4}
\end{aligned}$$

$$\geq -\frac{2+(A+B+2AB)\omega^2}{384(B-A)^4}$$
$$\geq -\frac{1+\omega^2}{192(B-A)^4}$$
$$> -\frac{1}{96}. \tag{17}$$

In the third to last inequality, we use the fact that

$$\sup_{\substack{A+B<2\\0<A<B}} A+B-2AB = 2. \tag{18}$$

Furthermore, in the last two inequalities, we use the assumption that $(B-A) > \omega > 1$.

Notice that outside of the domain of integration in (17), we will have $v \geq \frac{1}{4}$, and so the integrand will be strictly positive should $\cos(\omega v) > 0$. In the remainder of the domain of integration, the integrand can only be negative provided that $\cos(\omega v) < 0$. In this instance, the numerator is bounded below by

$$\begin{aligned}A^2 - A + B^2 - B + (A+B-2AB)\cos(\omega v) + v^2 &\geq A^2 - A + B^2 - B - (A+B-2AB) + v^2\\ &= (A+B)(A+B-2) + v^2. \end{aligned} \tag{19}$$

Next, it is a straightforward optimization exercise to show that

$$\inf_{\substack{A+B<2\\0<A<B}} (A+B)(A+B-2) = -1.$$

Even though this infimum cannot be attained in the triangular domain of A and B, if we fix any choice of A and B, the quantity $A^2 - A + B^2 - B + (A+B-2AB)\cos(\omega v)$ attains its largest negative magnitude when $v = \frac{n\pi}{\omega}$, with n an odd integer, and this quantity is bounded below by -1 over all valid A and B. However, we have to add the positive term v^2 to get the total numerator, and so it is necessary for $v^2 < 1$ in order for the total numerator, and thus the integrand, to be negative.

The domains for v such that $\cos(\omega v) \leq 0$ are the intervals $\left[\frac{n\pi}{\omega} - \frac{\pi}{2\omega}, \frac{n\pi}{\omega} + \frac{\pi}{2\omega}\right]$, with n an odd integer. Each interval has length $\frac{\pi}{\omega}$. However, we also require $\left(\frac{n\pi}{\omega} - \frac{\pi}{2\omega}\right)^2 < 1$, . There are only finitely many of these intervals, which is the number of odd integers n satisfying $n < \frac{1}{2} + \frac{\omega}{\pi}$. Hence, there are no more than $\frac{1}{4} + \frac{\omega}{2\pi}$ such intervals. On the n^{th} interval, the positive v^2 term is bounded below by $\left(\frac{(2n+1)\pi}{2\omega} - \frac{\pi}{2\omega}\right)^2$, and $\left(\frac{\pi}{2\omega}\right)^2$ independent of n. Hence to bound the denominator's magnitude from below to overestimate the negative contribution, observe

$$\begin{aligned}A^2 - 2AB\cos(\omega v) + B^2 + v^2 &\geq A^2 + B^2 + v^2\\ &= (B-A)^2 + 2AB + v^2\\ &\geq \omega^2 + \left(\frac{\pi}{2\omega}\right)^2. \end{aligned} \tag{20}$$

Thus the magnitude of the denominator is bounded below by $\omega^4 + \frac{\pi^2}{2} + \frac{\pi^4}{16\omega^4}$. Therefore, the negative contribution to the integral when $\cos(\omega v) < 0$ is bounded below by the Riemann sum

$$\left(\frac{\pi}{\omega}\right)\sum_{n=0}^{\lceil\frac{1}{4}+\frac{\omega}{2\pi}\rceil}\left[\frac{-1+\left[\frac{(2n+1)\pi}{\omega}-\frac{\pi}{2\omega}\right]^2}{\omega^4+\frac{\pi^2}{2}+\frac{\pi^4}{16\omega^4}}\right] = \left(\frac{\pi}{\omega^5+\frac{\pi^2\omega}{2}+\frac{\pi^4}{16\omega^3}}\right)\sum_{n=0}^{\lceil\frac{1}{4}+\frac{\omega}{2\pi}\rceil}\left[-1+\frac{\pi^2}{4\omega^2}+\frac{2\pi^2}{\omega^2}n+\frac{4\pi^2}{\omega^2}n^2\right]$$
$$\geq \left(\frac{\pi}{\omega^5+\frac{\pi^2\omega}{2}+\frac{\pi^4}{16\omega^3}}\right)\left(\frac{1}{4}+\frac{\omega}{2\pi}\right)\left(-1+\frac{\pi^2}{4\omega^2}\right)$$
$$+\left(\frac{2\pi^3}{\omega^7+\frac{\pi^2\omega^3}{2}+\frac{\pi^4}{16\omega}}\right)\sum_{n=0}^{\lceil\frac{1}{4}+\frac{\omega}{2\pi}\rceil}\left(n+2n^2\right)$$
$$\geq \frac{\omega(\pi-2\omega)(\pi+2\omega)^2}{(\pi^2+4\omega^4)^2}$$
$$+\left(\frac{2\pi^3}{\omega^7+\frac{\pi^2\omega^3}{2}+\frac{\pi^4}{16\omega}}\right)\left[\frac{1}{2}\left(\frac{1}{4}+\frac{\omega}{2\pi}\right)\left(\frac{5}{4}+\frac{\omega}{2\pi}\right)\right.$$
$$\left.+\frac{1}{3}\left(\frac{1}{4}+\frac{\omega}{2\pi}\right)\left(\frac{5}{4}+\frac{\omega}{2\pi}\right)\left(\frac{3}{2}+\frac{\omega}{\pi}\right)\right]$$
$$= \frac{\omega(\pi-2\omega)(\pi+2\omega)^2}{(\pi^2+4\omega^4)^2}+\frac{2\omega(3\pi+\omega)(\pi+2\omega)(5\pi+2\omega)}{3\left(\pi^2+4\omega^4\right)^2}$$
$$= \frac{\omega(\pi+2\omega)(33\pi^2+22\pi\omega-8\omega^2)}{3\left(\pi^2+4\omega^4\right)^2}. \tag{21}$$

This lower bound is a rational function in ω of degree -4 dominated by the term $-\frac{1}{3\omega^4}$ in its Laurent expansion about $\frac{1}{\omega}$. It is straightforward to verify that (21) has a global minimum over all $\omega > 1$ of approximately -1.77×10^{-6} at $\omega \approx 14.8$. Now it is our goal to compute a lower bound on the positive contribution of the integrand, expressed as rational functions in ω, and verify the contributions of the positive parts exceeds the combined negative contributions of (17) and (21). As we saw, for $v \geq 1$, the integrand is strictly positive. So consider the contribution of the tail

$$\int_1^\infty \frac{A^2-A+B^2-B+(A+B-2AB)\cos(\omega v)+v^2}{(A^2-2AB\cos(\omega v)+B^2+v^2)^2}dv > 0.$$

We see that $A^2-2AB\cos(\omega v)+B^2+v^2 \geq (A+B)^2+v^2$. Combining this with (18), we can conclude that the denominator is bounded above by $(4+v^2)^2$. Likewise, we again observe by (19) that the numerator is bounded below by $(A+B)(A+B-2)+v^2$. Therefore, the numerator is bounded below by v^2-1. Putting all this together, we have

$$\int_1^\infty \frac{A^2-A+B^2-B+(A+B-2AB)\cos(\omega v)+v^2}{(A^2-2AB\cos(\omega v)+B^2+v^2)^2}dv \geq \int_1^\infty \frac{v^2-1}{(4+v^2)^2}dv$$
$$= \frac{1}{32}\left(4+3\pi-6\operatorname{arccot}(2)\right)$$
$$\geq \frac{3}{10}. \tag{22}$$

By adding our bounds from (17),(21), and (22), we can conclude the integral is positive. $\square$

The assumption that $(B-A) > \omega$ was crucial in our proof, as we otherwise would not be able to obtain the bound in (17), as the coefficient $-\frac{1}{(B-A)^4}$ tends to $-\infty$ as $(B-A) \to 0^+$, which makes the bound useless in limiting the contribution of the negative parts of the integrand. We also used this assumption in (20) in order to force the Riemann sum approximation to be of order $O(\omega^{-4})$. Were we to naively make the use the inequality $A^2 + B^2 + v^2 \geq 0 + \left(\frac{\pi}{2\omega}\right)^2$, our Riemann sum approximation would be of order $O(\omega^4)$ with a leading negative coefficient, which tends to $-\infty$ as $\omega \to \infty$, which would again prevent us from bounding the negative contribution.

One can find numerical solutions to (14), though in practice, we found that the relationship between ω and $B-A$ is differs by a few orders of magnitude. Indeed, numerically solving (14) for ω given $\alpha = \beta = 1, A = 0.001, B = 0.002$ results in $\omega \approx 37.0171$.

So in contrast with the case of helicoidal screws, whose variational equation has solutions under mild assumptions, helicoidal ribbons can only be stable under the Möbius-Plateau energy under strict assumptions on the parameters.

Acknowledgements

The authors would like to thank Xin Zhou for his numerous helpful discussions with this work, and in particular making the initial suggestion to consider the Möbius-Plateau energy. The authors would like to thank Steven Strogatz for his helpful comments.

This work is partially funded by an NSF RTG grant entitled Dynamics, Probability and PDEs in Pure and Applied Mathematics, DMS-1645643.

Appendices

A The Plateau Energy is Continuous

Given a C^2 knot γ, the Plateau energy $E_P(\gamma)$ is defined as the area of the minimal immersed disc with boundary γ. One possible difficulty in studying the Plateau energy is that minimal surfaces are nonlocal with respect to perturbations of the boundary. That is, introducing a perturbation of γ on an arbitrarily small neighborhood can and will perturb the minimal surface everywhere, and not just on a neighborhood of the perturbation. Even though the minimal surfaces must lie in the convex hulls of the perturbed knots, the areas of said minimal surfaces could still jump discontinuously. However, this is not the case.

Lemma 12. *Fix a knot C^2 knot γ and let η be a C^2 curve from S^1 to $\mathbb{R}^3$. Consider the family of perturbed knots $\gamma_\varepsilon = \gamma + \varepsilon\eta$ for $\varepsilon \geq 0$. Then $E_P(\gamma_\varepsilon) \to E_P(\gamma)$ as $\varepsilon \to 0$.*

Proof. Observe that the union of the convex hulls of $\{\gamma_\varepsilon : 0 \leq \varepsilon \leq 1\}$ is bounded, and hence all the minimal surfaces are contained in a bounded set. For a given γ_ε, we can parametrize the minimally immersed disc with boundary γ_ε by a smooth function $u_\varepsilon : D^2 \to \mathbb{R}^3$, with D^2 being the closed unit disc and $u_\varepsilon|_{\partial D^2}$ mapping onto γ_ε monotonically. Writing u_ε as a triplet or real-valued functions $u_\varepsilon = (u_{\varepsilon,1}, u_{\varepsilon,2}, u_{\varepsilon,3})$, we have that each of these three component functions must be harmonic, and thus the real part of holomorphic functions on the unit disc. Furthermore, we can assume these maps are almost conformal, which means $\left|\frac{\partial}{\partial x}u_\varepsilon\right| = \left|\frac{\partial}{\partial y}u_\varepsilon\right|$ and $\langle \frac{\partial}{\partial x}u_\varepsilon, \frac{\partial}{\partial y}u_\varepsilon\rangle = 0$ for all ε.

As each $u_{\varepsilon,i}$ is uniformly bounded for $\varepsilon \in [0,1]$, so are their holomorphic counterparts. Hence by Montel's Theorem, these holomorphic functions converge uniformly on compact subsets to a holomorphic limit as $\varepsilon \to 0$. Hence, the convergence of $u_\varepsilon \to u_0$ is uniform via real-valued harmonic functions. By the Harnack interior gradient estimates, we also have that all derivatives of u_ε converge uniformly on compact subsets, and in particular we have convergence in L^1. We can therefore conclude

$$\begin{aligned}
|E_P(\gamma) - E_P(\gamma_\varepsilon)| &= \left| \int_{D^2} \left\| \frac{\partial u_0}{\partial x} \times \frac{\partial u_0}{\partial y} \right\| dxdy - \int_{D^2} \left\| \frac{\partial u_\varepsilon}{\partial x} \times \frac{\partial u_\varepsilon}{\partial y} \right\| dxdy \right| \\
&= \left| \int_{D^2} \left| \frac{\partial u_0}{\partial x} \right|^2 dxdy - \int_{D^2} \left| \frac{\partial u_\varepsilon}{\partial x} \right|^2 dxdy \right| \\
&\leq C \int_{D^2} \left| \frac{\partial u_\varepsilon}{\partial x} - \frac{\partial u_0}{\partial x} \right| dxdy \to 0,
\end{aligned}$$

where the last inequality follows from the fact $u \mapsto |u|^2$ is Lipschitz on D^2.

□

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Max Lipton, Department of Mathematics, Cornell University
E-mail address: ml2437@cornell.edu

Gokul Nair, Center for Applied Mathematics, Cornell University
E-mail address: gn234@cornell.edu