18.05 Solution 7

① 20.2

(a) $MSE(T) = (E(T)-\theta)^2 + Var(T) = 9+4 = 13$

$MSE(S) = (E(S)-\theta)^2 + Var(S) = 40$

so we prefer T.

(b) Similarly, $MSE(T) = a^2+4$, $MSE(S) = 40$

When $|a| < 6$, we prefer T

when $|a| > 6$, we prefer S.

② 21.5

(a) The likelihood function is

$$L(\mu) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

so $l(\mu) = \log L(\mu) = -\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2 - n\log(\sqrt{2\pi})$

Let $\frac{d\,l(\mu)}{d\mu} = 0$, we can find the MLE of μ

$$\hat{\mu}_{MLE} = \frac{x_1+\cdots+x_n}{n}$$

(b)

$L(\sigma) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \sigma^{-n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}x_i^2}$, so

$l(\sigma) = \log L(\sigma) = -n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}x_i^2 + n\log\left(\frac{1}{\sqrt{2\pi}}\right)$

$\frac{d\,l(\sigma)}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}x_i^2$.

Let $\frac{d\,l(\sigma)}{d\sigma} = 0$, we know that $\hat{\sigma}_{MLE} = \sqrt{\frac{x_1^2+\cdots+x_n^2}{n}}$

③ 21.6

(a) Likelihood.

$$L(\delta) = \prod_{i=1}^{n} f_\delta(x_i) = \prod_{i=1}^{n} e^{-(x_i-\delta)} \cdot I(x_i \geq \delta)$$

$$= e^{n\delta - \sum_{i=1}^{n} x_i} \cdot I(\delta \leq \min\{x_1, \cdots, x_n\})$$

(b) $e^{n\delta - \sum_{i=1}^{n} x_i}$ is an increasing function of δ

So $L(\delta)$ is increasing in δ when $\delta \leq \min\{x_1, \cdots, x_n\}$

ad $L(\delta) = 0$ when $\delta > \min\{x_1, \cdots, x_n\}$

Hence $\hat{\delta}_{MLE} = \min\{x_1, \cdots, x_n\}$

④ 21.15

Likelihood: $L(a,b) = \binom{6}{2}\left(\frac{e^{a+53b}}{1+e^{a+53b}}\right)^2 \left(\frac{1}{1+e^{a+53b}}\right)^4 \binom{6}{1}\left(\frac{e^{a+57b}}{1+e^{a+57b}}\right)$

$\left(\frac{\cancel{e^{a+57b}}\,1}{1+e^{a+57b}}\right)^5 \cdots\cdots \quad \binom{6}{0}\left(\frac{e^{a+81b}}{1+e^{a+81b}}\right)^0 \left(\frac{1}{1+e^{a+81b}}\right)^6$

log likelihood:

$$\ell(a,b) = \log L(a,b) = \sum_t \left[\log\binom{6}{N_t} + N_t \log\left(\frac{e^{a+bt}}{1+e^{a+bt}}\right) + (6-N_t) \log\left(\frac{1}{1+e^{a+bt}}\right) \right]$$