18.05 homework 5 solution

1 (11.2)

(a) $$P(X+Y=k) = \sum_{i=0}^{k} P(X=i, Y=k-i)$$

X and Y are independent:

$$\begin{aligned}
P(X+Y=k) &= \sum_{i=0}^{k} P(X=i)\,P(Y=k-i) \\
&= \sum_{i=0}^{k} \frac{1^i}{i!} e^{-1} \frac{1^{k-i}}{(k-i)!} e^{-1} \\
&= \sum_{i=0}^{k} \frac{e^{-2}}{i!(k-i)!} \\
&= \sum_{i=0}^{k} e^{-2} \frac{1}{k!} \binom{k}{i} \\
&= \frac{2^k}{k!} e^{-2}
\end{aligned}$$

(b) $$P(X+Y=k) = \sum_{i=0}^{k} P(X=i, Y=k-i)$$

X and Y are independent:

$$\begin{aligned}
P(X+Y=k) &= \sum_{i=0}^{k} P(X=i)\,P(Y=k-i) \\
&= \sum_{i=0}^{k} \frac{\lambda^i}{k!} e^{-\lambda} \frac{\mu^{k-i}}{(k-i)!} e^{-\mu} \\
&= \sum_{i=0}^{k} \frac{\lambda^i \mu^{k-i}}{i!(k-i)!} e^{-(\lambda+\mu)} \\
&= e^{-(\lambda+\mu)} \sum_{i=0}^{k} \frac{\left(\frac{\lambda}{\lambda+\mu}\right)^i \left(\frac{\mu}{\lambda+\mu}\right)^{k-i} (\lambda+\mu)^k}{i!(k-i)!} \\
&= e^{-(\lambda+\mu)} \sum_{i=0}^{k} \binom{k}{i} \left(\frac{\lambda}{\lambda+\mu}\right)^i \left(1-\frac{\lambda}{\lambda+\mu}\right)^{k-i} \frac{(\lambda+\mu)^k}{k!} \\
&= \frac{(\lambda+\mu)^k}{k!} e^{-(\lambda+\mu)}
\end{aligned}$$

2 (11.4)

(a) $E[Z] = E[3X-2Y+1]$

$= 3E[X] - 2E[Y] + 1$

$= 3\times 2 - 2\times 5 + 1$

$= -3$

$Var(Z) = Var(3X-2Y+1)$

$= 9\,Var(X) + 4\,Var(Y)$

$= 9\times 5 + 4\times 9$

$= 81$

(b) Because X and Y are independent, Z is also normal.

$Z \sim N(-3, 81)$

(c) Let $U = \frac{Z+3}{9}$

$U \sim N\left(\frac{-3+3}{9}, \frac{81}{9^2}\right) = N(0,1)$ is standard normal.

$P(Z \le 6) = P(U \le 1)$

$= 1 - 0.1587$

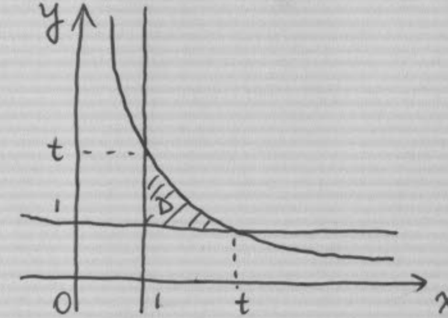
$= 0.8413$

3 (11.9)

(b) $X \sim Par(\alpha) \Rightarrow f_X(x) = \frac{\alpha}{x^{\alpha+1}} \quad (x \geq 1)$

$Y \sim Par(\beta) \Rightarrow f_Y(y) = \frac{\beta}{y^{\beta+1}} \quad (y \geq 1)$

We have



$$F_Z(t) = P(Z \leq t)$$
$$= P(xy \leq t)$$
$$= \iint_D f_X(x) f_Y(y)\, dx\, dy$$

where

$$D = \{ (x,y) \mid xy \leq t,\ x \geq 1,\ y \geq 1 \}$$

$$F_Z(t) = \int_1^t dy \int_1^{\frac{t}{y}} dx\, f_X(x) f_Y(y)$$

$$\Rightarrow \quad f_Z(t) = \frac{d}{dt} F_Z(t) = \frac{d}{dt} \int_1^t dy \int_1^{\frac{t}{y}} dx\, f_X(x) f_Y(y)$$

Since $\frac{d}{dt} \int^t g(t,y)\, dy = g(t,t) + \int^t \frac{\partial g(t,y)}{\partial t} dy$

we have

$$f_Z(t) = \int_1^{\frac{t}{t}} dx\, f_X(x) f_Y(t) + \int_1^t dy\, f_Y(y) \frac{\partial}{\partial t} \int_1^{\frac{t}{y}} dx\, f_X(x)$$
$$= 0 + \int_1^t dy\, f_Y(y) \frac{1}{y} f_X\left(\frac{t}{y}\right)$$
$$= \int_1^t dy\, \frac{1}{y} f_X\left(\frac{t}{y}\right) f_Y(y)$$
$$= \int_1^t dy\, \frac{1}{y} \frac{\alpha}{\left(\frac{t}{y}\right)^{\alpha+1}} \frac{\beta}{y^{\beta+1}}$$
$$= \frac{\alpha\beta}{t^{\alpha+1}} \int_1^t dy\, y^{\alpha-\beta-1}$$

If $\alpha = \beta$ $\quad \int_1^t dy\, y^{\alpha-\beta-1} = \ln t$

$$f_Z(t) = \frac{\alpha^2}{t^{\alpha+1}} \ln t \quad (t \geq 1)$$

If $\alpha \neq \beta$ $\quad \int_1^t dy\, y^{\alpha-\beta-1} = \frac{t^{\alpha-\beta}}{\alpha-\beta} - \frac{1}{\alpha-\beta}$

$$f_Z(t) = \frac{\alpha\beta}{\alpha-\beta} \left(\frac{1}{t^{\beta+1}} - \frac{1}{t^{\alpha+1}} \right) \quad (t \geq 1)$$

(a) $\alpha = 3.\ \beta = 1.$

$$f_Z(t) = \frac{3}{2} \left(\frac{1}{t^2} - \frac{1}{t^4} \right) \quad (t \geq 1)$$

4. (13.9)

(a) T_n is the average of n independent random variables X_i^2

Let $a = E[X_i^2]$ $\quad$ $\sigma^2 = E[X_i^4] - E[X_i^2]^2$

If a and σ^2 are finite, according to the law of large numbers

$$\lim_{n \to \infty} P\left(|T_n - a| > \varepsilon\right) = 0$$

(b) $$f_X(x) = \begin{cases} \frac{1}{2}, & x \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} a &= E[X^2] \\ &= \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx \\ &= \int_{-1}^{1} x^2 \frac{1}{2}\, dx \\ &= \boxed{\frac{1}{3}} \end{aligned}$$

$$\begin{aligned} \sigma^2 &= E[X^4] - E[X^2]^2 \\ &= \int_{-1}^{1} x^4 \frac{1}{2}\, dx - \left(\frac{1}{3}\right)^2 \\ &= \frac{1}{5} - \frac{1}{9} \\ &= \frac{4}{45} \end{aligned}$$

Because a and σ^2 are finite, the law of large numbers hold.