Decoupling seminar – problem set after lectures 5-6

We have been studying using the multilinear restriction estimate to prove decoupling estimates. Here are two problems to think about. These problems fill in steps from the argument in lecture.

1. In the proof of the first decoupling theorem in class, we used the following result as part of the inductive argument covering the narrow contribution.

Suppose that we have a decoupling estimate in $\mathbb{R}^{n-1}$. If $g : \mathbb{R}^{n-1} \to \mathbb{C}$ and supp $\hat{f} \subset N_{1/R} S^{n-2}$, for a certain $p \geq 2$, and if θ denote $R^{-1/2}$ caps covering $N_{1/R} S^{n-1}$, then

$$\|g\|_{L^p_{avg}(B_R^{n-1})} \leq M \left(\sum_\theta \|g_\theta\|^2_{L^p_{avg}(B_R^{n-1})} \right)^{1/2}.$$

Suppose that $f : \mathbb{R}^n \to \mathbb{C}$ and that supp $\hat{f} \subset N_{1/R} S^{n-1} \cap N_{R^{-1/2}} S^{n-2}$. Prove that we get the following similar decoupling estimate for f:

$$\|f\|_{L^p_{avg}(B_R^n)} \leq CM \left(\sum_\theta \|f_\theta\|^2_{L^p_{avg}(B_R^n)} \right)^{1/2}.$$

Here θ denote $R^{-1/2}$ caps covering $N_{1/R} S^{n-1} \cap N_{R^{-1/2}} S^{n-2}$.

Here is a hint to get started. Choose coordinates so that $S^{n-2} \subset S^{n-1}$ is defined by the equation $\omega_n = 0$. Consider the plane $x_n = h$, and define $f_h : \mathbb{R}^{n-1} \to \mathbb{C}$ by

$$f_h(x_1, ..., x_{n-1}) := f(x_1, ..., x_{n-1}, h).$$

Next check that $\hat{f}_h$ has support in $N_{10/R} S^{n-2}$. So our first estimate applies to f_h. Choose caps θ covering $N_{10/R} S^{n-2}$. Cover $N_{1/R} S^{n-1} \cap N_{R^{-1/2}} S^{n-2}$ with corresponding caps θ. If we do this carefully, then $f_{\theta,h}$ and $f_{h,\theta}$ will correspond. So we will get for each h,

$$\|f_h\|_{L^p_{avg}(B_R^{n-1})} \leq M \left(\sum_\theta \|f_{\theta,h}\|^2_{L^p_{avg}(B_R^{n-1})} \right)^{1/2}.$$

Next integrate in h and use Minkowski's inequality to control $\|f\|_{L^p_{avg}(B_R^n)}$.

(1'. If S is a positively curved compact C^2 surface, but not a sphere, find the right analogue of an equator, and prove a generalization of this result.)

2. In class, we proved a decoupling theorem for $2 \leq p \leq \frac{2n}{n-1}$. A similar argument shows that the linear decoupling problem is essentially equivalent to a multilinear decoupling problem.

Define $D_{n,p}(R)$ to be the smallest constant so that the following decoupling inequality holds:

If supp $\hat{f} \subset N_{1/R}S = \cup\theta$, where θ are disjoint $R^{-1/2}$-caps, then

$$\|f\|_{L^p_{avg}(B^n_R)} \leq D_{n,p}(R) \left(\sum_\theta \|f_\theta\|^2_{L^p_{avg}(\mu_{B_R})} \right)^{1/2}.$$

Define $\tilde{D}_{n,p}(R)$ to be the smallest constant so that the following multilinear decoupling inequality holds:

Whenever

- For $i = 1, ..., n$, supp $\hat{f}_i \subset N_{1/R}S_i$
- $S_i \subset \mathbb{R}^n$ are compact positively curved C^2 hypersurfaces (with implicit fixed bounds on the curvature and the second fundamental form).
- The normal vector to S_i at any point makes an angle of at most $(10n)^{-1}$ with the i^{th} coordinate axis.

Then we have the following inequality:

$$\| \prod_{i=1}^n |f_i|^{1/n} \|_{L^p_{avg}(B_R)} \leq \tilde{D}_{n,p}(R) \prod_{i=1}^n \left(\sum_\theta \|f_{i,\theta}\|^2_{L^p_{avg}(\mu_{B_R})} \right)^{\frac{1}{2n}}.$$

Prove the following result connecting linear decoupling and multilinear decoupling. (You should use the standard white lies and ignore the weights, at least on the first version.)

Theorem 1. *(Bourgain-Demeter) Suppose $D_{n-1,p}(R) \lesssim_\delta R^\delta$ for all $\delta > 0$. Then for any $\epsilon > 0$,*

$$D_{n,p}(R) \lesssim_\epsilon R^\epsilon \tilde{D}_{n,p}(R).$$

Hint: Decompose $N_{1/R}S$ into K^{-1}-caps τ for $K = R^\epsilon$. Cover B_R by cubes Q of side length K. Define significant indices for each cube, broad cubes, and narrow cubes. Control the contribution of the broad region by using the multilinear decoupling inequality. (It may require a coordinate change to move the $100nK^{-1}$-transverse caps into a position where $\tilde{D}_{n,p}$ applies, but this will introduce a factor of the form K^C.) Control the contribution of the narrow region by bringing in $\tilde{D}_{n-1,p}(K)$ on each cube Q.

We will use this Theorem in the proof of the sharp l^2-decoupling conjecture.