The Polynomial Method, Fall 2012, Problem Set 1

Problem Set 1 is due on Friday September 28 in class.

1. Introductory email. Send me an email introducing yourself. Let me know about your background and about your interests. If you could send me the introductory email by next Wednesday (Sep. 19), that would be great. I'll use these to put together a class email list, which I'll distribute to everybody. Even if you're just auditing the course, you're encouraged to send an email and introduce yourself (and I'll put you on the list).

2. Give a short summary of the proof for each of the main examples of the polynomial method that we've seen: the Berlekamp-Welch algorithm, finite field Nikodym and/or Kakeya, and the joints problem. The summary should be shorter than the whole proof. Since these proofs are already pretty short, probably 1-2 sentences is a good length for each summary. But you should try to include whatever seems to you to be the most essential points and/or the points that you would be most likely to forget. You might imagine that in a few weeks, you would try to reconstruct the proofs just based on these outlines. What is the key information that you should record for yourself?

3. Better estimates for finite field Kakeya. In this problem, we will look for (non-polynomial method) lower bounds for the size of a Kakeya set in a finite field that improve on the bush estimate. Let $K \subset \mathbb{F}_q^n$ be a Kakeya set. Write $K = \bigcup L_\alpha$, where $\{L_\alpha\}$ are lines in each direction. The hairbrush of a line $L \subset K$ is the union of all the lines in the Kakeya set which intersect L. Modify the bush argument, using hairbrushes in place of bushes. Prove that each Kakeya set $K \subset \mathbb{F}_q^3$ contains $\gtrsim q^{2+\epsilon}$ points for some $\epsilon > 0$. What is the best ϵ you can get? Also, what estimates do you get in higher dimensions?

Remark. We observed in class that q^2 distinct lines fit in a plane in $\mathbb{F}_q^3$. Therefore, to prove that a Kakeya set in $\mathbb{F}_q^3$ has $\gtrsim q^{2+\epsilon}$ points, we need to use that the lines of a Kakeya set point in different directions. How do you use this information in your argument? What is the weakest hypothesis you need for your argument to run?

4. Consider the map $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by

$$\gamma(t) = (t^{17} + t^5 + 3, t^{14} + t^9 - 7t^2 - 1).$$

Prove that there is some non-zero polynomial $P(x, y)$ so that image of γ is contained in the zero-set of P. Can you give some estimate for the degree of P?

5. Variant of finite field Nikodym theorem. Let γ_a be polynomials of degree $\leq d$ from $\mathbb{F}_q$ to $\mathbb{F}_q^{n-1}$. Let Γ_a be the graph of γ_a, which is a curve in $\mathbb{F}_q^n$. To be clear, $\Gamma_a = \{(t, \gamma_a(t)) \in \mathbb{F}^n | t \in \mathbb{F}\}$. The index a takes values in $\mathbb{F}^{n-1}$, and suppose that for each a, $\gamma_a(0) = a$. We have q^{n-1} curves $\Gamma_a \subset \mathbb{F}^n$, each curve containing q points. Prove that the union of the Γ_a contains at least $c(d, n)q^n$ points, for some constants $c(d, n) > 0$.

Optional. If you want to look further into the problem, what dependence on d do you get? Is it sharp in examples?