THE REGULUS DETECTION LEMMA

In this lecture we prove the regulus detection lemma, our last result about incidence of lines in $\mathbb{R}^3$.

Regulus detection lemma. *For any polynomial P in $\mathbb{R}[x_1, x_2, x_3]$, we can associate a list of polynomials RP with the following properties.*

(1) $DegRP \le CDegP$.
(2) If x is contained in two lines in $Z(P)$, then $RP(x) = 0$.
(3) If P is irreducible and RP vanishes on $Z(P)$, and if there is a non-special point x contained in two lines in $Z(P)$, then $Z(P)$ is a regulus.

The key fact about a regulus that we will use is that it is doubly ruled. A surface $Z(P)$ is called doubly ruled if each point of $Z(P)$ lies in two distinct lines in $Z(P)$. The regulus and the plane are the only irreducible doubly ruled algebraic surfaces in $\mathbb{R}^3$ (as we will prove). The job of the polynomial RP is roughly to detect whether there are two distinct directions in which the polynomial P vanishes to high order. (However, there is no polynomial that would do exactly this job. We discuss the problems below and we will see that RP almost does the job.)

Our first task is to define RP. Suppose that $v = (v_1, v_2, v_3)$. Suppose that $Q_s(v)$ is a homogenenous polynomial of degree s, for $s = 1, 2, 3$. Let I be the ideal generated by Q_1, Q_2, Q_3 in $\mathbb{R}[v]$. Recall that $I_{=3}$ denotes the homogeneous degree 3 polynomials in I and that $H_{=d}$ denotes the homogeneous degree d polynomials in $\mathbb{R}[v]$.

Lemma 0.1. *The set $\{(Q_1, Q_2, Q_3) \in H_{=1} \times ... \times H_{=3} | dimI_{=3} \le 8\}$ is an algebraic set. It is equal to $Z(R)$, where R is a finite list of polynomials in the coefficients of the Q_s. Each polynomial in R has degree ≤ 9.*

(This is just a special case of a previous lemma. Our set is given by the vanishing of some 9×9 subdeterminants of a multiplication matrix, whose coefficients are coefficients of Q_s.)

We define $Q_{s,x}(v) = \nabla_v^s P(x) = \sum_{|I|=s} I! \partial_I P(x) v^I$, a homogenenous polynomial in v of degree s. The coefficients of $Q_{s,x}(v)$ are polynomials in x of degree $\le degP$. We let RP be $R(Q_{1,x}, Q_{2,x}, Q_{3,x})$. Therefore, RP is a finite list of polynomials of degree $\le 9degP$. We have now checked property 1 of the regulus detection lemma.

We let $I(x)$ be the ideal generated by $Q_{1,x}, Q_{2,x}, Q_{3,x}$. We have $RP(x) = 0$ if and only if $I(x)_{=3}$ has dimension ≤ 8.

The next task is to discuss the geometric meaning of the condition $dimI(x)_{=3} \le 8$. The most important fact is contained in the following lemma.

Lemma 0.2. *Suppose that x is a regular point of $Z(P)$. Suppose that $\nabla^2 P(x) : T_x Z \times T_x Z \to \mathbb{R}$ has signature $(1,1)$. In this case there are two linearly-independent directions, $\nu_1, \nu_2 \in TZ$ so that $\nabla^2_{\nu_i} P(x) = 0$. Given these assumptions,*

$$RP(x) = 0 \text{ if and only if } \nabla^3_{\nu_1} P(x) = \nabla^3_{\nu_2} P(x) = 0.$$

This lemma says that under some fairly mild conditions, RP detects whether there are two linearly independent vectors which solve the equations $0 = \nabla^s_\nu P(x)$ for $s = 1, 2, 3$.

Proof. We start by understanding the ideal $I_{1,2}$ generated by $Q_{1,x}$ and $Q_{2,x}$. We claim that $I_{1,2}$ is exactly the ideal of polynomials that vanishes on the multiples of ν_1 and ν_2. In other words, for any degree d, $I_{1,2,=d}$ is the space of degree d polynomials that vanish on ν_1 and ν_2.

We prove the claim as follows. Since x is a regular point of $Z(P)$, $\nabla P(x)$ is non-zero. The ideal generated by $Q_{1,x}$ is exactly the set of polynomials that vanish on TZ. After performing a linear transformation, we can arrange that TZ is spanned by $(1,0,0) = \nu_1$ and $(0,1,0) = \nu_2$. Now $\mathbb{R}[v_1, v_2, v_3]/(Q_{1,x})$ is isomorphic to $\mathbb{R}[v_1, v_2]$. Next, we consider the image of $Q_{2,x}$ in $\mathbb{R}[v_1, v_2, v_3]/(Q_{1,x}) = \mathbb{R}[v_1, v_2]$. This image is non-zero, because $\nabla^2 P(x) : T_x Z \times T_x Z \to \mathbb{R}$ is non-degenerate. It vanishes on $(1,0,0)$ and on $(0,1,0)$, so it must be a non-zero multiple of $v_1 v_2$. Therefore, $I(x)$ is the ideal generated by v_3 and $v_1 v_2$. The rest of the claim is easy to check.

We see that $I_{1,2,=d}$ is the kernel of the evaluation map from $\mathbb{R}[v]_{=d}$ to the two points ν_1 and ν_2. For each $d \geq 1$, this map is surjective, and so for all $d \geq 1$, $dim I_{1,2,=d} = dim \mathbb{R}[v]_{=d} - 2$. In particular, for $d = 3$, we get $dim I_{1,2,=3} = 8$.

Now we are ready to show the the conclusion of the lemma. We know that $RP(x) = 0$ if and only if the dimension of $I_{=3}$ is ≤ 8. Now $I_{=3}$ is spanned by $I_{1,2,=3}$ and $Q_{3,x}$, and the dimension of $I_{1,2,=3}$ is already 8. So $dim I_{=3} \leq 8$ if and only if $Q_{3,x} \in I_{1,2}$ if and only if $Q_{3,x}(\nu_1) = Q_{3,x}(\nu_2) = 0$. Since $Q_{3,x}(v) = \nabla^3_v P(x)$, this last equation is equivalent to $\nabla^3_{\nu_1} P(x) = \nabla^3_{\nu_2} P(x) = 0$. □

We talk briefly about other situations. If x is a critical point of P, then $RP(x) = 0$. If x is a flat point of $Z(P)$ then $RP(x) = 0$. These are the only situations we will actually need. We put the write-up in the appendix.

For context, we talk a little more generally. The basic issue is that we are trying to detect whether some equations have two distinct roots. But having two distinct roots is not an algebraic condition - which we can see already by considering quadratic polynomials. Roughly speaking, if $RP(x) = 0$ then there are either two independent directions which satisfy the flecnodal equation, or else there may be one direction that satisfies the equation "with multiplicity 2". I believe that this happens for Gaussian flat surfaces. So I believe that there are lots of irreducible P where $RP = 0$ on $Z(P)$: planes and reguli and also Gaussian flat algebraic surfaces such as cylinders...

Now we are ready to verify the second property in the regulus detection lemma.

Lemma 0.3. *If x lies in two lines in $Z(P)$, then $RP(x) = 0$.*

Proof. If x is critical or flat, then we have seen that $RP(x) = 0$. Suppose that x is not critical or flat. Let ν_1 and ν_2 be the tangent directions of the two lines. We know that $\nabla^s_{\nu_i} P(x) = 0$ for $i = 1, 2$ and for any s. In particular, $\nabla^2 P(x) : T_x Z \times T_x Z \to \mathbb{R}$ is a non-zero quadratic form (in two variables) that vanishes on two independent vectors, and so it must have signature $(1, 1)$. Now Lemma 0.2 implies that $RP(x) = 0$. □

Finally, we are ready to prove the third property - that under some conditions $RP = 0$ implies that $Z(P)$ is a regulus. We state the result as a lemma.

Lemma 0.4. *If P is irreducible and RP vanishes on $Z(P)$, and if there is a non-special point x_0 contained in two lines in $Z(P)$, then $Z(P)$ is a regulus.*

The proof is based on local-to-global results for ruled surfaces. In particular, we will use the following result from last lecture:

Proposition 0.5. *Suppose that $P \in \mathbb{R}[x_1, x_2, x_3]$. Let $O \subset Z(P)$ be an open subset of $Z(P)$. Suppose that V is a smooth, non-zero vector field on O, obeying the flecnodal equation:*

$$0 = \nabla^s_V P(x), \text{ for all } x \in O, s = 1, 2, 3.$$

Suppose that at each point $x \in O$, $\nabla P(x) \neq 0$ and $\nabla^2 P(x) : TZ \times TZ \to \mathbb{R}$ is non-degenerate.

Then the integral curves of V are straight line segments.

Proof. We know that $\nabla^2 P(x_0)$ vanishes in the tangent directions to the two lines. Since x_0 is not flat, $\nabla^2 P(x_0) : T_x Z \times T_x Z \to \mathbb{R}$ is non-zero, and we see that it must have signature $(1, 1)$. We can choose an open neighborhood $O \subset Z(P)$ around x_0, so that $\nabla P \neq 0$ and $\nabla^2 P : TZ \times TZ \to \mathbb{R}$ has signature $(1, 1)$ in O. (In particular, $\nabla^2 P$ is non-degenerate on O.)

At each point of O, there are two independent vectors $V_1, V_2 \in TZ$ with $\nabla^2_V P(x) = 0$. We can normalize them to get two smooth vector fields V_1 and V_2. Since $RP = 0$ on O, Lemma 0.2 implies that V_1 and V_2 each satisfy the flecnodal equation: $\nabla^s_{V_i} P(x) = 0$ for $s = 1, 2, 3$. Now by the proposition above, the integral curves of V_1 and V_2 are each straight line segments. We call the integeral curves of V_1 "horizontal" lines, and we call the integral curves of V_2 "vertical lines".

In a small neighborhood of x_0, we will check that each horizontal line intersects each vertical line. Then we will find a plane or regulus that contains infinitely many horizontal lines, and we will conclude that $Z(P)$ is a plane or a regulus. (Finally the assumption that x_0 is not flat means that $Z(P)$ can only be a regulus.)

The set $O \subset Z(P)$ is given by a graph. After a rotation and possibly shrinking O, we can assume that O is given by equation $h(x_1, x_2) = x_3$ for a smooth function h, and that x_0 is the origin $(0, 0, 0)$. After a linear change of coordinates, we can assume that at x_0, the direction V_1 is $(1, 0, 0)$ and V_2 is $(0, 1, 0)$. Let L_1 be the horizontal line through x_0, and let L_2 be the vertical line through x_0. Notice that L_1 is just the line $x_2 = x_3 = 0$. For each point $(t, 0, 0)$ in L_1, let $L_2(t)$ be the vertical line through $(t, 0, 0)$. Notice that $L_2(t)$ is the graph of h restricted to a line $l_2(t)$ in the $x_1 - x_2$ plane. The line $l_2(t)$ passes through $(t, 0)$, and if t is small, it has slope close to $(0, 1)$. Similarly, let $L_1(u)$ be the horizontal line through $(0, u, 0)$, which is the graph of h restricted to $l_1(u)$ - a line in the plane thru $(0, u)$ with slope close to $(1, 0)$. If t, u are small enough, then $l_1(u)$ and $l_2(t)$ intersect in a small neighborhood of 0, and so $L_1(u)$ and $L_2(t)$ interect in O.

By shrinking O, we can arrange that no two vertical lines intesect in O. Now fix three vertical lines close to L_2. There are infinitely many horizontal lines that intersect all three of the vertical lines in O. If the three vertical lines are skew, then infinitely many horizontal lines lie in a regulus. Now $Z(P)$ intersects the regulus in infinitely many lines - and since P is irreducible, $Z(P)$ is a regulus. If two of the vertical lines are coplanar, then infinitely many horizontal lines lie in a plane, and so $Z(P)$ would be a plane. □

0.1. **On RP at critical and flat points.**

Lemma 0.6. *If $\nabla P(x) = 0$, then $RP(x) = 0$.*

Proof. Since $\nabla P(x) = 0$, we have $Q_{1,x}(v) = 0$. Therefore, $I(x)$ is the ideal generated by $Q_{2,x}$ and $Q_{3,x}$. Therefore, the dimension of $I(x)_{=3}$ is at most $3 + 1 = 4 \leq 8$. □

Lemma 0.7. *Assume x is a regular point of $Z(P)$. Then x is flat if and only if $\nabla^2 P(x) : T_x Z \times T_x Z \to \mathbb{R}$ is equal to zero, if and only if $Q_{2,x}$ is a multiple of $Q_{1,x}$.*

Proof. The first equivalence is an exercise in multivariable calculus. Rotate and translate space so that $x = 0$, and $\partial_1 P(0) = \partial_2 P(0) = 0$ but $\partial_3 P(0) \neq 0$. Without loss of generality we can work with these coordinates for the rest of the proof.

Locally near 0, the surface $Z(P)$ is given by a graph of a function h: $x_3 = h(x_1, x_2)$. Therefore $P(x_1, x_2, h(x_1, x_2)) = 0$ for all (x_1, x_2) in a neighborhood of 0. Differentiating once, we see that $\partial_1 h(0) = \partial_2 h(0) = 0$. Using this information and differentiating twice, we see that

$$\partial_{ij} P(0) = \partial_3 P(0) \partial_{ij} h(0), \text{ for } i, j \in \{1, 2\}.$$

This proves the first equivalence. In these coordinates, we have at $x = 0$, $Q_{1,x}(v) = cv_3$ for a non-zero constant c. Also, $Q_{2,x}(v) = \sum_{|I|=2} I! v^I \partial_I P(x)$. So $Q_{2,x}(v)$ is a multiple of v_3 if and only if $\partial_{1,1}P(x) = \partial_{1,2}P(x) = \partial_{2,2}P(x) = 0$, if and only if x is a flat point of $Z(P)$. □

Lemma 0.8. *If x is a flat point of $Z(P)$, then $RP(x) = 0$.*

Proof. By the last lemma, $Q_{2,x}$ is in the ideal generated by $Q_{1,x}$. Therefore, $I(x)$ is the ideal generated by $Q_{1,x}$ and $Q_{3,x}$. Therefore, the dimension of $I(x)_{=3}$ is at most $6 + 1 = 7 \leq 8$. □

1. Incidence estimates

Using the regulus detection lemma, and the ideas in the proof of the P_3 estimate (lecture 15), it's straightforward to prove the following.

Theorem 1.1. *Suppose that $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^3$ with $\leq B$ lines in any plane or regulus, and suppose that $B \geq L^{1/2}$. Then $|P_2(\mathfrak{L})| \lesssim BL$.*

Remark: It's not clear at all what happens for B smaller than $L^{1/2}$ - for example $B = 10$.

This finishes our work on incidences of lines in $\mathbb{R}^3$. For large k, the number of k-rich points is covered by the incidence estimate using polynomial ham sandwich (lecture 20). All together we get the following result.

Theorem 1.2. *Suppose that $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^3$ with $\leq B$ lines in any plane or regulus. Suppose that $B \geq L^{1/2}$ and $2 \leq k \leq L^{1/2}$.*

Then $|P_k(\mathfrak{L})| \lesssim BLk^{-2}$.

Remark. The incidence estimate in lecture 20 gives the slightly sharper but more complicated estimate $\lesssim L^{3/2}k^{-2} + BLk^{-3} + Lk^{-1}$, which holds for all $2 \leq k \leq L$.

This incidence estimate gives enough information to carry out the program of Elekes and Sharir on distinct distances (lecture 11).

At the beginning of next lecture, we'll talk briefly about how everything fits together, and then we'll close this chapter of the course.