1. Show fixed point functor is left exact, but not right exact in general. You may prove more generally that: for an $R$-module $M$,

$$\mathrm{Hom}_R(M, -) : R - \mathrm{mod} \to \mathbb{Z} - \mathrm{mod}$$

is left exact.

2. Suppose $H$ is diagonalizable. Prove that taking $H$ fixed points is an exact functor.

3. Let $G$ be a connected, reductive linear algebraic group defined over an algebraically closed field, and $M$ a finite dimensional $G$-module. (a) Show $M^B = M^G$. (b) Since $M$ is a $G$-module, we may restrict the action to $T$, a maximal torus and decompose $M$ into weight spaces: $M = \bigoplus_{\lambda \in X(T)} M_\lambda$. Define the character of $M$ as follows:

$$\mathrm{ch}M = \sum_{\lambda \in X(T)} \mathrm{dim}M_\lambda e^\lambda \in \mathbb{Z}[X(T)].$$

Prove that $\mathrm{ch}M \in \mathbb{Z}[X(T)]^W$.

4. (a, b) Springer, 8.2.11 exercise 3. (c) Find the fundamental weights of $\mathrm{SL}_3$.

5. Springer, 7.5.3 exercise 1.

6. Springer, 7.5.3 exercise 2.

7. (a) Let $\lambda \in X(T)_+$, and suppose that $M$ is a $G$-module with $\mathrm{dim}M_\lambda = 1$ and all weights of $M$ are of the form $w(\lambda)$ for some $w \in W$. Prove that $M$ is simple, and so isomorphic to $L(\lambda)$.

(b) A dominant weight is called minuscule if $\langle \alpha^\vee, \lambda \rangle \leq 1$ for all positive coroots $\alpha^\vee$. Prove that if $\lambda$ is minuscule, then $H^0(\lambda)$ has all weights in a single Weyl group orbit, and so must be simple.

(c) Find all minuscule weights of $\mathrm{SL}_3$.