18.022: Multivariable calculus — The implicit function theorem

The following version of the implicit function theorem was stated in class. It is a litte different from the version in the book.

THEOREM. *Let $A \subset \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$ be an open subset and $\mathbf{F}\colon A \to \mathbb{R}^m$ a function of class C^1. Let $(\mathbf{a}, \mathbf{b}) \in A$ and consider the system of equations*

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{F}(\mathbf{a}, \mathbf{b}).$$

Assume that the $(m \times m)$-matrix of partial derivatives

$$\left(\frac{\partial F_i}{\partial y_k}(\mathbf{a}, \mathbf{b})\right)$$

is invertible. Then there exists an open subset $U \subset \mathbb{R}^n$ containing $\mathbf{a}$ and an open subset $V \subset \mathbb{R}^m$ containing $\mathbf{b}$ such that $U \times V \subset A$ and such that, for all $\mathbf{x} \in U$, there exists a unique $\mathbf{y} = \mathbf{f}(\mathbf{x}) \in V$ such that $(\mathbf{x}, \mathbf{y})$ is a solution to the above system of equations. Moreover, the function $\mathbf{f}\colon U \to V$ is of class C^1 and the derivative at $\mathbf{x} \in U$ satisfies the linear equation

$$\left(\frac{\partial F_i}{\partial x_j}(\mathbf{x}, \mathbf{f}(\mathbf{x}))\right) + \left(\frac{\partial F_i}{\partial y_k}(\mathbf{x}, \mathbf{f}(\mathbf{x})\right) \cdot \left(\frac{\partial f_k}{\partial x_j}(\mathbf{x})\right) = \mathbf{0}.$$

Let me try to explain the statement of the theorem. We are given the m constraints

$$\begin{aligned} F_1(x_1, \ldots, x_n, y_1, \ldots, y_m) &= F_1(a_1, \ldots, a_n, b_1, \ldots, b_m) \\ F_2(x_1, \ldots, x_n, y_1, \ldots, y_m) &= F_2(a_1, \ldots, a_n, b_1, \ldots, b_m) \\ &\vdots \\ F_m(x_1, \ldots, x_n, y_1, \ldots, y_m) &= F_m(a_1, \ldots, a_n, b_1, \ldots, b_m). \end{aligned}$$

on the $n + m$ variables $x_1, \ldots, x_n, y_1, \ldots, y_m$. We would generally expect that this leaves us n independent variables $x_1, \ldots, x_n$ and m dependent variables $y_1, \ldots, y_m$. The theorem tell us that this is true if $(x_1, \ldots, x_n)$ is close to $(a_1, \ldots, a_n)$ and $(y_1, \ldots, y_m)$ close to $(b_1, \ldots, b_m)$ and if the $(m \times m)$-matrix of partial derivatives

$$\left(\frac{\partial F_i}{\partial y_k}(a_1, \ldots, a_n, b_1, \ldots, b_m)\right)$$

is invertible. Indeed, we get an open neighborhood $U \subset \mathbb{R}^n$ of $(a_1, \ldots, a_n)$ and an open neighborhood $V \subset \mathbb{R}^m$ of $(b_1, \ldots, b_m)$ such that, for $(x_1, \ldots, x_n) \in U$, the m equations have a unique solution

$$\begin{aligned} y_1 &= f_1(x_1, \ldots, x_n) \\ y_2 &= f_2(x_1, \ldots, x_n) \\ &\vdots \\ y_m &= f_m(x_1, \ldots, x_n) \end{aligned}$$

with $(y_1, \ldots, y_m) \in V$. Moreover, the function $\mathbf{f} = (f_1, \ldots, f_m)\colon U \to V$ so defined is of class C^1. In practice it is usually impossible to actually find the solution and determine the functions $f_1, \ldots, f_m$. However, it is easy to find the partial derivatives of these functions. Indeed, let $\mathbf{G}\colon U \to \mathbb{R}^n \times \mathbb{R}^m$ be the function defined by

$$\mathbf{G}(\mathbf{x}) = (\mathbf{x}, \mathbf{f}(\mathbf{x})).$$

Then $\mathbf{F} \circ \mathbf{G} \colon U \to \mathbb{R}^m$ is constant equal to $\mathbf{F}(\mathbf{a}, \mathbf{b})$ and hence

$$D(\mathbf{F} \circ \mathbf{G})(\mathbf{x}) = \mathbf{0}.$$

But, on the other hand, the chain rule shows that

$$D(\mathbf{F} \circ \mathbf{G})(\mathbf{x}) = D\mathbf{F}(\mathbf{G}(\mathbf{x})) \cdot D\mathbf{G}(\mathbf{x}),$$

and the right-hand side is equal to the sum of $(m \times n)$-matrices

$$\left(\frac{\partial F_i}{\partial x_j}(\mathbf{x}, \mathbf{f}(\mathbf{x}))\right) + \left(\frac{\partial F_i}{\partial y_k}(\mathbf{x}, \mathbf{f}(\mathbf{x}))\right) \cdot \left(\frac{\partial f_k}{\partial x_j}(\mathbf{x})\right).$$

Finally, we note that the continuity of the partial derivatives $\partial F_i / \partial x_j$ and $\partial F_i / \partial y_k$ and the continuity of the function $\mathbf{f}$ implies that, for all $\mathbf{x}$ in a possibly smaller open neighborhood $U' \subset U$ of $\mathbf{a}$, the $(m \times m)$-matrix

$$\left(\frac{\partial F_i}{\partial y_k}(\mathbf{x}, \mathbf{f}(\mathbf{x}))\right)$$

is invertible. Hence, for all $\mathbf{x} \in U'$, we find that

$$\left(\frac{\partial f_k}{\partial x_j}(\mathbf{x})\right) = -\left(\frac{\partial F_i}{\partial y_k}(\mathbf{x}, \mathbf{f}(\mathbf{x}))\right)^{-1} \cdot \left(\frac{\partial F_i}{\partial x_j}(\mathbf{x}, \mathbf{f}(\mathbf{x}))\right).$$