18.022: Multivariable calculus — Points and vectors

The difference between vectors and points is basically that there is a particular vector—the zero vector—which stands out among all other vectors while no point is special compared with other points. Here are the mathematical definitions.

Definition 1. A *real vector space* consists of the following:

(i) A set V whose elements are called vectors.

(ii) An element $\mathbf{0} \in V$ called the *zero vector*.

(iii) An operation that to two vectors $\mathbf{a}, \mathbf{b} \in V$ assigns a vector $\mathbf{a} + \mathbf{b} \in V$ called the *vector sum* of $\mathbf{a}$ and $\mathbf{b}$.

(iv) An operation that to a real number $k \in \mathbb{R}$ and a vector $a \in V$ assigns a vector $k\mathbf{a} \in V$ called the *scalar multiple* of $\mathbf{a}$ by k.

The operations of vector sum and scalar multiplication are required to satify the following axioms:

(V1) For all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$,

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}).$$

(V2) For all $\mathbf{a} \in V$,

$$\mathbf{a} + \mathbf{0} = \mathbf{a} = \mathbf{0} + \mathbf{a}.$$

(V3) For all $\mathbf{a}, \mathbf{b} \in V$,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

(V4) For all $\mathbf{a} \in V$,

$$0\mathbf{a} = \mathbf{0}.$$

(V5) For all $\mathbf{a} \in V$,

$$1\mathbf{a} = \mathbf{a}.$$

(V6) For all $k, \ell \in \mathbf{R}$ and all $\mathbf{a} \in V$,

$$(k + \ell)\mathbf{a} = k\mathbf{a} + \ell\mathbf{a}.$$

(V7) For all $k \in \mathbf{R}$ and all $\mathbf{a}, \mathbf{b} \in V$,

$$k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}.$$

(V8) For all $k, \ell \in \mathbf{R}$ and all $\mathbf{a} \in V$,

$$(k\ell)\mathbf{a} = k(\ell\mathbf{a}).$$

Example 2. (The vector space $\mathbb{R}^n$) The set $\mathbb{R}^n$ of n-tuples of real numbers with zero vector $\mathbf{0} = \langle 0, 0, \ldots, 0\rangle$, with vector sum

$$\langle a_1, a_2, \ldots, a_n\rangle + \langle b_1, b_2, \ldots, b_n\rangle = \langle a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n\rangle,$$

and with scalar multiplication

$$k\langle a_1, a_2, \ldots, a_n\rangle = \langle ka_1, ka_2, \ldots, ka_n\rangle$$

is a real vector space.

DEFINITION 3. A *real affine space* consists of the following:

(i) A set E whose elements are called points.

(ii) A real vector space V.

(iii) An operation that to a point $P \in E$ and a vector $\mathbf{a} \in V$ assigns a point $P + \mathbf{a} \in E$ called the *translation* of P by $\mathbf{a}$.

The operation of translation is required to satisfy the following axioms.

(A1) For all $P \in E$ and all $\mathbf{a}, \mathbf{b} \in V$,

$$(P + \mathbf{a}) + \mathbf{b} = P + (\mathbf{a} + \mathbf{b}).$$

(A2) For all $P \in E$,

$$P + \mathbf{0} = P.$$

(A3) For all $P, Q \in E$, there exists a *unique* vector $\vec{PQ} \in V$ such that

$$P + \vec{PQ} = Q.$$

EXAMPLE 4. (The affine space $\mathbb{R}^n$) The set $\mathbb{R}^n$ of n-tuples of real numbers with translation by the vector space $\mathbb{R}^n$ of Example 2 defined by

$$(x_1, x_2, \ldots, x_n) + \langle a_1, a_2, \ldots, a_n \rangle = (x_1 + a_1, x_2 + a_2, \ldots, x_n + a_n)$$

is a real affine space. If $P = (x_1, x_2, \ldots, x_n)$ and $Q = (y_1, y_2, \ldots, y_n)$ are two points, then $\vec{PQ} = \langle y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n \rangle$.

The mathematical distinction between the vector space $\mathbb{R}^n$ and the affine $\mathbb{R}^n$ lies in the algebraic structure that the set $\mathbb{R}^n$ of n-tuples of real numbers is equipped with in the two cases. The set $\mathbb{R}^n$ is the same in the two cases. In this course, we will use pointy brackets $\langle a_1, a_2, \ldots, a_n \rangle$ to indicate that we are considering the n-tuple as a vector and round brackets $(a_1, a_2, \ldots, a_n)$ to indicate that we are considering the n-tuple as a point.

LEMMA 5. *Let E be a real affine space, and let $P, Q, R \in E$ be three points. Then*

$$\vec{PQ} + \vec{QR} = \vec{PR}.$$

PROOF. We have

$$P + (\vec{PQ} + \vec{QR}) = (P + \vec{PQ}) + \vec{QR} = Q + \vec{QR} = R$$

where the left-hand equality holds by A1, and where the middle and right-hand equalities hold by the definition of $\vec{PQ}$ and $\vec{QR}$. Now, by definition, the vector $\vec{PR}$ is the *unique* vector such that

$$P + \vec{PR} = R.$$

Hence, $\vec{PQ} + \vec{QR} = \vec{PR}$ as stated. $\square$