18.022: Multivariable calculus — Smooth manifolds

The mathematical term for a curved m-dimensional space is the notion of a smooth manifold of dimension m. We give a brief discussion following John W. Milnor, *Topology from the differentiable viewpoint*, The University Press of Virginia.

We recall that a map $\mathbf{F}\colon U \to V$ from an open subset $U \subset \mathbb{R}^k$ to an open subset $V \subset \mathbb{R}^l$ is *smooth* if all higher order partial derivatives $\partial^n \mathbf{F}/\partial x_{i_1} \dots \partial x_{i_n}$ exist and are continuous. More generally, a map

$$\mathbf{f}\colon X \to Y$$

between arbitrary subsets $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$ is *smooth* if, for every $\mathbf{a} \in X$, there exists an open subset $U \subset \mathbb{R}^k$ that contains $\mathbf{a}$ and a smooth map $\mathbf{F}\colon U \to \mathbb{R}^l$ such that $\mathbf{F}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$, for all $\mathbf{x} \in U \cap X$. We then define the *derivative* of $\mathbf{f}\colon X \to Y$ at $\mathbf{x} \in U \cap X$ by $D\mathbf{f}(\mathbf{x}) = D\mathbf{F}(\mathbf{x})$. We define $\mathbf{f}\colon X \to Y$ to be a *diffeomorphism* if $\mathbf{f}$ is a bijection and both $\mathbf{f}\colon X \to Y$ and $\mathbf{f}^{-1}\colon Y \to X$ are smooth maps.

Example. A map $\mathbf{x}\colon [a,b] \to \mathbb{R}^l$ is smooth if and only if there exists $\epsilon > 0$ and a smooth map $\mathbf{X}\colon (a-\epsilon, b+\epsilon) \to \mathbb{R}^l$ such that $\mathbf{X}(t) = \mathbf{x}(t)$, for all $t \in [a,b]$.

Definition. A subset $M \subset \mathbf{R}^k$ is a *smooth manifold* of *dimension* m if, for every $\mathbf{a} \in M$, there exists an open subset $W \subset \mathbb{R}^k$ with $\mathbf{a} \in W$, an open subset $U \subset \mathbb{R}^m$, and a diffeomorphism $\mathbf{g}\colon U \to M \cap W$. We call $\mathbf{g}\colon U \to M \cap W$ a *parametrization* of the region $M \cap W$ and $\mathbf{g}^{-1}\colon M \cap W \to U$ a system of *coordinates* on $M \cap W$.

Example. We show that the 2-sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$$

is a smooth manifold of dimension 2. The subset

$$U = \{(u,v) \mid u^2 + v^2 < 1\} \subset \mathbb{R}^2$$

and the subsets

$$\begin{aligned} W_1^+ &= \{(x,y,z) \mid x > 0\} \subset \mathbb{R}^3 & W_1^- &= \{(x,y,z) \mid x < 0\} \subset \mathbb{R}^3 \\ W_2^+ &= \{(x,y,z) \mid y > 0\} \subset \mathbb{R}^3 & W_2^- &= \{(x,y,z) \mid y < 0\} \subset \mathbb{R}^3 \\ W_3^+ &= \{(x,y,z) \mid z > 0\} \subset \mathbb{R}^3 & W_3^- &= \{(x,y,z) \mid z < 0\} \subset \mathbb{R}^3 \end{aligned}$$

are all open and the maps

$$\begin{aligned} \mathbf{g}_1^\pm &\colon U \to S^2 \cap W_1^\pm, & \mathbf{g}_1^\pm(u,v) &= (\pm\sqrt{1-u^2-v^2}, u, v), \\ \mathbf{g}_2^\pm &\colon U \to S^2 \cap W_1^\pm, & \mathbf{g}_2^\pm(u,v) &= (u, \pm\sqrt{1-u^2-v^2}, v), \\ \mathbf{g}_3^\pm &\colon U \to S^2 \cap W_1^\pm, & \mathbf{g}_3^\pm(u,v) &= (u, v, \pm\sqrt{1-u^2-v^2}), \end{aligned}$$

are diffeomorphisms. Every point $(x,y,z) \in S^2$ belongs to at least one of the subsets $S^2 \cap W_i^\pm$. Indeed, at least one of x, y, and z is non-zero, and hence, either positive or negative. This shows that S^2 is a smooth manifold of dimension 2.

We also discuss the more general notion of a smooth manifold with boundary. Let

$$\mathbb{H}^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_m \geqslant 0\}$$

be the closed halfspace with *boundary* the subset $\partial\mathbb{H}^m \subset \mathbb{H}^m$ where $x_m = 0$.

Definition. A subset $M \subset \mathbf{R}^k$ is a *smooth m-manifold with boundary* if, for every $\mathbf{a} \in M$, there exists an open subset $W \subset \mathbb{R}^k$ with $\mathbf{a} \in W$, an open subset $U \subset \mathbb{R}^m$, and a diffeomorphism $\mathbf{g} \colon \mathbb{H}^m \cap U \to M \cap W$. The boundary $\partial M \subset M$ is the subset of all points that correspond to $\partial \mathbb{H}^m$ under such a diffeomorphism.

If M is a smooth m-manifold with boundary, then ∂M is a smooth $(m-1)$-manifold and $M \smallsetminus \partial M$ is a smooth m-manifold both without boundary.

Example. Let $\mathbf{x} \colon [a, b] \to \mathbb{R}^k$ be a smooth map, and suppose that the velocity vector $\mathbf{x}'(t)$ is non-zero, for all $t \in [a, b]$. Then the image $C \subset \mathbb{R}^k$ is a smooth 1-manifold with boundary. The boundary ∂C consists of the points $\mathbf{x}(a)$ and $\mathbf{x}(b)$.