The big de Rham-Witt complex

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Introduction

The big de Rham-Witt complex was introduced by the author and Madsen in [12] with the purpose of giving an algebraic description of the equivariant homotopy groups in low degrees of Bökstedt's topological Hochschild spectrum of a commutative ring. This functorial algebraic description, in turn, is essential for understading algebraic K-theory by means of the cyclotomic trace map of Bökstedt-Hsiang-Madsen [2]; compare [13, 11, 8]. The original construction, which relied on the adjoint functor theorem, was very indirect and a direct construction has been lacking. In this paper, we give a new and explicit construction of the big de Rham-Witt complex and we also correct the 2-torsion which was not quite correct in the original construction. The new construction is based on a theory, which is developed first, of modules and derivations over a λ-ring. The main result of this first part of the paper is that the universal derivation of a λ-ring is given by the universal derivation of the underlying ring together with an additional structure that depends directly on the λ-ring structure in question. In the case of the universal λ-ring, which is given by the ring of big Witt vectors, this additional structure consists in divided Frobenius operators on the module of Kähler differentials. It is the existence of these divided Frobenius operators that makes possible the new direct construction of the big de Rham-Witt complex. This is carried out in the second part of the paper, where we also show that the big de Rham-Witt complex behaves well with respect to étale morphisms. Finally, we explicitly evaluate the big de Rham-Witt complex of the ring of integers.

In more detail, let A be a ring, always assumed to be commutative and unital. The ring $\mathbb{W}(A)$ of big Witt vectors in A is equipped with a natural action by the multiplicative monoid $\mathbb{N}$ of positive integers with the action by $n \in \mathbb{N}$ given by the nth Frobenius map

$$F_n \colon \mathbb{W}(A) \to \mathbb{W}(A).$$

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The Frobenius maps give rise to a natural ring homomorphism

$$\Delta\colon \mathbb{W}(A) \to \mathbb{W}(\mathbb{W}(A))$$

whose Witt components $\Delta_e\colon \mathbb{W}(A) \to \mathbb{W}(A)$ are characterized by the formula

$$F_n(a) = \sum_{e \mid n} e\Delta_e(a)^{n/e}.$$

The triple $(\mathbb{W}(-), \Delta, \varepsilon)$ is a comonad on the category of rings and a λ-ring in the sense of Grothendieck [9] is precisely a coalgebra (A, λ_A) over this comonad.

Recently, Borger [5] has proposed that a λ-ring structure $\lambda_A\colon A \to \mathbb{W}(A)$ on the ring A be considered as descent data for the extension of $\mathbb{Z}$ over a deeper base $\mathbb{F}_1$. This begs the question as to the natural notions of modules and derivations over (A, λ_A). We show that the general approach of Quillen [20] leads to the following answer. Recall that if (A, λ_A) is a λ-ring, then the ring A is equipped with an $\mathbb{N}$-action with the action by $n \in \mathbb{N}$ given by the nth associated Adams operation

$$\psi_{A,n}\colon A \to A$$

that is defined by the formula

$$\psi_{A,n}(a) = \sum_{e \mid n} e\lambda_{A,e}(a)^{n/e}$$

where $\lambda_{A,e}\colon A \to A$ is the eth Witt component of $\lambda_A\colon A \to \mathbb{W}(A)$. Now, the category of (A, λ_A)-modules is identified with the category of left modules over the twisted monoid algebra $A^{\psi}[\mathbb{N}]$ whose product is defined by the formula

$$n \cdot a = \psi_{A,n}(a) \cdot n.$$

Hence, an (A, λ_A)-module is a pair (M, λ_M) that consists of an A-module M and an $\mathbb{N}$-indexed family of maps $\lambda_{M,n}\colon M \to M$ such that $\lambda_{M,n}$ is $\psi_{A,n}$-linear, $\lambda_{M,1} = \mathrm{id}_M$, and $\lambda_{M,m}\lambda_{M,n} = \lambda_{M,mn}$. Moreover, we identify the derivations

$$D\colon (A, \lambda_A) \to (M, \lambda_M)$$

with the derivations $D\colon A \to M$ that satisfy

$$\lambda_{M,n}(Da) = \sum_{e \mid n} \lambda_{A,e}(a)^{(n/e)-1} D\lambda_{A,e}(a).$$

It is now easy to show that there is a universal derivation

$$d\colon (A, \lambda_A) \to (\Omega_{(A,\lambda_A)}, \lambda_{\Omega_{(A,\lambda_A)}}).$$

We prove the following result.

Theorem A. *For every λ-ring (A, λ_A), the canonical map*

$$\Omega_A \to \Omega_{(A,\lambda_A)}$$

is an isomorphism of A-modules.

It follows that for a λ-ring (A,λ_A), the A-module of differentials Ω_A carries the richer structure of an (A,λ_A)-module. In the case of $(\mathbb{W}(A),\Delta_A)$, this implies that there are natural F_n-linear maps $F_n\colon \Omega_{\mathbb{W}(A)} \to \Omega_{\mathbb{W}(A)}$ defined by

$$F_n(da) = \sum_{e\mid n} \Delta_e(a)^{(n/e)-1} d\Delta_e(a)$$

such that $F_1 = \mathrm{id}$, $F_mF_n = F_{mn}$, $dF_n(a) = nF_n(da)$, and $F_n(d[a]) = [a]^{n-1}d[a]$. The p-typical analog of F_p was also constructed by Borger and Wieland in [6, 12.8].

The construction of the de Rham-Witt complex begins with the following variant of the de Rham complex. The ring $\mathbb{W}(\mathbb{Z})$ contains exactly the four units $\pm[\pm 1]$ all of which are square roots of $[1]$ and the 2-torsion element

$$d\log[-1] = [-1]^{-1}d[-1] = [-1]d[-1] \in \Omega_{\mathbb{W}(A)}$$

plays a special role. We define the graded $\mathbb{W}(A)$-algebra

$$\hat{\Omega}^{\cdot}_{\mathbb{W}(A)} = T^{\cdot}_{\mathbb{W}(A)}\Omega_{\mathbb{W}(A)}/J^{\cdot}$$

to be the quotient of the tensor algebra of the $\mathbb{W}(A)$-module $\Omega_{\mathbb{W}(A)}$ by the graded ideal $J^{\cdot}$ generated by all elements of the form

$$da \otimes da - d\log[-1] \otimes F_2 da$$

with $a \in \mathbb{W}(A)$. It is an anti-symmetric graded ring which carries a unique graded derivation d that extends $d\colon \mathbb{W}(A) \to \Omega_{\mathbb{W}(A)}$ and satisfies

$$dd\omega = d\log[-1] \cdot d\omega.$$

Moreover, the maps $F_n\colon \mathbb{W}(A) \to \mathbb{W}(A)$ and $F_n\colon \Omega_{\mathbb{W}(A)} \to \Omega_{\mathbb{W}(A)}$ extend uniquely to a map of graded rings $F_n\colon \Omega^{\cdot}_{\mathbb{W}(A)} \to \Omega^{\cdot}_{\mathbb{W}(A)}$ which satisfies $dF_n = nF_nd$. Next, we show that the maps d and F_n both descend to the further quotient

$$\check{\Omega}^{\cdot}_{\mathbb{W}(A)} = \hat{\Omega}^{\cdot}_{\mathbb{W}(A)}/K^{\cdot}$$

by the graded ideal generated by all elements of the form

$$F_pdV_p(a) - da - (p-1)d\log[-1] \cdot a$$

with p a prime number and $a \in \mathbb{W}(A)$. We now recall the Verschiebung maps

$$V_n\colon \mathbb{W}(A) \to \mathbb{W}(A)$$

which are additive and satisfy the projection formula

$$aV_n(b) = V_n(F_n(a)b).$$

These maps, however, do not extend to $\check{\Omega}^{\cdot}_{\mathbb{W}(A)}$, and the de Rham-Witt complex is roughly speaking the largest quotient

$$\eta\colon \check{\Omega}^{\cdot}_{\mathbb{W}(A)} \to \mathbb{W}\Omega^{\cdot}_A$$

such that the Verschiebung maps extend to $\mathbb{W}\Omega_A^{\cdot}$ and such that the extended maps F_n and V_n satisfy the projection formula. The precise definition given in Section 4 below is by recursion with respect to the quotients $\mathbb{W}_S(A)$ of $\mathbb{W}(A)$ where S ranges over the finite subsets $S \subset \mathbb{N}$ that are stable under division. We further prove the following result to the effect that the de Rham-Witt complex may be characterized as the universal example of an algebraic structure called a Witt complex, the precise definition of which is given in Definition 4.1.

Theorem B. *There exists an initial Witt complex $S \mapsto \mathbb{W}_S\Omega_A^{\cdot}$ over the ring A. In addition, the canonical maps*

$$\eta_S\colon \check{\Omega}^q_{\mathbb{W}_S(A)} \to \mathbb{W}_S\Omega^q_A$$

are surjective, and the diagrams

$$\begin{array}{ccc} \check{\Omega}^q_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & \mathbb{W}_S\Omega^q_A \\ \downarrow{\scriptstyle R^S_T} & & \downarrow{\scriptstyle R^S_T} \\ \check{\Omega}^q_{\mathbb{W}_T(A)} & \xrightarrow{\eta_T} & \mathbb{W}_T\Omega^q_A \end{array} \qquad \begin{array}{ccc} \check{\Omega}^q_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & \mathbb{W}_S\Omega^q_A \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\ \check{\Omega}^{q+1}_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & \mathbb{W}_S\Omega^{q+1}_A \end{array} \qquad \begin{array}{ccc} \check{\Omega}^q_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & \mathbb{W}_S\Omega^q_A \\ \downarrow{\scriptstyle F_m} & & \downarrow{\scriptstyle F_m} \\ \check{\Omega}^q_{\mathbb{W}_{S/m}(A)} & \xrightarrow{\eta_{S/m}} & \mathbb{W}_{S/m}\Omega^q_A \end{array}$$

commute.

If A is an $\mathbb{F}_p$-algebra and $S = \{1, p, \dots, p^{n-1}\}$, then $\mathbb{W}_S\Omega_A^{\cdot}$ agrees with the original p-typical de Rham-Witt complex $W_n\Omega_A^{\cdot}$ of Bloch-Deligne-Illusie [16]. More generally, if A is a $\mathbb{Z}_{(p)}$-algebra and $S = \{1, p, \dots, p^{n-1}\}$, then $\mathbb{W}_S\Omega_A^{\cdot}$ agrees with the p-typical de Rham-Witt complex $W_n\Omega_A^{\cdot}$ constructed by the author and Madsen [14] for p odd and by Costeanu [7] for $p = 2$. Finally, if 2 is either invertible or zero in A and S is arbitrary, then $\mathbb{W}_S\Omega_A^{\cdot}$ agrees with the big de Rham-Witt complex introduced by the author and Madsen [12]. We also note that if $f\colon R \to A$ is a map of $\mathbb{Z}_{(p)}$-algebras and $S = \{1, p, \dots, p^{n-1}\}$, then the relative p-typical de Rham-Witt complex $W_n\Omega_{A/R}^{\cdot}$ of Langer-Zink [18] agrees with the quotient of $\mathbb{W}_S\Omega_A^{\cdot}$ by the differential graded ideal generated by the image of $\mathbb{W}_S\Omega^1_R \to \mathbb{W}_S\Omega^1_A$.

We recall that van der Kallen [22, Theorem 2.4] and Borger [3, Theorem B] have proved independently that for every étale morphism $f\colon A \to B$ and every finite subset $S \subset \mathbb{N}$ stable under division, the induced morphism

$$\mathbb{W}_S(f)\colon \mathbb{W}_S(A) \to \mathbb{W}_S(B)$$

again is étale. Based on this theorem, we prove the following result.

Theorem C. *Let $f\colon A \to B$ be an étale map and let $S \subset \mathbb{N}$ be a finite subset stable under division. Then the induced map*

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^q_A \to \mathbb{W}_S\Omega^q_B$$

is an isomorphism.

To prove Theorem C, we verify that the left-hand terms form a Witt complex over the ring B and use Theorem B to obtain the inverse of the map in the statement. The verification of the Witt complex axioms, in turn, is significantly simplified by the existence of the divided Frobenius on $\Omega_{\mathbb{W}(A)}$ as follows from Theorem A.

Finally, we evaluate the de Rham-Witt complex of $\mathbb{Z}$. The result is that $\mathbb{W}\Omega^q_{\mathbb{Z}}$ is non-zero for $q \leqslant 1$ only. Moreover, we may consider $\mathbb{W}\Omega^{\cdot}_{\mathbb{Z}}$ as the quotient

$$\Omega^{\cdot}_{\mathbb{W}(\mathbb{Z})} \to \mathbb{W}\Omega^{\cdot}_{\mathbb{Z}}$$

of the de Rham complex of $\mathbb{W}(\mathbb{Z})$ by a differential graded ideal generated by elements of degree 1. Hence, following Borger [5], we may interpret $\mathbb{W}\Omega^{\cdot}_{\mathbb{Z}}$ as the complex of differentials along the leaves of a foliation of $\operatorname{Spec}(\mathbb{Z})$ considered as an $\mathbb{F}_1$-space. Moreover, this foliation is compatible with the canonical action of the multiplicative monoid $\mathbb{N}$ on the $\mathbb{F}_1$-space $\operatorname{Spec}(\mathbb{Z})$.

As mentioned earlier, the big de Rham-Witt complex was introduced in [12] with the purpose of giving an algebraic description of the equivariant homotopy groups

$$\mathrm{TR}^r_q(A) = [S^q \wedge (\mathbb{T}/C_r)_+, T(A)]_{\mathbb{T}}$$

of the topological Hochschild $\mathbb{T}$-spectrum $T(A)$ of the ring A. Here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle group, $C_r \subset \mathbb{T}$ is the subgroup of order r, and $[-,-]_{\mathbb{T}}$ is the abelian group of maps in the homotopy category of orthogonal $\mathbb{T}$-spectra. We proved in [10, Section 1] that the groups $\mathrm{TR}^r_q(A)$ give rise to a Witt complex over the ring A in the sense of Definition 4.1 below. Therefore, by Theorem B, there is a unique map

$$\mathbb{W}_{\langle r\rangle}\Omega^q_A \to \mathrm{TR}^r_q(A)$$

of Witt complexes over A, where $\langle r\rangle$ denotes the set of divisors of r. We will show elsewhere that this map is an isomorphism for all r and all $q \leqslant 1$.

1 Witt vectors

We begin with a brief review of Witt vectors and λ-rings. In the approach taken here, all necessary congruences are isolated in Dwork's lemma. The reader is also referred to the very readable account by Bergman [19, Appendix] and to the more modern and general exposition by Borger [3].

Let $\mathbb{N}$ be the set of positive integers. We say that $S \subset \mathbb{N}$ is a truncation set if whenever $n \in S$ and d is a divisor in n, then $d \in S$. The big Witt ring $\mathbb{W}_S(A)$ is defined to be the set A^S equipped with a ring structure such that the ghost map

$$w\colon \mathbb{W}_S(A) \to A^S$$

that takes the vector $(a_n \mid n \in S)$ to the sequence $\langle w_n \mid n \in S\rangle$ with

$$w_n = \sum_{d\mid n} d a_d^{n/d}$$

is a natural transformation of functors from the category of rings to itself. Here the target A^S is considered a ring with componentwise addition and multiplication. To prove that there exists a unique ring structure on $\mathbb{W}_S(A)$ characterized in this way, we first prove the following result. We write $v_p(n)$ for the p-adic valuation of n.

Lemma 1.1 (Dwork). *Suppose that for every prime number p, there exists a ring homomorphism $\phi_p \colon A \to A$ with the property that $\phi_p(a) \equiv a^p$ modulo pA. Then the sequence $\langle x_n \mid n \in S \rangle$ is in the image of the ghost map*

$$w \colon \mathbb{W}_S(A) \to A^S$$

if and only if $x_n \equiv \phi_p(x_{n/p})$ modulo $p^{v_p(n)}A$ for every prime number p every $n \in S$ with $v_p(n) \geqslant 1$.

Proof. We first show that if $a \equiv b$ modulo pA, then $a^{p^{v-1}} \equiv b^{p^{v-1}}$ modulo $p^v A$. If we write $a = b + p\varepsilon$, then

$$a^{p^{v-1}} = b^{p^{v-1}} + \sum_{1 \leqslant i \leqslant p^{v-1}} \binom{p^{v-1}}{i} b^{p^{v-1}-i} p^i \varepsilon^i.$$

In general, the p-adic valuation of the binomial coefficient $\binom{m+n}{n}$ is equal to the number of carrys in the addition of m and n in base p. So

$$v_p\left(\binom{p^{v-1}}{i} p^i\right) = v - 1 + i - v_p(i) \geqslant v$$

which proves the claim. Now, since ϕ_p is a ring-homomorphism,

$$\phi_p(w_{n/p}(a)) = \sum_{d \mid (n/p)} d\phi_p(a_d^{n/pd})$$

which is congruent to $\sum_{d \mid (n/p)} d a_d^{n/d}$ modulo $p^{v_p(n)}A$. If d divides n but not n/p, then $v_p(d) = v_p(n)$ and hence this sum is congruent to $\sum_{d \mid n} d a_d^{n/d} = w_n(a)$ modulo $p^{v_p(n)}A$ as stated. Conversely, if $\langle x_n \mid n \in S \rangle$ is a sequence such that $x_n \equiv \phi_p(x_{n/p})$ modulo $p^{v_p(n)}A$, we find a vector $a = (a_n \mid n \in S)$ with $w_n(a) = x_n$ as follows. We let $a_1 = x_1$ and assume, inductively, that a_d has been chosen, for all d that divides n, such that $w_d(a) = x_d$. The calculation above shows that the difference

$$x_n - \sum_{d \mid n, d \neq n} d a_d^{n/d}$$

is congruent to zero modulo $p^{v_p(n)}A$. Hence, we can find $a_n \in A$ such that $n a_n$ is equal to this difference. □

Proposition 1.2. *There exists a unique ring structure such that the ghost map*

$$w \colon \mathbb{W}_S(A) \to A^S$$

is a natural transformation of functors from rings to rings.

Proof. Let A be the polynomial ring $\mathbb{Z}[a_n, b_n \mid n \in S]$. The unique ring homomorphism

$$\phi_p \colon A \to A$$

that maps a_n to a_n^p and b_n to b_n^p satisfies $\phi_p(f) = f^p$ modulo pA. Let a and b be the vectors $(a_n \mid n \in S)$ and $(b_n \mid n \in S)$. Then Lemma 1.1 shows that the sequences $w(a) + w(b)$, $w(a) \cdot w(b)$, and $-w(a)$ are in the image of the ghost map. Hence, there are sequences of polynomials $s = (s_n \mid n \in S)$, $p = (p_n \mid n \in S)$, and $\iota = (\iota_n \mid n \in S)$ such that $w(s) = w(a) + w(b)$, $w(p) = w(a) \cdot w(b)$, and $w(\iota) = -w(a)$. Moreover, as A is torsion free, the ghost map is injective, so these polynomials are unique.

Let now A' be any ring, and let $a' = (a'_n \mid n \in S)$ and $b' = (b'_n \mid n \in S)$ be two vectors in $\mathbb{W}_S(A')$. Then there is a unique ring homomorphism $f \colon A \to A'$ such that $\mathbb{W}_S(f)(a) = a'$ and $\mathbb{W}_S(f)(b) = b'$. We define $a' + b' = \mathbb{W}_S(f)(s)$, $a \cdot b = \mathbb{W}_S(f)(p)$, and $-a = \mathbb{W}_S(f)(\iota)$. It remains to prove that the ring axioms are verified. Suppose first that A' is torsion free. Then the ghost map is injective, and hence, the ring axioms are satisfied in this case. In general, we choose a surjective ring homomorphism $g \colon A'' \to A'$ from a torsion free ring A''. Then

$$\mathbb{W}_S(g) \colon \mathbb{W}_S(A'') \to \mathbb{W}_S(A')$$

is again surjective, and since the ring axioms are satisfied on the left-hand side, they are satisfied on the right-hand side. $\square$

If $T \subset S$ are two truncation sets, then the forgetful map

$$R_T^S \colon \mathbb{W}_S(A) \to \mathbb{W}_T(A)$$

is a natural ring homomorphism called the restriction from S to T. If $n \in \mathbb{N}$ and $S \subset \mathbb{N}$ is a truncation set, then $S/n = \{d \in \mathbb{N} \mid nd \in S\}$ again is a truncation set.

Lemma 1.3. *The nth Verschiebung map $V_n \colon \mathbb{W}_{S/n}(A) \to \mathbb{W}_S(A)$, whose mth component is a_d, if $m = nd$, and 0, otherwise, is an additive map.*

Proof. The following diagram, where V_n^w takes the sequence $\langle x_d \mid d \in S/n \rangle$ to the sequence whose mth component is nx_d, if $m = nd$, and 0, otherwise, commutes.

$$\begin{tikzcd}
\mathbb{W}_{S/n}(A) \arrow[r, "w"] \arrow[d, "V_n"] & A^{S/n} \arrow[d, "V_n^w"] \\
\mathbb{W}_S(A) \arrow[r, "w"] & A^S
\end{tikzcd}$$

Since V_n^w is additive, so is V_n. Indeed, if A is torsion free, the horizontal maps are both injective, and hence V_n is additive in this case. In general, we choose a surjective ring homomorphism $g \colon A' \to A$ and argue as in the proof of Proposition 1.2. $\square$

Lemma 1.4. *There exists a unique natural ring homomorphism*

$$F_n \colon \mathbb{W}_S(A) \to \mathbb{W}_{S/n}(A)$$

called the nth Frobenius that makes the following diagram, where the map F_n^w takes the sequence $\langle x_m \mid m \in S \rangle$ to the sequence whose dth component is x_{nd}, commute.

$$\begin{tikzcd}
\mathbb{W}_S(A) \arrow[r, "w"] \arrow[d, "F_n"] & A^S \arrow[d, "F_n^w"] \\
\mathbb{W}_{S/n}(A) \arrow[r, "w"] & A^{S/n}
\end{tikzcd}$$

Proof. The construction of the map F_n is similar to the proof of Proposition 1.2. We let $A = \mathbb{Z}[a_n \mid n \in S]$ and let a be the vector $(a_n \mid n \in S)$. Then Lemma 1.1 shows that the sequence $F_n^w(w(a)) \in A^{S/n}$ is the image of a (unique) vector

$$F_n(a) = (f_{n,d} \mid d \in S/n) \in \mathbb{W}_{S/n}(A)$$

by the ghost map. If A' is any ring and $a' = (a'_n \mid n \in S)$ a vector in $\mathbb{W}_S(A')$, then we define $F_n(a') = \mathbb{W}_{S/n}(g)(F_n(a))$ where $g \colon A \to A'$ is the unique ring homomorphism that maps a to a'. Finally, since F_n^w is a ring homomorphism, an argument similar to the proof of Lemma 1.3 shows that also F_n is a ring homomorphism. □

The Teichmüller representative is the map

$$[-]_S \colon A \to \mathbb{W}_S(A)$$

whose nth component is a for $n = 1$, and 0 for $n > 1$. It is a multiplicative map. Indeed, the following diagram, where $[a]_S^w$ is the sequence with nth component a^n, commutes, and $[-]_S^w$ is a multiplicative map.

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "{[-]_S}"] & A \arrow[d, "{[-]_S^w}"] \\
\mathbb{W}_S(A) \arrow[r, "w"] & A^S
\end{tikzcd}$$

Lemma 1.5. *The following relations holds.*
(i) $a = \sum_{n \in S} V_n([a_n]_{S/n})$.
(ii) $F_n V_n(a) = na$.
(iii) $aV_n(a') = V_n(F_n(a)a')$.
(iv) $F_m V_n = V_n F_m \quad$ *if* $(m,n) = 1$.
(v) $F_n([a]_S) = [a^n]_{S/n}$.

Proof. One easily verifies that both sides of each equation have the same image by the ghost map. This shows that the relations hold, if A is torsion free, and hence, in general. □

Proposition 1.6. *The ring of Witt vectors in $\mathbb{Z}$ is equal to the product*

$$\mathbb{W}_S(\mathbb{Z}) = \prod_{n\in S} \mathbb{Z}\cdot V_n([1]_{S/n})$$

and the product given by $V_m([1]_{S/m})\cdot V_n([1]_{S/n}) = c\cdot V_e([1]_{S/e})$ with $c=(m,n)$ the greatest common divisor and $e=[m,n]$ the least common multiple.

Proof. The formula for the product follows from Lemma 1.5 (ii)-(iv). For finite S, we prove the statement by induction beginning from the case $S=\emptyset$ which is trivial. So suppose that S is non-empty, let $m\in S$ be maximal, and let $T=S\smallsetminus\{m\}$. The sequence of abelian groups

$$0\to \mathbb{W}_{\{1\}}(\mathbb{Z}) \xrightarrow{V_m} \mathbb{W}_S(\mathbb{Z}) \xrightarrow{R^S_T} \mathbb{W}_T(\mathbb{Z}) \to 0$$

is exact, and we wish to show that it is equal to the sequence

$$0\to \mathbb{Z}\cdot[1]_{\{1\}} \xrightarrow{V_m} \prod_{n\in S}\mathbb{Z}\cdot V_n([1]_{S/n}) \xrightarrow{R^S_T} \prod_{n\in T}\mathbb{Z}\cdot V_n([1]_{T/n}) \to 0.$$

The latter sequence is a sub-sequence of the former, and by induction, the right-hand terms of the two sequences are equal. Since also the left-hand terms are equal, so are the middle terms. This completes the proof for S finite. Finally, every truncation set S is the union of its finite sub-truncation sets $S_\alpha\subset S$ and $\mathbb{W}_S(\mathbb{Z}) = \lim_\alpha \mathbb{W}_{S_\alpha}(\mathbb{Z})$. □

The values of the restriction, Frobenius, and Verschiebung maps on the generators $V_n([1]_{S/n})$ are readily evaluated by using Lemma 1.5 (ii)–(iv). To give a formula for the Teichmüller representative, we recall the Möbius inversion formula. Let $g\colon \mathbb{N}\to\mathbb{Z}$ be a function and define the function $f\colon\mathbb{N}\to\mathbb{Z}$ by $f(n)=\sum_{d|n} g(d)$. Then the original function is given by $g(n)=\sum_{d|n}\mu(d)f(n/d)$, where $\mu\colon\mathbb{N}\to\{-1,0,1\}$ is the Möbius function defined by $\mu(d)=(-1)^r$, if d is a product of $r\geqslant 0$ distinct prime numbers, and $\mu(d)=0$, otherwise.

Addendum 1.7. *Let m be an integer. Then*

$$[m]_S = \sum_{n\in S}\frac{1}{n}\big(\sum_{d|n}\mu(d)m^{n/d}\big)V_n([1]_{S/n}).$$

If $m=q$ is a prime power, the coefficient of $V_n([1]_{S/n})$ is equal to the number of monic irreducible polynomials of degree n over the finite field $\mathbb{F}_q$.

Proof. It suffices to prove that the formula holds in $\mathbb{W}_S(\mathbb{Z})$. By Proposition 1.6, there are unique integers r_d, $d\in S$ such that

$$[m]_S = \sum_{d\in S} r_d V_d([1]_{S/d}).$$

Evaluating the nth ghost component of this equation, we find that

$$m^n = \sum_{d|n} d r_d.$$

The Möbius inversion formula completes the proof. □

Lemma 1.8. *If A is an $\mathbb{F}_p$-algebra, then*

$$F_p = R^S_{S/p} \circ \mathbb{W}_S(\varphi)\colon \mathbb{W}_S(A) \to \mathbb{W}_{S/p}(A),$$

where $\varphi\colon A \to A$ is the Frobenius endomorphism.

Proof. We let $A = \mathbb{Z}[a_n \mid n \in S]$ and recall from the proof of Lemma 1.4 the family of polynomials $f_{p,n} \in A$, $n \in S$, defined by the following system of equations.

$$\sum_{d \mid n} d f_{p,d}^{n/d} = \sum_{d \mid pn} d a_d^{pn/d}$$

The lemma is equivalent to the statement that for all $n \in S/p$, $f_{p,n} \equiv a_n^p$ modulo pA. We proceed by induction on $n \in S$. Since $f_{p,1} = a_1^p + pa_p$, the statement holds for $n = 1$. So let $n > 1$ and assume that for all proper divisors d of n, $f_{p,d} \equiv a_d^p$ modulo pA. As in the proof of Lemma 1.1, it follows that $d f_{p,d}^{n/d} \equiv d a_d^{pn/d}$ modulo $p^{v_p(n)+1}A$. We now rewrite the defining system of equations as

$$\sum_{d \mid n} d f_{p,d}^{n/d} = \sum_{d \mid n} d a_d^{pn/d} + \sum_{d \mid pn, d \nmid n} d a_d^{pn/d}$$

and note that if $d \mid pn$ and $d \nmid n$, then $v_p(d) = v_p(n) + 1$. Therefore, we may conclude that $n f_{p,n} \equiv n a_n^p$ modulo $p^{v_p(n)+1}A$. But A is torsion free, so $f_{p,n} \equiv a_n^p$ modulo pA as desired. This completes the proof. $\square$

Lemma 1.9. *Suppose that the integer m is invertible (resp. a non-zero-divisor) in A. Then m is invertible (resp. a non-zero-divisor) in $\mathbb{W}_S(A)$.*

Proof. As in the proof of Proposition 1.6, we may assume that S is finite. We proceed by induction on S beginning from the trivial case $S = \emptyset$. So let S be non-empty and assume the statement for all proper sub-truncation sets of S. We let $n \in S$ be maximal, and let $T = S \smallsetminus \{n\}$. Then we have the exact sequence

$$0 \to \mathbb{W}_{\{1\}}(A) \xrightarrow{V_n} \mathbb{W}_S(A) \xrightarrow{R^S_T} \mathbb{W}_T(A) \to 0$$

from which the induction step readily follows. $\square$

We consider the truncation set $P = \{1, p, p^2, \dots\} \subset \mathbb{N}$ that consists of all powers of a fixed prime number p. The proper non-empty sub-truncation sets of P all are of the form $\{1, p, \dots, p^{n-1}\}$, for some positive integer n. The rings

$$W(A) = \mathbb{W}_P(A), \qquad W_n(A) = \mathbb{W}_{\{1,p,\dots,p^{n-1}\}}(A)$$

are called the ring of p-typical Witt vectors in A and the ring of p-typical Witt vectors of length n in A, respectively. We have the following p-typical decomposition.

Proposition 1.10. *Let p be a prime number, S a truncation set, and $I(S)$ the set of $k \in S$ not divisible by p. If A is a ring in which every $k \in I(S)$ is invertible, then the ring of Witt vectors $\mathbb{W}_S(A)$ admits the idempotent decomposition*

$$\mathbb{W}_S(A) = \prod_{k \in I(S)} \mathbb{W}_S(A)e_k, \qquad e_k = \prod_{l \in I(S) \smallsetminus \{1\}} \left(\frac{1}{k} V_k([1]_{S/k}) - \frac{1}{kl} V_{kl}([1]_{S/kl}) \right).$$

Moreover, the composite map

$$\mathbb{W}_S(A)e_k \hookrightarrow \mathbb{W}_S(A) \xrightarrow{F_k} \mathbb{W}_{S/k}(A) \xrightarrow{R^{S/k}_{S/k\cap P}} \mathbb{W}_{S/k\cap P}(A)$$

is an isomorphism of rings.

Proof. We first show that e_k, $k \in I(S)$, are orthogonal idempotents in $\mathbb{W}_S(A)$. It suffices to consider $A = \mathbb{Z}[I(S)^{-1}]$, and since the ghost map is injective in this case, it further suffices to show that $w(e_k)$, $k \in I(S)$, are orthogonal idempotents in A^S. This, in turn, follows immediately from the calculation that

$$w_n(e_k) = \begin{cases} 1 & \text{if } n \in S \cap kP \\ 0 & \text{otherwise.} \end{cases}$$

Finally, to prove the second part of the statement, we consider the composite map

$$\mathbb{W}_{S/k\cap P}(A) \xrightarrow{\sigma} \mathbb{W}_{S/k}(A) \xrightarrow{\frac{1}{k}V_k} \mathbb{W}_S(A) \xrightarrow{\mathrm{pr}} \mathbb{W}_S(A)e_k,$$

where σ is any set theoretic section of $R^{S/k}_{S/k\cap P}$ and pr the canonical projection. We claim that this map is the inverse of the map in the statement. Indeed, this is readily verified by evaluating the induced map in ghost coordinates. $\square$

Example 1.11. If $S = \{1, 2, \dots, n\}$, then $S/k \cap P = \{1, p, \dots, p^{s-1}\}$ where $s = s(n,k)$ is the unique integer with $p^{s-1}k \leqslant n < p^s k$. Hence, if every integer $1 \leqslant k \leqslant n$ not divisible by p is invertible in A, then Proposition 1.10 shows that

$$\mathbb{W}_{\{1,2,\dots,n\}}(A) \xrightarrow{\sim} \prod W_s(A)$$

where the product ranges over integers $1 \leqslant k \leqslant n$ not divisible by p and $s = s(n,k)$.

We consider the rings $W_n(A)$ in more detail. The p-typical ghost map

$$w \colon W_n(A) \to A^n$$

takes the vector $(a_0, \dots, a_{n-1})$ to the sequence $\langle w_0, \dots, w_{n-1} \rangle$ with

$$w_i = a_0^{p^i} + p a_1^{p^{i-1}} + \dots + p^i a_i.$$

If A is equipped with a ring endomorphism $\phi \colon A \to A$ with $\phi(a) \equiv a^p$ modulo pA, then Lemma 1.1 identifies the image of the ghost map with the subring of sequences $\langle x_0, \dots x_{n-1} \rangle$ such that $x_i \equiv \phi(x_{i-1})$ modulo $p^i A$, for all $1 \leqslant i \leqslant n-1$. We write

$$[-]_n \colon A \to W_n(A)$$

for the Teichmüller representative and

$$F\colon W_n(A) \to W_{n-1}(A)$$
$$V\colon W_{n-1}(A) \to W_n(A)$$

for the pth Frobenius and pth Verschiebung.

Lemma 1.12. *If A is an $\mathbb{F}_p$-algebra, then $VF = p \cdot \mathrm{id}$.*

Proof. By Lemma 1.5 (iii), $VF = V([1]_{n-1}) \cdot \mathrm{id}$. We show by induction on n that for an $\mathbb{F}_p$-algebra A, $V([1]_{n-1}) = p \cdot [1]_n$; the case $n = 1$ is trivial. The exact sequences

$$0 \longrightarrow A \xrightarrow{V^{n-1}} W_n(A) \xrightarrow{R} W_{n-1}(A) \longrightarrow 0$$

furnish an induction argument which shows that $W_n(A)$ is annihilated by p^n. In particular, $V([1]_{n-1})$ is annihilated by p^{n-1}. Now, Addendum 1.7 shows that

$$[p]_n = p[1]_n + \sum_{0<s<n} \frac{p^{p^s} - p^{p^{s-1}}}{p^s} V^s([1]_{n-s}).$$

Here $[p]_n = 0$ since A is an $\mathbb{F}_p$-algebra, and $V^s([1]_{n-s}) = p^s V([1]_{n-1})$ by the inductive hypothesis. Therefore, the formula becomes

$$0 = p[1]_n + (p^{p^{n-1}-1} - 1)V([1]_{n-1}).$$

But $p^{n-1} - 1 \geqslant n - 1$, so the induction step follows. $\square$

Let A be a p-torsion free ring equipped with a ring homomorphism $\phi\colon A \to A$ such that $\phi(a) \equiv a^p$ modulo pA. By Lemma 1.1, there is a unique ring homomorphism

$$\lambda_\phi\colon A \to W(A)$$

whose nth ghost component is ϕ^n. We define $\tau_\phi\colon A \to W(A/pA)$ to be the composite of λ_ϕ and the map induced by the canonical projection of A onto A/pA. We recall that A/pA is said to be perfect, if the Frobenius $\varphi\colon A/pA \to A/pA$ is an automorphism.

Proposition 1.13. *Let A be a p-torsion free ring equipped with a ring homomorphism $\phi\colon A \to A$ such that $\phi(a) \equiv a^p$ modulo pA, and suppose that A/pA is perfect. Then for every positive integer n, the map τ_ϕ induces an isomorphism*

$$\tau_\phi\colon A/p^nA \xrightarrow{\sim} W_n(A/pA).$$

Proof. The map τ_ϕ induces a map as stated since

$$V^n W(A/pA) = V^n W(\phi^n(A/pA)) = V^n F^n W(A/pA) = p^n W(A/pA);$$

compare Lemmas 1.8 and 1.12. The proof is completed by an induction argument based on the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A/pA & \xrightarrow{p^{n-1}} & A/p^nA & \xrightarrow{\mathrm{pr}} & A/p^{n-1}A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi^{n-1}} & & \downarrow{\scriptstyle \tau_\phi} & & \downarrow{\scriptstyle \tau_\phi} & & \\
0 & \longrightarrow & A/pA & \xrightarrow{V^{n-1}} & W_n(A/pA) & \xrightarrow{R} & W_{n-1}(A/pA) & \longrightarrow & 0.
\end{array}$$

Here the top horizontal sequence is exact, since A is p-torsion free, and the left-hand vertical map is an isomorphism, since A/pA is perfect. ☐

We return to the ring of big Witt vectors. We write $(1+tA[[t]])^*$ for the multiplicative group of power series over A with constant term 1.

Proposition 1.14. *There is a natural commutative diagram*

$$\begin{array}{ccc}
\mathbb{W}(A) & \xrightarrow{\gamma} & (1+tA[[t]])^* \\
\downarrow{\scriptstyle w} & & \downarrow{\scriptstyle t\frac{d}{dt}\log} \\
A^{\mathbb{N}} & \xrightarrow{\gamma^w} & tA[[t]]
\end{array}$$

where the horizontal maps are defined by

$$\gamma(a_1,a_2,\dots) = \prod_{n\geqslant 1}(1-a_nt^n)^{-1}, \qquad \gamma^w\langle x_1,x_2,\dots\rangle = \sum_{n\geqslant 1} x_nt^n.$$

The maps γ and γ^w are isomorphisms of abelian groups.

Proof. It is clear that γ^w is an isomorphism of abelian groups. We show that the map γ is a bijection. To this end, we write

$$\prod_{n\geqslant 1}(1-a_nt^n)^{-1} = (1+b_1t+b_2t^2+\dots)^{-1}$$

where the coefficient b_n is given by the sum

$$b_n = \sum(-1)^r a_{i_1}\dots a_{i_r}$$

that ranges over all $1\leqslant i_1<\dots<i_r\leqslant n$ such that $i_1+2i_2+\dots+ri_r=n$. It follows that the coefficients a_n are determined recursively by the coefficients b_n, and hence, that γ is a bijection. It remains to prove that γ is a homomorphism from the additive group $\mathbb{W}(A)$ to the multiplicative group $(1+tA[[t]])^*$. We may assume that A is torsion free. Then the vertical maps in the diagram of the statement are both injective, and it suffices to show that this diagram commutes. The calculation

$$\begin{aligned}
t\frac{d}{dt}\log(\prod_{d\geqslant 1}(1-a_dt^d)^{-1}) &= -\sum_{d\geqslant 1} t\frac{d}{dt}\log(1-a_dt^d) = \sum_{d\geqslant 1}\frac{da_dt^d}{1-a_dt^d} \\
&= \sum_{d\geqslant 1}\sum_{s\geqslant 0} da_dt^d\cdot a_d^st^{sd} = \sum_{d\geqslant 1}\sum_{q\geqslant 1} da_d^qt^{qd} = \sum_{n\geqslant 1}\big(\sum_{d\mid n} da_d^{n/d}\big)t^n
\end{aligned}$$

completes the proof. ☐

Addendum 1.15. *The map γ induces an isomorphism of abelian groups*

$$\gamma_S \colon \mathbb{W}_S(A) \xrightarrow{\sim} \Gamma_S(A)$$

where $\Gamma_S(A)$ is the quotient of the multiplicative group $\Gamma(A) = (1+tA[[t]])^*$ by the subgroup $I_S(A)$ of all power series of the form $\prod_{n\in\mathbb{N}\smallsetminus S}(1-a_nt^n)^{-1}$.

Proof. The kernel of the restriction map $R^{\mathbb{N}}_S \colon \mathbb{W}(A) \to \mathbb{W}_S(A)$ is equal to the subset of all vectors $a = (a_n \mid n \in \mathbb{N})$ such that $a_n = 0$, if $n \in S$. The image of this subset by the map γ is the subset $I_S(A) \subset \Gamma$. □

Example 1.16. If $S = \{1,2,\dots,n\}$, then $I_S(A) = (1+t^{n+1}A[[t]])^*$. Hence, in this case, Addendum 1.15 gives an isomorphism of abelian groups

$$\gamma_S \colon \mathbb{W}_{\{1,2,\dots,n\}}(A) \xrightarrow{\sim} (1+tA[[t]])^*/(1+t^{n+1}A[[t]])^*.$$

For A a $\mathbb{Z}_{(p)}$-algebra, the structure of this group was examined in Example 1.11.

Lemma 1.17. *Let p be a prime number, and let A be any ring. Then the ring homomorphism $F_p \colon \mathbb{W}(A) \to \mathbb{W}(A)$ satisfies that $F_p(a) \equiv a^p$ modulo $p\mathbb{W}(A)$.*

Proof. We first let $A = \mathbb{Z}[a_1,a_2,\dots]$ and $a = (a_1,a_2,\dots)$. If suffices to show that there exists $b \in \mathbb{W}(A)$ such that $F_p(a) - a^p = pb$. By Lemma 1.9, the element is necessarily unique; we use Lemma 1.1 to prove that it exists. We have

$$w_n(F_p(a)-a^p) = \sum_{d\mid pn} da_d^{pn/d} - \big(\sum_{d\mid n} da_d^{n/d}\big)^p$$

which is clearly congruent to zero modulo pA. So let $x = \langle x_n \mid n \in \mathbb{N}\rangle$ with

$$x_n = \frac{1}{p}(F_p(a)-a^p).$$

We wish to show that $x = w(b)$, for some $b \in \mathbb{W}(A)$. The unique ring homomorphism $\phi_\ell \colon A \to A$ that maps a_n to a_n^ℓ satisfies that $\phi_\ell(f) = f^\ell$ modulo ℓA, and hence, Lemma 1.1 shows that x is in the image of the ghost map if and only if

$$x_n \equiv \phi_\ell(x_{n/\ell})$$

modulo $\ell^{v_\ell(n)}A$, for all primes ℓ and all $n \in \ell\mathbb{N}$. This is equivalent to showing that

$$w_n(F_p(a)-a^p) \equiv \phi_\ell(w_{n/p}(F_p(a)-a^p))$$

modulo $\ell^{v_\ell(n)}A$, if $\ell \neq p$ and $n \in \ell\mathbb{N}$, and modulo $\ell^{v_\ell(n)+1}A$, if $\ell = p$ and $n \in \ell\mathbb{N}$. If $\ell \neq p$, the statement follows from Lemma 1.1, and if $\ell = p$ and $n \in \ell\mathbb{N}$, we calculate

$$\begin{aligned}&w_n(F_p(a)-a^p) - \phi_p(w_{n/p}(F_p(a)-a^p))\\ &= \sum_{d\mid pn, d\nmid n} da_d^{pn/d} - \big(\sum_{d\mid n} da_d^{n/d}\big)^p + \big(\sum_{d\mid (n/p)} da_d^{n/d}\big)^p.\end{aligned}$$

If $d \mid pn$ and $d \nmid n$, then $v_p(d) = v_p(n) + 1$, so the first summand is congruent to zero modulo $p^{v_p(n)+1}A$. Similarly, if $d \mid n$ and $d \nmid (n/p)$, then $v_p(d) = v_p(n)$, and hence,

$$\sum_{d|n} da_d^{n/d} \equiv \sum_{d|(n/p)} da_d^{n/d}$$

modulo $p^{v_p(n)}A$. But then

$$\Big(\sum_{d|n} da_d^{n/d}\Big)^p \equiv \Big(\sum_{d|(n/p)} da_d^{n/d}\Big)^p$$

modulo $p^{v_p(n)+1}A$; compare the proof of Lemma 1.1. This completes the proof. $\square$

We write $\varepsilon = \varepsilon_A \colon \mathbb{W}(A) \to A$ for the natural ring homomorphism $w_1 \colon \mathbb{W}(A) \to A$.

Proposition 1.18. *There exists a unique natural ring homomorphism*

$$\Delta = \Delta_A \colon \mathbb{W}(A) \to \mathbb{W}(\mathbb{W}(A))$$

such that $w_n(\Delta(a)) = F_n(a)$ for all positive integers n. Moreover, the diagrams

$$\begin{tikzcd} \mathbb{W}(A) & \mathbb{W}(\mathbb{W}(A)) \arrow[l, "\varepsilon_{\mathbb{W}(A)}"'] \arrow[r, "\mathbb{W}(\varepsilon_A)"] & \mathbb{W}(A) \\ & \mathbb{W}(A) \arrow[ul, equal] \arrow[u, "\Delta_A"'] \arrow[ur, equal] & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{W}(\mathbb{W}(\mathbb{W}(A))) & \mathbb{W}(\mathbb{W}(A)) \arrow[l, "\Delta_{\mathbb{W}(A)}"'] \\ \mathbb{W}(\mathbb{W}(A)) \arrow[u, "\mathbb{W}(\Delta_A)"'] & \mathbb{W}(A) \arrow[l, "\Delta_A"'] \arrow[u, "\Delta_A"'] \end{tikzcd}$$

commute.

Proof. By naturality, we may assume that A is torsion free. Lemma 1.9 shows that also $\mathbb{W}(A)$ is torsion free such that the ghost map

$$w \colon \mathbb{W}(\mathbb{W}(A)) \to \mathbb{W}(A)^{\mathbb{N}}$$

is injective. Lemma 1.17 and Lemma 1.1 show that the sequence $\langle F_n(a) \mid a \in \mathbb{N} \rangle$ is in the image of the ghost map. Hence, the natural ring homomorphism Δ exists. The commutativity of the diagrams is readily verified by evaluating the corresponding maps in ghost coordinates. $\square$

Remark 1.19. The map $\Delta_n \colon \mathbb{W}(A) \to \mathbb{W}(A)$ given by the nth Witt component of the map Δ is generally not a ring homomorphism. For example, for a prime number p, the map Δ_p is the unique natural solution to the equation $F_p(a) = a^p + p\Delta_p(a)$.

The definition of a λ-ring due to Grothendieck [9] may be stated as follows.

Definition 1.20. A λ-ring is a pair (A, λ_A) of a ring A and a ring homomorphism $\lambda_A \colon A \to \mathbb{W}(A)$ that the following diagrams commute.

$$\begin{tikzcd} A & \mathbb{W}(A) \arrow[l, "\varepsilon_A"'] \\ & A \arrow[ul, equal] \arrow[u, "\lambda_A"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{W}(\mathbb{W}(A)) & \mathbb{W}(A) \arrow[l, "\Delta_A"'] \\ \mathbb{W}(A) \arrow[u, "\mathbb{W}(\lambda_A)"'] & A \arrow[l, "\lambda_A"'] \arrow[u, "\lambda_A"'] \end{tikzcd}$$

A morphism of λ-rings $f\colon (A,\lambda_A) \to (B,\lambda_B)$ is a ring homomorphism $f\colon A \to B$ with the property that $\lambda_B \circ f = \mathbb{W}(f) \circ \lambda_A$.

Remark 1.21. The commutative diagrams in Proposition 1.18 and Definition 1.20 express that $(\mathbb{W}(-),\Delta,\varepsilon)$ is a comonad on the category of rings and that (A,λ_A) is a coalgebra over this comonad, respectively. The composite ring homomorphism

$$A \xrightarrow{\lambda_A} \mathbb{W}(A) \xrightarrow{w_n} A$$

is called the nth associated Adams operation. We write $\psi_{A,n}\colon A \to A$ for this map. We further write $\lambda_{A,n}\colon A \to A$ for the nth component of the map $\lambda_A\colon A \to \mathbb{W}(A)$. It is related to the traditional λ-operation $\lambda_A^n\colon A \to A$ [1, Exposé V] by the formula

$$\prod_{n\geqslant 1}(1-\lambda_{A,n}(a)t^n) = 1 + \sum_{n\geqslant 1}(-1)^n\lambda_A^n(a)t^n.$$

We note that $\lambda_{A,n}\colon A \to A$ and $\lambda_A^n\colon A \to A$ are generally not ring homomorphisms.

Finally, we recall the following general theorem which was proved independently by Borger [3, Theorem B], [4, Corollary 15.4] and van der Kallen [22, Theorem 2.4].

Theorem 1.22. *Let $f\colon A \to B$ be an étale morphism, let S be a finite truncation set, and let n be a positive integer. Then the morphism*

$$\mathbb{W}_S(f)\colon \mathbb{W}_S(A) \to \mathbb{W}_S(B)$$

is étale and the diagram

$$\begin{tikzcd}
\mathbb{W}_S(A) \arrow[r, "\mathbb{W}_S(f)"] \arrow[d, "F_n"] & \mathbb{W}_S(B) \arrow[d, "F_n"] \\
\mathbb{W}_{S/n}(A) \arrow[r, "\mathbb{W}_{S/n}(f)"] & \mathbb{W}_{S/n}(B)
\end{tikzcd}$$

is cartesian. □

We remark that in loc. cit., the theorem above is stated only for those finite truncation sets that consist of all divisors of a given positive integer. These truncation sets, however, include all finite p-typical truncation sets, and therefore, the general case follows immediately by applying Proposition 1.10. Indeed, since S is finite, the canonical maps $\mathbb{W}_S(A_{(p)}) \to \mathbb{W}_S(A_{(p)})_{(p)} \leftarrow \mathbb{W}_S(A)_{(p)}$ are isomorphisms.

2 Modules and derivations over λ-rings

In general, given a category $\mathscr{C}$ in which finite limits exist and an object X of $\mathscr{C}$, Quillen [20] defines the category of X-modules to be the category $(\mathscr{C}/X)_{\mathrm{ab}}$ of abelian group objects in the category over X. Quillen further defines the set of derivations from X to the X-module $(Y/X,+_Y,0_Y,-_Y)$ to be the set of morphisms

$$\mathrm{Der}(X,(Y/X,+_Y,0_Y,-_Y)) = \mathrm{Hom}_{\mathscr{C}/X}(X/X,Y/X)$$

in the category $\mathscr{C}/X$. This set inherits an abelian group structure from the abelian group object structure on Y/X. In this section, we identify and study these notions in the case of the category $\mathscr{A}_\lambda$ of λ-rings.

To begin, we first recall the case of the category $\mathscr{A}$ of rings, which we always assume to be commutative and unital. Let A be a ring and let $\mathscr{M}(A)$ be the usual category of A-modules. We define an equivalence of categories

$$(\mathscr{A}/A)_{\mathrm{ab}} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{M}(A)$$

as follows. Let $f\colon B \to A$ be an object of $\mathscr{A}/A$, and let

$$\begin{tikzcd} B \times_A B \arrow[r, "+"] \arrow[d] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "0"] \arrow[d, equal] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "-"] \arrow[d, "f"] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

be abelian group object structure maps. Then the functor F associates to the abelian group object $(f,+,0,-)$ the A-module M defined by the kernel of the ring homomorphism f with the A-module structure map given by $a \cdot x = 0(a)x$. Conversely, the functor G associates to the A-module M the abelian group object $(f,+,0,-)$ where $f\colon A \oplus M \to A$ is the canonical projection from the ring given by the direct sum equipped with the multiplication $(a,x)\cdot(a',x') = (aa', ax' + a'x)$ together with the abelian group object structure maps

$$\begin{tikzcd} A \oplus M \times_A A \oplus M \arrow[r, "+"] \arrow[d] & A \oplus M \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "0"] \arrow[d, equal] & A \oplus M \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd} \qquad \begin{tikzcd} A \oplus M \arrow[r, "-"] \arrow[d, "f"] & A \oplus M \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

defined by $(a,x)+(a,x') = (a,x+x')$, $0(a) = (a,0)$, and $-(a,x) = (a,-x)$. We claim that the natural transformations $\eta\colon \mathrm{id} \Rightarrow FG$ and $\varepsilon\colon GF \Rightarrow \mathrm{id}$ given by

$$\begin{aligned} \eta(x) &= (0,x) \\ \varepsilon(a,x) &= 0(a) + x \end{aligned}$$

are well-defined and isomorphisms. This is clear for the map η. It is also clear that the map $\varepsilon\colon A \oplus M \to B$ is an isomorphism of abelian groups; we must show that it is a ring homomorphism and a map of abelian group objects. Since $+_B\colon B \times_A B \to B$ is a ring homomorphism, the compositions $+_B$ and $+$ on M are mutually distributive. They further have the common unit $0 = 0(0)$, and hence, agree. Therefore,

$$\begin{aligned} &\varepsilon(a,x) +_B \varepsilon(a,x') = (0(a)+x) +_B (0(a)+x') \\ &= (0(a) +_B 0(a)) + (x +_B x') = 0(a) + (x+x') = \varepsilon(a, x+x'). \end{aligned}$$

It remains to prove that ε is multiplicative, or equivalently, that $M \subset B$ is a square-zero ideal. Since $+_B\colon B \times_A B \to B$ is a ring homomorphism, we have

$$\varepsilon(a,x)\varepsilon(b,y) +_B \varepsilon(a,x')\varepsilon(b,y') = (\varepsilon(a,x) +_B \varepsilon(a,x'))(\varepsilon(b,y') +_B \varepsilon(b,y'))$$

which, by the formula just proved, becomes

$$\begin{aligned}&0(ab)+0(a)y+0(b)x+xy+0(a)y'+0(b)x'+x'y'\\&=0(ab)+0(a)(y+y')+0(b)(x+x')+(x+x')(y+y').\end{aligned}$$

This shows that $\varepsilon\colon A\oplus M\to B$ is a ring homomorphism as desired.

We next consider the ring of Witt vectors in the ring $A\oplus M$. We note that the polynomials s_n, p_n, and ι_n that defined the sum, product, and opposite in the ring of Witt vectors have zero constant terms. Therefore, the ring of Witt vectors is defined also for non-unital rings. Since M has zero multiplication, it follows that $\mathbb{W}(M)$ has zero multiplication and that the addition is componentwise addition.

Lemma 2.1. *The canonical inclusions induce a ring isomorphism*

$$\mathrm{in}_{1*}+\mathrm{in}_{2*}\colon \mathbb{W}(A)\oplus\mathbb{W}(M)\xrightarrow{\sim}\mathbb{W}(A\oplus M)$$

where, on the left-hand side, the ring structure is induced from the $\mathbb{W}(A)$-module structure on $\mathbb{W}(M)$ defined by $(ax)_n=w_n(a)x_n$.

Proof. In the following diagram of rings and ring homomorphisms, the sequence of map directed from left to right is an exact sequence of additive abelian groups.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\mathrm{in}_2"] & A\oplus M \arrow[r, shift right, "\mathrm{pr}_1"'] & A \arrow[l, shift right, "\mathrm{in}_1"'] \arrow[r] & 0 \end{tikzcd}$$

It induces a diagram of rings and ring homomorphisms where again the sequence of maps directed from left to right is an exact sequence of abelian groups.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{W}(M) \arrow[r, "\mathrm{in}_{2*}"] & \mathbb{W}(A\oplus M) \arrow[r, shift right, "\mathrm{pr}_{1*}"'] & \mathbb{W}(A) \arrow[l, shift right, "\mathrm{in}_{1*}"'] \arrow[r] & 0 \end{tikzcd}$$

Hence, the map $\mathrm{in}_{1*}+\mathrm{in}_{2*}$ is an isomorphism of abelian groups. Moreover, since $\mathbb{W}(M)$ has zero multiplication, this map is a ring isomorphism for the ring structure on the domain induced from the $\mathbb{W}(A)$-module structure on $\mathbb{W}(M)$ defined by the formula $\mathrm{in}_{2*}(ax)=\mathrm{in}_{1*}(a)\,\mathrm{in}_{2*}(x)$. Finally, we wish to show that ax is equal to the Witt vector y with nth component $w_n(a)x_n$. Since every ring admits a surjective ring homomorphism from a torsion free ring, we may further assume that A and M are both torsion free. Since the ghost map is injective in this case, it will suffice to show that $w_n(ax)=w_n(y)$, or equivalently, that $\mathrm{in}_2(w_n(ax))=\mathrm{in}_2(w_n(y))$, for all $n\geqslant 1$. Now, since w_n is a natural ring homomorphism, we find that for all $n\geqslant 1$,

$$\begin{aligned}\mathrm{in}_2(w_n(ax))&=w_n(\mathrm{in}_{2*}(ax))=w_n(\mathrm{in}_{1*}(a)\,\mathrm{in}_{2*}(x))=w_n(\mathrm{in}_{1*}(a))w_n(\mathrm{in}_{2*}(x))\\&=\mathrm{in}_1(w_n(a))\,\mathrm{in}_2(w_n(x))=\mathrm{in}_2(w_n(a)w_n(x))=\mathrm{in}_2(nw_n(a)x_n)\\&=\mathrm{in}_2(ny_n)=\mathrm{in}_2(w_n(y))\end{aligned}$$

as desired. Here the fifth equality follows from the definition of the multiplication on the direct sum ring $A\oplus M$. $\square$

We give an explicit recursive formula for the ring isomorphism $\mathrm{in}_{1*}+\mathrm{in}_{2*}$.

Lemma 2.2. *Let $b = \mathrm{in}_{1*}(a) + \mathrm{in}_{2*}(x)$ and write $b_n = (a_n, y_n)$. Then for all $n \geqslant 1$,*

$$\sum_{d|n} a_d^{(n/d)-1} y_d = x_n.$$

Proof. As before, we may assume that A and M are torsion free. We now calculate $w_n(b)$ in two different ways. First, since w_n is a natural ring homomorphism, we have

$$\begin{aligned} w_n(b) &= w_n(\mathrm{in}_{1*}(a)) + w_n(\mathrm{in}_{2*}(x)) = \mathrm{in}_1(w_n(a)) + \mathrm{in}_2(w_n(x)) \\ &= (w_n(a), w_n(x)) = (\sum_{d|n} d a_d^{n/d}, n x_n). \end{aligned}$$

Second, by the definition of the multiplication in $A \oplus M$, we have

$$w_n(b) = \sum_{d|n} d b_d^{n/d} = \sum_{d|n} d(a_d, y_d)^{n/d} = (\sum_{d|n} d a_d^{n/d}, \sum_{d|n} n a_d^{(n/d)-1} y_d).$$

The stated formula follows as M was assumed to be torsion free. ☐

Example 2.3. Let p be a prime number. Then $y_p = x_p - a_1^{p-1} x_1$.

We proceed to identify the category $(\mathscr{A}_\lambda/(A, \lambda_A))_{\mathrm{ab}}$ of abelian group objects in the category of λ-rings over (A, λ_A). By category theory, there is an adjunction

$$\mathscr{A}_\lambda \underset{R}{\overset{u}{\rightleftarrows}} \mathscr{A}$$

with the left adjoint functor defined by $u(A, \lambda_A) = A$ and with the right adjoint functor defined by $R(A) = (\mathbb{W}(A), \Delta_A)$. The unit and counit of the adjunction are defined to be the maps $\lambda_A \colon (A, \lambda_A) \to (\mathbb{W}(A), \Delta_A)$ and $\varepsilon_A \colon \mathbb{W}(A) \to A$, respectively. Since $\mathbb{W}(-)$ preserves limits, u creates limits. In particular, the category $\mathscr{A}_\lambda$ has all small limits. It follows, that for (A, λ_A) an object of $\mathscr{A}_\lambda$, there is an adjoint pair of functors

$$\mathscr{A}_\lambda/(A, \lambda_A) \underset{R_{(A,\lambda_A)}}{\overset{u_{(A,\lambda_A)}}{\rightleftarrows}} \mathscr{A}/A$$

where the left adjoint functor $u_{(A,\lambda_A)}$ maps $f \colon (B, \lambda_B) \to (A, \lambda_A)$ to $f \colon B \to A$, and where the right adjoint functor $R_{(A,\lambda_A)}$ maps $f \colon B \to A$ to $\mathrm{pr}_2 \colon (C, \lambda_C) \to (A, \lambda_A)$ defined by the following pull-back diagram.

$$\begin{array}{ccc} (C, \lambda_C) & \xrightarrow{\mathrm{pr}_1} & (\mathbb{W}(B), \Delta_B) \\ \big\downarrow{\scriptstyle \mathrm{pr}_2} & & \big\downarrow{\scriptstyle \mathbb{W}(f)} \\ (A, \lambda_A) & \xrightarrow{\lambda_A} & (\mathbb{W}(A), \Delta_A) \end{array}$$

The unit and counit maps are defined to be the maps (λ_B, f), and $\varepsilon_B \circ \mathrm{pr}_1$, respectively.

If M is an A-module, we view $\mathbb{W}(M)$ as a $\mathbb{W}(A)$-module as in Lemma 2.1. We now make the following definition of the category $\mathscr{M}(A, \lambda_A)$ of (A, λ_A)-modules.

Definition 2.4. Let (A,λ_A) be a λ-ring. An (A,λ_A)-module is a pair (M,λ_M) of an A-module M and a λ_A-linear map

$$\lambda_M \colon M \to \mathbb{W}(M)$$

with the property that the diagrams

$$\begin{tikzcd} M & \mathbb{W}(M) \arrow[l, "\varepsilon_M"'] \\ & M \arrow[u, "\lambda_M"'] \arrow[ul, equal] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{W}(\mathbb{W}(M)) & \mathbb{W}(M) \arrow[l, "\Delta_M"'] \\ \mathbb{W}(M) \arrow[u, "\mathbb{W}(\lambda_M)"'] & M \arrow[l, "\lambda_M"'] \arrow[u, "\lambda_M"'] \end{tikzcd}$$

commute. A morphism $h\colon (M,\lambda_M) \to (N,\lambda_N)$ of (A,λ_A)-modules is an A-linear map $h\colon M \to N$ such that $\lambda_N \circ h = \mathbb{W}(h) \circ \lambda_M$.

Remark 2.5. Let (A,λ_A) be a λ-ring, M an A-module, and $\lambda_M\colon M \to \mathbb{W}(M)$ a map. Then (M,λ_M) is an (A,λ_A)-module if and only if the components $\lambda_{M,n}\colon M \to M$ of the map λ_M satisfy the following conditions: The map $\lambda_{M,n}$ is linear with respect to the nth Adams operation $\psi_{A,n} = w_n \circ \lambda_A$; the map $\lambda_{M,1}$ is the identity map; and the composite map $\lambda_{M,m} \circ \lambda_{M,n}$ is equal to the map $\lambda_{M,mn}$.

Example 2.6. Let (A,λ_A) be a λ-ring. The functor that to an (A,λ_A)-module (M,λ_M) assigns the underlying set of M has a left adjoint functor that to the set S assigns the free (A,λ_A)-module $(F(S),\lambda_{F(S)})$ defined as follows. The A-module $F(S)$ is defined to be the free A-module generated by the symbols $\lambda_{F(S),n}(s)$, where $s \in S$ and $n \in \mathbb{N}$, and $\lambda_{F(S)}\colon F(S) \to \mathbb{W}(F(S))$ is defined to be the map with mth component

$$\lambda_{F(S),m}(\textstyle\sum a_{s,n}\lambda_{F(S),n}(s)) = \sum \psi_{A,m}(a_{s,n})\lambda_{F(S),mn}(s).$$

It follows from Remark 2.5 that the pair $(F(S),\lambda_{F(S)})$ is an (A,λ_A)-module. The unit of the adjunction maps $s \in S$ to $\lambda_{F(S),1}(s) \in F(S)$, and the counit of the adjunction maps $\sum a_{x,n}\lambda_{F(M),n}(x) \in F(M)$ to $\sum a_{x,n}\lambda_{M,n}(x) \in M$.

Example 2.7. Let (A,λ_A) be a λ-ring. Then (A,λ_A) is not an (A,λ_A)-module except in trivial cases. However, the Adams operations $\psi_{A,n}\colon A \to A$ define an (A,λ_A)-module structure on A considered as an A-module. The (A,λ_A)-module defined in this way is not a free module except in trivial cases.

Example 2.8. Let A be a ring, unital and commutative as usual, and let $K_*(A)$ be the graded ring given by the Quillen K-groups. Then the ring $K_0(A)$ has a canonical λ-ring structure defined by Grothendieck [9], and for all $q \geqslant 1$, the group $K_q(A)$ has a canonical structure of a module over this λ-ring defined by Kratzer [17] and Quillen [15]. The $(K_0(A),\lambda_{K_0(A)})$-module structure maps are determined by the maps

$$\lambda^n_{K_q(A)}\colon K_q(A) \to K_q(A)$$

defined in [17, Théorème 5.1]. We remark that the corresponding Adams operation is given by $\psi_{K_q(A),n} = (-1)^{n-1}n\lambda^n_{K_q(A)}$; compare Remark 1.21.

We show that the categories $\mathscr{M}(A,\lambda_A)$ and $(\mathscr{A}_\lambda/(A,\lambda_A))_{\mathrm{ab}}$ are equivalent. First note that since the adjoint functors $u_{(A,\lambda_A)}$ and $R_{(A,\lambda_A)}$ considered earlier both preserve limits, they give rise to the following adjoint pair of functors.

$$\begin{tikzcd} (\mathscr{A}_\lambda/(A,\lambda_A))_{\mathrm{ab}} \arrow[r, shift left, "u^{\mathrm{ab}}_{(A,\lambda_A)}"] & (\mathscr{A}/A)_{\mathrm{ab}} \arrow[l, shift left, "R^{\mathrm{ab}}_{(A,\lambda_A)}"] \end{tikzcd}$$

Next, we define a pair of functors

$$\begin{tikzcd} (\mathscr{A}_\lambda/(A,\lambda_A))_{\mathrm{ab}} \arrow[r, shift left, "F_\lambda"] & \mathscr{M}(A,\lambda_A) \arrow[l, shift left, "G_\lambda"] \end{tikzcd}$$

as follows. If $f\colon (B,\lambda_B)\to(A,\lambda_A)$ is an object of $\mathscr{A}_\lambda/(A,\lambda_A)$ and if

$$\begin{tikzcd} (B,\lambda_B)\times_{(A,\lambda_A)}(B,\lambda_B) \arrow[r, "+"] \arrow[d] & (B,\lambda_B) \arrow[d, "f"] \\ (A,\lambda_A) \arrow[r, equal] & (A,\lambda_A) \end{tikzcd}$$

$$\begin{tikzcd} (A,\lambda_A) \arrow[r, "0"] \arrow[d, equal] & (B,\lambda_B) \arrow[d, "f"] \\ (A,\lambda_A) \arrow[r, equal] & (A,\lambda_A) \end{tikzcd} \qquad \begin{tikzcd} (B,\lambda_B) \arrow[r, "-"] \arrow[d, "f"] & (B,\lambda_B) \arrow[d, "f"] \\ (A,\lambda_A) \arrow[r, equal] & (A,\lambda_A) \end{tikzcd}$$

are abelian group object structure maps, then the functor F^λ associates to the abelian group object $(f,+,0,-)$ the (A,λ_A)-module (M,λ_M), where $M=\ker(f)$ considered as an A-module via $0\colon A\to B$, and where $\lambda_M\colon M\to\mathbb{W}(M)$ is the induced map of kernels of the vertical maps in the following diagram.

$$\begin{tikzcd} B \arrow[r, "\lambda_B"] \arrow[d, "f"] & \mathbb{W}(B) \arrow[d, "\mathbb{W}(f)"] \\ A \arrow[r, "\lambda_A"] & \mathbb{W}(A) \end{tikzcd}$$

Conversely, if (M,λ_M) is an (A,λ_A)-module, then $G^\lambda(M,\lambda_M)$ is the unique abelian group object such that $u^{\mathrm{ab}}_{(A,\lambda_A)}(G^\lambda(M,\lambda_M))=G(M)$ and such that the λ-ring structure on the ring $A\oplus M$ is given by the map $\lambda_{A\oplus M}$ defined by the composition

$$A\oplus M \xrightarrow{\lambda_A\oplus\lambda_M} \mathbb{W}(A)\oplus\mathbb{W}(M) \xrightarrow{\mathrm{in}_{1*}+\mathrm{in}_{2*}} \mathbb{W}(A\oplus M).$$

Finally, it follows from what was proved earlier that there are well-defined natural isomorphisms $\eta^\lambda\colon \mathrm{id}\Rightarrow F^\lambda G^\lambda$ and $\varepsilon^\lambda\colon G^\lambda F^\lambda\Rightarrow\mathrm{id}$ given by

$$\eta^\lambda(x)=(0,x)$$
$$\varepsilon^\lambda(a,x)=0(a)+x.$$

This shows that the categories $(\mathscr{A}_\lambda/(A,\lambda_A))_{\mathrm{ab}}$ and $\mathscr{M}(A,\lambda_A)$ are equivalent.

We summarize with the following diagram of categories and functors, where the horizontal pairs of functors are equivalences of categories, and where the vertical pairs of functors are adjoint pairs of functors.

$$\begin{tikzcd}
(\mathscr{A}/A)_{\mathrm{ab}} \arrow[r, shift left, "F"] \arrow[d, shift left, "R^{\mathrm{ab}}_{(A,\lambda_A)}"] & \mathscr{M}(A) \arrow[l, shift left, "G"] \arrow[d, shift left, "R'"] \\
(\mathscr{A}_\lambda/(A,\lambda_A))_{\mathrm{ab}} \arrow[r, shift left, "F^\lambda"] \arrow[u, shift left, "u^{\mathrm{ab}}_{(A,\lambda_A)}"] & \mathscr{M}(A,\lambda_A) \arrow[l, shift left, "G^\lambda"] \arrow[u, shift left, "u'"]
\end{tikzcd}$$

The right-hand adjunction is defined as follows. The left adjoint functor u' takes the (A,λ_A)-module (M,λ_M) to the A-module M, and the right adjoint functor R' takes the A-module N to the (A,λ_A)-module $(\lambda_A^*\mathbb{W}(N),\Delta_N)$, where $\lambda_A^*\mathbb{W}(N)$ denotes the $\mathbb{W}(A)$-module $\mathbb{W}(N)$ considered as an A-module via $\lambda_A\colon A\to\mathbb{W}(A)$. The unit and counit of the adjunction are given by the maps $\lambda_M\colon (M,\lambda_M)\to(\lambda_A^*\mathbb{W}(M),\Delta_M)$ and $\varepsilon_N\colon \lambda_A^*\mathbb{W}(N)\to N$, respectively.

Proposition 2.9. *The category $\mathscr{M}(A,\lambda_A)$ of modules over the λ-ring (A,λ_A) is an abelian category. Moreover, the forgetful functor $u'\colon \mathscr{M}(A,\lambda_A)\to\mathscr{M}(A)$ creates all small limits and colimits.*

Proof. We may identify $\mathscr{M}(A,\lambda_A)$ with the category $\mathscr{M}(A^\Psi[\mathbb{N}])$ of left modules over the twisted monoid algebra $A^\Psi[\mathbb{N}]$ with multiplication given by $n\cdot a=\psi_{A,n}(a)\cdot n$. $\square$

Remark 2.10. A map of λ-rings $f\colon (B,\lambda_B)\to(A,\lambda_A)$ gives rise to a functor

$$f^*\colon \mathscr{M}(A,\lambda_A)\to\mathscr{M}(B,\lambda_B)$$

defined by viewing an (A,λ_A)-module (N,λ_N) as a (B,λ_B)-module $f^*(N,\lambda_N)$ via the map f. The functor f^* has a left adjoint functor that to the (B,λ_B)-module (M,λ_M) associates the (A,λ_A)-module defined by

$$(A,\lambda_A)\otimes_{(B,\lambda_B)}(M,\lambda_M)=(A\otimes_B M,\lambda_{A\otimes_B M})$$

where $\lambda_{A\otimes_B M}$ is given by the composition of $\lambda_A\otimes_{\lambda_B}\lambda_M$ and the map

$$\mathbb{W}(A)\otimes_{\mathbb{W}(B)}\mathbb{W}(M)\to\mathbb{W}(A\otimes_B M)$$

that to $a\otimes x$ associates the vector whose nth Witt component is $w_n(a)\otimes x_n$.

Using Lemma 2.2, one similarly verifies that Quillen's notion of a derivation of the λ-ring (A,λ_A) into the (A,λ_A)-module may be characterized as follows.

Definition 2.11. Let (A,λ_A) be a λ-ring and let (M,λ_M) be an (A,λ_A)-module. A derivation from (A,λ_M) to (M,λ_M) is a map

$$D\colon (A,\lambda_A)\to(M,\lambda_M)$$

such that the following (1)–(3) hold.

(1) For all $a,b \in A$, $D(a+b) = D(a)+D(b)$.
(2) For all $a,b \in A$, $D(ab) = bD(a)+aD(b)$.
(3) For all $a \in A$ and $n \in \mathbb{N}$, $\lambda_{M,n}(D(a)) = \sum_{e|n} \lambda_{A,e}(a)^{(n/e)-1} D(\lambda_{A,e}(a))$.

The set of derivations from (A,λ_A) to (M,λ_M) is denoted by $\mathrm{Der}((A,\lambda_A),(M,\lambda_M))$.

Lemma 2.12. *Let (A,λ_A) be a λ-ring. There exists a derivation*

$$d\colon (A,\lambda_A) \to (\Omega_{(A,\lambda_A)}, \lambda_{\Omega_{(A,\lambda_A)}})$$

with the property that for every (A,λ_A)-module (M,λ_M), the map

$$\delta\colon \mathrm{Hom}_{(A,\lambda_A)}((\Omega_{(A,\lambda_A)}, \lambda_{\Omega_{(A,\lambda_A)}}),(M,\lambda_M)) \to \mathrm{Der}((A,\lambda_A),(M,\lambda_M))$$

defined by $\delta(f)(a) = f(d(a))$ is a bijection.

Proof. We define the target of the map d to be the quotient of the free (A,λ_A)-module (F,λ_F) generated by $\{d(a) \mid a \in A\}$ by the sub-(A,λ_A)-module $(R,\lambda_R) \subset (F,\lambda_F)$ generated by $d(a+b)-d(a)-d(b)$ with $a,b \in a$; by $d(ab)-bd(a)-ad(b)$ with $a,b \in A$; and by $\lambda_{F,n}(da) - \sum_{e|n} \lambda_{A,e}(a)^{(n/e)-1} d(\lambda_{A,e}(a))$ with $a \in A$ and $n \in \mathbb{N}$. The map d takes $a \in A$ to the class of $d(a)$ in $\Omega_{(A,\lambda_A)}$. By construction, the map d is a derivation and the map δ is a bijection. $\square$

The inclusion of $\mathrm{Der}((A,\lambda_A),(M,\lambda_M))$ in $\mathrm{Der}(A,M)$ in general is strict. It is represented by a map of A-modules $\Omega_A \to \Omega_{(A,\lambda_A)}$ that we call the canonical map. We now prove Theorem A which states that this map is an isomorphism.

Proof of Theorem A. We consider the following diagram of adjunctions.

$$\begin{tikzcd}
\mathscr{A}/A \arrow[r, shift left, "(-)_{\mathrm{ab}}"] \arrow[d, shift left, "R_{(A,\lambda_A)}"] & (\mathscr{A}/A)_{\mathrm{ab}} \arrow[l, shift left, "i"] \arrow[r, shift left, "F"] \arrow[d, shift left, "R^{\mathrm{ab}}_{(A,\lambda_A)}"] & \mathscr{M}(A) \arrow[l, shift left, "G"] \arrow[d, shift left, "R'"] \\
\mathscr{A}_\lambda/(A,\lambda_A) \arrow[u, shift left, "u_{(A,\lambda_A)}"] \arrow[r, shift left, "(-)^{\lambda}_{\mathrm{ab}}"] & (\mathscr{A}_\lambda/(A,\lambda_A))_{\mathrm{ab}} \arrow[u, shift left, "u^{\mathrm{ab}}_{(A,\lambda_A)}"] \arrow[l, shift left, "i^\lambda"] \arrow[r, shift left, "F^\lambda"] & \mathscr{M}(A,\lambda_A) \arrow[u, shift left, "u'"] \arrow[l, shift left, "G^\lambda"]
\end{tikzcd}$$

Here the functors i and i^λ are the forgetful functors that forget the abelian group object structure maps, and the functors $(-)_{\mathrm{ab}}$ and $(-)^{\lambda}_{\mathrm{ab}}$ are the respective left adjoint functors which we now define. It suffices to define the left adjoint functors $F \circ (-)_{\mathrm{ab}}$ and $F^\lambda \circ (-)^{\lambda}_{\mathrm{ab}}$ of the composite functors $i \circ G$ and $i^\lambda \circ G^\lambda$. The functor $i \circ G$ takes the A-module M to the canonical projection $f\colon A \oplus M \to A$, and the left adjoint functor $F \circ (-)_{\mathrm{ab}}$ takes $f\colon B \to A$ to the A-module

$$F((f\colon B \to A)_{\mathrm{ab}}) = A \otimes_B \Omega_B.$$

The unit and counit maps of the adjunction are defined by $\eta(b) = (f(b), 1 \otimes b)$ and $\varepsilon(1 \otimes d(a,x)) = x$, respectively. Similarly, the functor $i^\lambda \circ G^\lambda$ takes the (A,λ_A)-module (M,λ_A) to the canonical projection $f\colon (A \oplus M, \lambda_{A\oplus M}) \to (A,\lambda_A)$. Here, we recall, $\lambda_{A\oplus M}$ is defined to be the composite map $(\mathrm{in}_{1*}+\mathrm{in}_{2*})\circ(\lambda_A\oplus\lambda_M)$. The left adjoint functor $F^{\lambda}\circ(-)^{\lambda}_{\mathrm{ab}}$ takes $f\colon (B,\lambda_B)\to(A,\lambda_A)$ to the (A,λ_A)-module

$$F^{\lambda}((f\colon (B,\lambda_B)\to(A,\lambda_A))_{\mathrm{ab}})=(A,\lambda_A)\otimes_{(B,\lambda_B)}\Omega_{(B,\lambda_B)}$$

and the unit and counit of the adjunction are defined by $\eta(b)=(f(b),1\otimes db)$ and $\varepsilon(1\otimes\lambda_{\Omega_{(B,\lambda_B)},n}(d(a,x)))=\lambda_{M,n}(x)$, respectively.

Now, in the diagram of adjunctions above, the diagram of right adjoint functors commutes up to natural isomorphism. Therefore, the diagram of left adjoint functors commutes up to natural isomorphisms. It follows that the natural transformation

$$A\otimes_B\Omega_B\to u'((A,\lambda_A)\otimes_{(B,\lambda_B)}\Omega_{(B,\lambda_B)})$$

is an isomorphism. Taking $(B,\lambda_B)=(A,\lambda_A)$, the theorem follows. $\square$

Theorem 2.13. *There are natural F_n-linear maps $F_n\colon \Omega_{\mathbb{W}(A)}\to\Omega_{\mathbb{W}(A)}$ defined by*

$$F_n(da)=\sum_{e|n}\Delta_{A,e}(a)^{(n/e)-1}d\Delta_{A,e}(a).$$

Moreover, the following (1)–(3) *hold.*

(1) *For all $m,n\in\mathbb{N}$, $F_mF_n=F_{mn}$, and $F_1=\mathrm{id}$.*
(2) *For all $n\in\mathbb{N}$ and $a\in\mathbb{W}(A)$, $dF_n(a)=nF_n(da)$.*
(3) *For all $n\in\mathbb{N}$ and $a\in A$, $F_nd[a]=[a]^{n-1}d[a]$.*

Proof. The canonical map $\Omega_{\mathbb{W}(A)}\to\Omega_{(\mathbb{W}(A),\Delta_A)}$ is an isomorphism by Theorem A, and the codomain is a $(\mathbb{W}(A),\Delta_A)$-module. We define

$$F_n=\lambda_{\Omega_{(\mathbb{W}(A),\Delta_A)},n}\colon \Omega_{(\mathbb{W}(A),\Delta_A)}\to\Omega_{(\mathbb{W}(A),\Delta_A)}$$

to be the nth Witt component of the $(\mathbb{W}(A),\Delta_A)$-module structure map. It follows from Remark 2.5 that it is an $F_n=w_n\circ\Delta_A$-linear map and from Definition 2.11 (3) that it is given by the stated formula. Properties (1) and (2) follow immediately from the definition of a $(\mathbb{W}(A),\Delta_A)$-module and from the calculation

$$dF_n(a)=d(\sum_{e|n}e\Delta_{A,e}(a)^{n/e})=\sum_{e|n}n\Delta_{A/e}^{(n/e)-1}d\Delta_{A,e}(a)=nF_n(da).$$

Finally, to prove property (3), it suffices to show that $\Delta_{A,e}([a])$ is equal to $[a]$ and 0, respectively, as $e=1$ and $e>1$, or equivalently, that $\Delta([a])=[[a]]$. We may assume that $A=\mathbb{Z}[a]$. In this case, the ghost map is injective, and the calculation

$$w_n(\Delta([a]))=F_n([a])=[a]^n=w_n([[a]])$$

shows that $\Delta([a])$ and $[[a]]$ have the same image by the ghost map. $\square$

3 The anti-symmetric graded algebras $\hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$ and $\check{\Omega}^{\cdot}_{\mathbb{W}(A)}$

We next introduce the anti-symmetric graded algebra $\hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$ which agrees with the alternating algebra $\Omega^{\cdot}_{\mathbb{W}(A)} = \bigwedge_{\mathbb{W}(A)} \Omega^1_{\mathbb{W}(A)}$ if the element $d\log[-1] \in \Omega^1_{\mathbb{W}(A)}$ is zero but which is different in general.

Definition 3.1. Let A be a ring. The graded $\mathbb{W}(A)$-algebra

$$\hat{\Omega}^{\cdot}_{\mathbb{W}(A)} = T^{\cdot}_{\mathbb{W}(A)}\Omega^1_{\mathbb{W}(A)}/J^{\cdot}$$

is defined to be the quotient of the tensor algebra of the $\mathbb{W}(A)$-module $\Omega^1_{\mathbb{W}(A)}$ by the graded ideal generated by all elements of the form

$$da \otimes da - d\log[-1] \otimes F_2 da$$

with $a \in \mathbb{W}(A)$.

We remark that the defining relation $da \cdot da = d\log[-1] \cdot F_2 da$ is analogous to the relation $\{a,a\} = \{-1,a\}$ in Milnor K-theory.

Lemma 3.2. *The graded $\mathbb{W}(A)$-algebra $\hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$ is anti-symmetric.*

Proof. It suffices to show that the sum $da \cdot db + db \cdot da \in \hat{\Omega}^2_{\mathbb{W}(A)}$ is equal to zero for all $a, b \in \mathbb{W}(A)$. Now, on the one hand, we have

$$d(a+b) \cdot d(a+b) = d\log[-1] \cdot F_2 d(a+b) = d\log[-1] \cdot F_2 da + d\log[-1] \cdot F_2 db,$$

since $F_2 d$ is additive, and on the other hand, we have

$$\begin{aligned} d(a+b) \cdot d(a+b) &= da \cdot da + da \cdot db + db \cdot da + db \cdot db \\ &= d\log[-1] \cdot F_2 da + da \cdot db + db \cdot da + d\log[-1] \cdot F_2 db \end{aligned}$$

This shows that $da \cdot db + db \cdot da$ is zero as desired. $\square$

Proposition 3.3. *There exists a unique graded derivation*

$$d \colon \hat{\Omega}^{\cdot}_{\mathbb{W}(A)} \to \hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$$

that extends the derivation $d \colon \mathbb{W}(A) \to \Omega^1_{\mathbb{W}(A)}$ and satisfies the formula

$$dd\omega = d\log[-1] \cdot d\omega$$

for all $\omega \in \hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$. Moreover, the element $d\log[-1]$ is a cycle.

Proof. The relation $dd\omega = d\log[-1] \cdot d\omega$ implies that $d\log[-1]$ is a cycle for the desired derivation d. Indeed,

$$\begin{aligned} d(d\log[-1]) &= d([-1]d[-1]) = d[-1] \cdot d[-1] + [-1] \cdot dd[-1] \\ &= d\log[-1] \cdot F_2 d[-1] + [-1] d\log[-1] \cdot d[-1] \\ &= d\log[-1] \cdot [-1]d[-1] + d\log[-1] \cdot [-1]d[-1] \end{aligned}$$

which is zero by Lemma 3.2. This proves that the desired derivation d is unique and given by the formula

$$d(a_0da_1\dots da_q) = da_0da_1\cdots da_q + qd\log[-1]\cdot a_0da_1\dots da_q$$

where $a_0,a_1,\dots,a_q \in \mathbb{W}(A)$. To complete the proof, it remains to prove that the map d given by this formula is (a) well defined, (b) a graded derivation, and (c) satisfies $dd\omega = d\log[-1]\cdot d\omega$. First, we have

$$\begin{aligned}
&d(a_0da_1\dots da_p\cdot b_0db_1\dots db_q) = d(a_0b_0da_1\dots da_pdb_1\dots db_q)\\
&= d(a_0b_0)da_1\dots da_pdb_1\dots db_q + (p+q)d\log[-1]\cdot a_0b_0da_1\dots da_pdb_1\dots db_q\\
&= da_0da_1\dots da_p\cdot b_0db_1\dots db_q + pd\log[-1]\cdot a_0da_1\dots da_p\cdot b_0db_1\dots db_q\\
&\quad + (-1)^p(a_0da_1\dots da_p\cdot db_0db_1\dots db_q + a_0da_1\dots da_p\cdot d\log[-1]\cdot b_0db_1\dots db_q)\\
&= d(a_0da_1\dots da_p)\cdot b_0db_1\dots db_q + (-1)^pa_0da_1\dots da_p\cdot d(b_0db_1\dots db_q)
\end{aligned}$$

which proves (b). Next, using that 2 divides q^2+q, we find

$$\begin{aligned}
dd(a_0da_1\dots da_q) &= d(da_0da_1\dots da_q + qd\log[-1]\cdot a_0da_1\dots da_q)\\
&= (q+1)d\log[-1]\cdot da_0da_1\dots da_q - qd\log[-1]\cdot da_0da_1\dots da_q\\
&\quad - qd\log[-1]\cdot qd\log[-1]\cdot a_0da_1\dots da_q\\
&= d\log[-1]\cdot(da_0da_1\dots da_q + qd\log[-1]\cdot a_0da_1\dots da_q)\\
&= d\log[-1]\cdot d(a_0da_1\dots da_q)
\end{aligned}$$

which proves (c). Finally, to prove (a), we must show that for all $a,b\in\mathbb{W}(A)$, the following two types of elements of $\hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$ are zero.

$$\begin{aligned}
&d(d(ab) - bda - adb)\\
&d(dada - d\log[-1]\cdot F_2da)
\end{aligned}$$

First, using Lemma 3.2 together with (b) and (c), we find that

$$\begin{aligned}
d(d(ab) - bda - adb) &= dd(ab) - dbda - bdda - dadb - addb\\
&= d\log[-1]\cdot d(ab) - d\log[-1]\cdot bda - d\log[-1]\cdot adb\\
&= d\log[-1]\cdot(d(ab) - bda - adb)
\end{aligned}$$

which is zero, since $d\colon \mathbb{W}(A)\to\hat{\Omega}^1_{\mathbb{W}(A)}$ is a derivation. This shows that the first type of elements are zero. Next, (b) and (c) show that

$$d(dada) = 2d\log[-1]\cdot dada$$

which is zero as is

$$\begin{aligned}
d(d\log[-1]\cdot F_2da) &= d\log[-1]\cdot dF_2da\\
&= d\log[-1]\cdot d(ada + d\Delta_2(a))\\
&= d\log[-1]\cdot(dada + d\log[-1]\cdot F_2da)
\end{aligned}$$

by the definition of $\hat{\Omega}^2_{\mathbb{W}(A)}$. Hence also the second type of elements are zero. This completes the proof of (a) and hence of the proposition. □

Remark 3.4. In general, there is no $\mathbb{W}(A)$-algebra map $f\colon \hat{\Omega}^{\cdot}_{\mathbb{W}(A)} \to \Omega^{\cdot}_{\mathbb{W}(A)}$ that is compatible with the derivations.

Proposition 3.5. *Let A be a ring and let n be a positive integer. There is a unique homomorphism of graded rings*

$$F_n\colon \hat{\Omega}^{\cdot}_{\mathbb{W}(A)} \to \hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$$

that is given by the maps $F_n\colon \mathbb{W}(A) \to \mathbb{W}(A)$ and $F_n\colon \Omega^1_{\mathbb{W}(A)} \to \Omega^1_{\mathbb{W}(A)}$ in degrees 0 and 1, respectively. In addition, the following formula holds.

$$dF_n = nF_n d$$

Proof. The uniqueness statement is clear: The map F_n is necessarily given by

$$F_n(a_0 da_1 \dots da_q) = F_n(a_0)F_n(da_1)\dots F_n(da_q)$$

where $a_0,\dots,a_q \in \mathbb{W}(A)$. We show that this formula gives a well-defined map. To prove this, we must show that for every $a \in \mathbb{W}(A)$,

$$F_n(da)F_n(da) = F_n(d\log[-1])F_n(F_2 da).$$

It will suffice to let $n = p$ be a prime number. In this case, we find that

$$\begin{aligned}
&F_p(da)F_p(da) = (a^{p-1}da + d\Delta_p(a))\cdot(a^{p-1}da + d\Delta_p(a))\\
&= (a^{p-1})^2 da\cdot da + d\Delta_p(a)\cdot d\Delta_p(a) = d\log[-1]\cdot((a^{p-1})^2 F_2 da + F_2 d\Delta_p(a))\\
&= d\log[-1]\cdot(F_2(a^{p-1})F_2 da + F_2 d\Delta_p(a)) = d\log[-1]\cdot F_2 F_p da\\
&= F_p(d\log[-1]\cdot F_2 da)
\end{aligned}$$

where we have used that $F_2(a)$ is congruent to a^2 modulo $2\mathbb{W}(A)$. This shows that the map F_n is well-defined. It is a graded ring homomorphism by definition.

We next prove the formula $dF_n = nF_n d$. Again, we may assume that $n = p$ is a prime number. We already know from the definition of $F_n\colon \Omega^1_{\mathbb{W}(A)} \to \Omega^1_{\mathbb{W}(A)}$ that for all $a \in \mathbb{W}(A)$, $dF_p(a) = pF_p(a)$. Now, for all $a \in \mathbb{W}(A)$,

$$dF_p(da) = d(a^{p-1}da + d\Delta_p(a)) = (p-1)a^{p-2}dada + d\log[-1]\cdot F_p da$$

which is equal to zero for $p = 2$, and equal to $d\log[-1]\cdot F_p da$ for p odd. Hence, for every prime p and every $a \in \mathbb{W}(A)$, we have

$$dF_p(da) = pd\log[-1]\cdot F_p da = pF_p(d\log[-1]\cdot da) = pF_p d(da)$$

as desired. Now, let $a_0,\dots,a_q \in \mathbb{W}(A)$. We find

$$\begin{aligned}
&dF_p(a_0 da_1 \dots da_q) = d(F_p(a_0)F_p da_1 \dots F_p da_q)\\
&= dF_p(a_0)F_p da_1 \dots F_p da_q + \sum_{1\leqslant i\leqslant q}(-1)^{i-1}F_p(a_0)F_p da_1 \dots dF_p da_i \dots F_p da_q\\
&= p(F_p da_0 F_p da_1 \dots F_p da_q + \sum_{1\leqslant i\leqslant q}(-1)^{i-1}F_p(a_0)F_p da_1 \dots F_p dda_i \dots F_p da_q)\\
&= pF_p d(a_0 da_1 \dots da_q)
\end{aligned}$$

as stated. This completes the proof. □

We next define the quotient graded algebra $\check{\Omega}^{\cdot}_{\mathbb{W}(A)}$ of the graded algebra $\hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$ and show that the Frobenius F_n and derivation d descend to this quotient.

Definition 3.6. Let A be a ring. The graded $\mathbb{W}(A)$-algebra

$$\check{\Omega}^{\cdot}_{\mathbb{W}(A)} = \hat{\Omega}^{\cdot}_{\mathbb{W}(A)}/K^{\cdot}$$

is defined to be the quotient by the graded ideal $K^{\cdot}$ generated by the elements

$$F_p dV_p(a) - da - (p-1)d\log[-1] \cdot a$$

where p ranges over all prime numbers and a over all elements of $\mathbb{W}(A)$.

We remark that if the prime number p is invertible in A and hence in $\mathbb{W}(A)$, then the element $F_p dV_p(a) - da - (p-1)d\log[-1] \cdot a$ already is zero in $\hat{\Omega}^1_{\mathbb{W}(A)}$.

Lemma 3.7. *The Frobenius $F_n \colon \hat{\Omega}^{\cdot}_{\mathbb{W}(A)} \to \hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$ induces a map of graded algebras*

$$F_n \colon \check{\Omega}^{\cdot}_{\mathbb{W}(A)} \to \check{\Omega}^{\cdot}_{\mathbb{W}(A)}.$$

Proof. It will suffice to show let $n = \ell$ be a prime number and show that for all prime numbers p and all $a \in \mathbb{W}(A)$, the element

$$F_\ell(F_p dV_p(a) - da - (p-1)d\log[-1] \cdot a) \in \Omega^1_{\mathbb{W}(A)}$$

maps to zero in $\hat{\Omega}^1_{\mathbb{W}(A)}$. Suppose first that $p = \ell$. For p odd, we find

$$\begin{aligned} &F_p(F_p dV_p(a) - da) = F_p(p^{p-2}V_p(a^{p-1})dV_p(a) - p^{p-2}dV_p(a^p)) \\ &= p^{p-1}a^{p-1}F_p dV_p(a) - p^{p-2}F_p dV_p(a^p) \end{aligned}$$

whose image in $\check{\Omega}^1_{\mathbb{W}(A)}$ is zero since $F_p dV_p = d \colon \mathbb{W}(A) \to \check{\Omega}^1_{\mathbb{W}(A)}$ and the common map is a derivation. Similarly, for $p = 2$, we find

$$\begin{aligned} &F_2(F_2 dV_2(a) - da - d\log[-1] \cdot a) = F_2(V_2(a)dV_2(a) - dV_2(a^2) - d\log[-1] \cdot a) \\ &= 2aF_2 dV_2(a) - F_2 dV_2(a^2) - d\log[-1] \cdot F_2(a) \end{aligned}$$

whose image in $\check{\Omega}^1_{\mathbb{W}(A)}$ is

$$2ada - d(a^2) - d\log[-1] \cdot (a^2 + F_2(a))$$

which is zero since d is a derivation and since $a^2 + F_2(a)$ divisible by 2. Suppose next that $p \neq \ell$. In this case, we recall, $F_\ell F_p = F_p F_\ell$ and ℓ divides $p^{\ell-1} - 1$. Now, if both p and ℓ are different from 2, we find

$$\begin{aligned} &F_\ell(F_p dV_p(a) - da) = F_p(F_\ell dV_p(a)) - F_\ell da \\ &= F_p(V_p(a)^{\ell-1}dV_p(a) + d(\frac{F_\ell V_p(a) - V_p(a)^\ell}{\ell})) - a^{\ell-1}da - d(\frac{F_\ell(a) - a^\ell}{\ell}) \\ &= F_p(p^{\ell-2}V_p(a^{\ell-1})dV_p(a) + dV_p(\frac{F_\ell(a) - p^{\ell-1}a^\ell}{\ell})) - a^{\ell-1}da - d(\frac{F_\ell(a) - a^\ell}{\ell}) \end{aligned}$$

whose image in $\check{\Omega}^1_{\mathbb{W}(A)}$ is

$$(p^{\ell-1}-1)a^{\ell-1}da - \frac{p^{\ell-1}-1}{\ell}d(a^\ell)$$

which is zero since d is a derivation. Similarly, if $p=2$ and $\ell \neq p$, we find

$$\begin{aligned}
&F_\ell(F_2 dV_2(a) - da - d\log[-1]\cdot a) = F_2(F_\ell dV_2(a)) - F_\ell da - d\log[-1]\cdot F_\ell(a)\\
&= F_2(V_2(a)^{\ell-1}dV_2(a) + d(\frac{F_\ell V_2(a) - V_2(a)^\ell}{\ell})\\
&\qquad - a^{\ell-1}da - d(\frac{F_\ell(a)-a^\ell}{\ell}) - d\log[-1]\cdot F_\ell(a)\\
&= 2^{\ell-1}a^{\ell-1}F_2 dV_2(a) + F_2 dV_2(\frac{F_\ell(a) - 2^{\ell-1}a^\ell}{\ell})\\
&\qquad - a^{\ell-1}da - d(\frac{F_\ell(a)-a^\ell}{\ell}) - d\log[-1]\cdot F_\ell(a)
\end{aligned}$$

whose image in $\check{\Omega}^1_{\mathbb{W}(A)}$ is

$$(2^{\ell-1}-1)a^{\ell-1}da - \frac{2^{\ell-1}-1}{\ell}d(a^\ell) + d\log[-1]\cdot(\frac{F_\ell(a)-2^{\ell-1}a^\ell}{\ell} - F_\ell(a))$$

which is zero since d is a derivation and since ℓ is congruent to 1 modulo 2. Finally, if $\ell = 2$ and $p \neq \ell$, we find

$$\begin{aligned}
&F_2(F_p dV_p(a) - da) = F_p(F_2 dV_p(a)) - F_2 da\\
&= F_p(V_p(a)dV_p(a) + d(\frac{F_2 V_p(a) - V_p(a)^2}{2})) - ada - d(\frac{F^2(a)-a^2}{2})\\
&= paF_p dV_p(a) + F_p dV_p(\frac{F_2(a) - pa^2}{2}) - ada - d(\frac{F^2(a)-a^2}{2})
\end{aligned}$$

whose image in $\check{\Omega}^1_{\mathbb{W}(A)}$ is

$$(p-1)ada - \frac{p-1}{2}d(a^2)$$

which is zero since d is a derivation. □

Lemma 3.8. *For all positive integers n and $a \in \mathbb{W}(A)$, the relation*

$$F_n dV_n(a) = da + (n-1)d\log[-1]\cdot a$$

holds in $\check{\Omega}^1_{\mathbb{W}(A)}$.

Proof. We argue by induction on the number r of prime factors in n that the stated relation holds for all $a \in \mathbb{W}(A)$. The case $r = 1$ follows from Definition 3.6. So we let n be a positive integer with $r > 1$ prime factors and assume that the relation has been proved for all positive integers with less than r prime factors. We write $n = pm$ with p a prime number. Then

$$\begin{aligned} F_n dV_n(a) &= F_p F_m dV_m V_p(a) = F_p(dV_p(a) - (m-1)d\log[-1] \cdot V_p(a)) \\ &= F_p dV_p(a) - (m-1)d\log[-1] \cdot F_p V_p(a) \\ &= da - (p-1)d\log[-1] \cdot a - p(m-1)d\log[-1] \cdot a \\ &= da - (n-1)d\log[-1] \cdot a \end{aligned}$$

which proves the induction step. □

Lemma 3.9. *The graded derivation* $d\colon \hat{\Omega}^{\cdot}_{\mathbb{W}(A)} \to \hat{\Omega}^{\cdot}_{\mathbb{W}(A)}$ *induces a graded derivation*

$$d\colon \check{\Omega}^{\cdot}_{\mathbb{W}(A)} \to \check{\Omega}^{\cdot}_{\mathbb{W}(A)}.$$

Proof. We must show that for all prime numbers p and $a \in \mathbb{W}(A)$, the element

$$d(F_p dV_p(a) - da - (p-1)d\log[-1] \cdot a) \in \hat{\Omega}^2_{\mathbb{W}(A)}$$

maps to zero in $\check{\Omega}^2_{\mathbb{W}(A)}$. First, for $p = 2$, we have

$$\begin{aligned} &d(F_2 dV_2(a) - da - d\log[-1] \cdot a) = dF_2 dV_2(a) - dda + d\log[-1] \cdot da \\ &= F_2 ddV_2(a) = 2F_2 ddV_2(a) = 2d\log[-1] \cdot F_2 dV_2(a) \end{aligned}$$

which is even zero in $\hat{\Omega}^2_{\mathbb{W}(A)}$. For p odd, we find

$$\begin{aligned} &d(F_p dV_p(a) - da) = d(V_p(a)^{p-1} dV_p(a) + d\Delta_p(a)) - dda \\ &= (p-1)V_p(a)^{p-2} dV_p(a) dV_p(a) + d\log[-1] \cdot F_p dV_p(a) - d\log[-1] \cdot da \\ &= (p-1)V_p(a)^{p-2} d\log[-1] \cdot F_2 dV_p(a) + d\log[-1] \cdot (F_p dV_p(a) - da) \\ &= d\log[-1] \cdot (F_p dV_p(a) - da) \end{aligned}$$

which maps to zero in $\check{\Omega}^2_{\mathbb{W}(A)}$ as desired. Here we have used that $p-1$ is even. □

Definition 3.10. Let A be a ring, let $S \subset \mathbb{N}$ be a truncation set, and let $I_S(A) \subset \mathbb{W}(A)$ be the kernel of the restriction map $R^{\mathbb{N}}_S\colon \mathbb{W}(A) \to \mathbb{W}_S(A)$. The maps

$$\begin{aligned} R^{\mathbb{N}}_S &\colon \hat{\Omega}^{\cdot}_{\mathbb{W}(A)} \to \hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \\ R^{\mathbb{N}}_S &\colon \check{\Omega}^{\cdot}_{\mathbb{W}(A)} \to \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \end{aligned}$$

are defined to be the quotient maps that annihilate the respective graded ideals generated by $I_S(A)$ and $dI_S(A)$.

Remark 3.11. The kernel $K_S^{\cdot}$ of the canonical projection $\hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \to \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)}$ is equal to the graded ideal generated by the elements

$$V_p(R^S_{S/p}(a)^{p-1})dV_pR^S_{S/p}(a) - p^{p-2}dV_pR^S_{S/p}(a^p) - (p-1)d\log[-1]_S \cdot a$$

with p a prime number and $a \in \mathbb{W}_S(A)$.

Remark 3.12. Let p be a prime number and let S be a p-typical truncation set. Then the ideal $V_p\mathbb{W}_{S/p}(A) \subset \mathbb{W}_S(A)$ has a canonical divided power structure defined by

$$V_p(a)^{[n]} = \frac{p^{n-1}}{n!}V_p(a^n).$$

Suppose, in addition, that the prime number p is *odd* and that A is a $\mathbb{Z}_{(p)}$-algebra. Then $d\colon \mathbb{W}_S(A) \to \check{\Omega}^1_{\mathbb{W}_S(A)}$ is a divided power derivation in the sense that

$$d(V_p(a)^{[n]}) = V_p(a)^{[n-1]}dV_p(a)$$

and it is universal with this property; see [18, Lemma 1.2].

Lemma 3.13. *The derivation, restriction, and Frobenius induce maps*

$$\begin{aligned}
d\colon \hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)} &\to \hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)} && (\textit{resp. } d\colon \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \to \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)})\\
R^S_T\colon \hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)} &\to \hat{\Omega}^{\cdot}_{\mathbb{W}_T(A)} && (\textit{resp. } R^S_T\colon \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \to \check{\Omega}^{\cdot}_{\mathbb{W}_T(A)})\\
F_n\colon \hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)} &\to \hat{\Omega}^{\cdot}_{\mathbb{W}_{S/n}(A)} && (\textit{resp. } F_n\colon \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \to \check{\Omega}^{\cdot}_{\mathbb{W}_{S/n}(A)})
\end{aligned}$$

Proof. To prove the statement for d, we note that as d is a derivation, it suffices to show that $R^{\mathbb{N}}_S(ddI_S(A)) \subset \hat{\Omega}^2_{\mathbb{W}_S(A)}$ is zero. But if $x \in I_S(A)$, then

$$R^{\mathbb{N}}_S(ddx) = R^{\mathbb{N}}_S(d\log[-1] \cdot dx) = R^{\mathbb{N}}_S(d\log[-1]) \cdot R^{\mathbb{N}}_S(dx)$$

which is zero as desired. It follows that $R^{\mathbb{N}}_T(dI_S(A)) = dR^{\mathbb{N}}_T(I_S(A)$. Hence, also the statement for R^S_T follows as $R^{\mathbb{N}}_T(I_S(A)$ is trivially zero. Finally, to prove the statement for F_n, we show that both $R^{\mathbb{N}}_{S/n}(F_n(I_S(A)))$ and $R^{\mathbb{N}}_{S/n}(F_n(dI_S(A)))$ are zero. For the former, this follows immediately from Lemma 1.4, and for the latter, it will suffice to show that for divisors e of n, $\Delta_e(I_S(A)) \subset I_{S/e}(A)$. Moreover, to prove this, we may assume that A is torsion free. So let e be a divisor of n and assume that for all proper divisors d of e, $\Delta_d(I_S(A)) \subset I_{S/d}(A)$. Since $F_e(I_S(A)) \subset I_{S/e}(A)$, the formula

$$F_e(a) = \sum_{d|e} d\Delta_d(a)^{e/d}$$

shows that $e\Delta_e(I_S(A)) \subset I_{S/e}(A)$. The lemma follows. □

Lemma 3.14. *For every ring A, the differential graded algebras $\hat{\Omega}^{\cdot}_A$ and $\Omega^{\cdot}_A$ are equal and the canonical projection $\hat{\Omega}^{\cdot}_A \to \check{\Omega}^{\cdot}_A$ is an isomorphism.*

Proof. Since $d\log[-1]_{\{1\}}$ is zero, $\hat{\Omega}^{\cdot}_A = \Omega^{\cdot}_A$ as stated. Moreover, Remark 3.11 shows that the kernel $K_{\{1\}}$ of the canonical projection $\hat{\Omega}^{\cdot}_A \to \check{\Omega}^{\cdot}_A$ is zero. □

4 The big de Rham-Witt complex

In this section, we construct the big de Rham-Witt complex. The set J of truncation sets is partially ordered under inclusion. We consider J as a category with one morphism from T to S if $T \subset S$. If A is a ring, the assignment $S \mapsto \mathbb{W}_S(A)$ defines a contravariant functor from the category J to the category of rings, and this functor takes colimits in the category J to limits in the category of rings. Moreover, the assignment $S \mapsto S/n$ is an endo-functor on the category J, and the ring homomorphism

$$F_n \colon \mathbb{W}_S(A) \to \mathbb{W}_{S/n}(A)$$

and the abelian group homomorphism

$$V_n \colon \mathbb{W}_{S/n}(A) \to \mathbb{W}_S(A)$$

are natural transformations with respect to the variable S.

We proceed to define the notion of a Witt complex over A. The original definition given in [12, Definition 1.1.1] is not quite correct unless the prime 2 is either invertible or zero in A. The correct definition of a 2-typical Witt complex was given first by Costeanu [7, Definition 1.1]. The definition given below was also inspired by [21].

Definition 4.1. A *Witt complex* over A is a contravariant functor

$$S \mapsto E_S^{\cdot}$$

that to every truncation set $S \subset \mathbb{N}$ assigns an anti-symmetric graded ring $E_S^{\cdot}$ and that takes colimits to limits together with a natural ring homomorphism

$$\eta_S \colon \mathbb{W}_S(A) \to E_S^0$$

and natural maps of graded abelian groups

$$\begin{aligned} d \colon E_S^q &\to E_S^{q+1} \\ F_n \colon E_S^q &\to E_{S/n}^q \qquad (n \in \mathbb{N}) \\ V_n \colon E_{S/n}^q &\to E_S^q \qquad (n \in \mathbb{N}) \end{aligned}$$

such that the following (i)–(v) hold.

(i) For all $x \in E_S^q$ and $x' \in E_S^{q'}$,

$$\begin{aligned} d(x \cdot x') &= d(x) \cdot x' + (-1)^q x \cdot d(x'), \\ d(d(x)) &= d\log \eta_S([-1]_S) \cdot d(x). \end{aligned}$$

(ii) For all positive integers m and n,

$$\begin{aligned} &F_1 = V_1 = \mathrm{id}, \quad F_m F_n = F_{mn}, \quad V_n V_m = V_{mn}, \\ &F_n V_n = n \cdot \mathrm{id}, \quad F_m V_n = V_n F_m \quad \text{if } (m,n) = 1, \\ &F_n \eta_S = \eta_{S/n} F_n, \qquad \eta_S V_n = V_n \eta_{S/n}. \end{aligned}$$

(iii) For all positive integers n, the map F_n is a ring homomorphism and the maps F_n and V_n satisfy the projection formula that for all $x \in E_S^q$ and $y \in E_{S/n}^{q''}$,

$$x \cdot V_n(y) = V_n(F_n(x)y).$$

(iv) For all positive integers n and all $y \in E_{S/n}^q$,

$$F_n dV_n(y) = d(y) + (n-1) d\log \eta_{S/n}([-1]_{S/n}) \cdot y.$$

(v) For all positive integers n and $a \in A$,

$$F_n d\eta_S([a]_S) = \eta_{S/n}([a]_{S/n}^{n-1}) d\eta_{S/n}([a]_{S/n}).$$

A map of Witt complexes is a natural map of graded rings

$$f \colon E_S^{\cdot} \to E_S'^{\cdot}$$

such that $f\eta = \eta' f$, $fd = d'f$, $fF_n = F_n' f$, and $fV_n = V_n' f$.

Remark 4.2. (a) For $T \subset S$ a pair of truncation sets, we write $R_T^S \colon E_S^{\cdot} \to E_T^{\cdot}$ for the map of graded rings that is part of the structure of a Witt complex.
(b) Every Witt functor is determined, up to canonical isomorphism, by its value on finite truncation sets.
(c) The element $d\log \eta_S([-1]_S)$ is annihilated by 2. Indeed, since d is a derivation,

$$2d\log \eta_S([-1]_S) = d\log \eta_S([1]_S) = 0.$$

Therefore, $d\log \eta_S([-1]_S)$ is zero if 2 is invertible in A and hence in $\mathbb{W}_S(A)$. It is also zero if $2 = 0$ in A since, in this case, $[-1]_S = [1]_S$. Finally, the general formula

$$[-1]_S = -[1]_S + V_2([1]_{S/2})$$

shows that $d\log \eta_S([-1]_S)$ is zero if every $n \in S$ is odd.

Lemma 4.3. *Let m and n be positive integers, let $c = (m,n)$ be the greatest common divisor, and let i and j be any pair of integers such that $mi + nj = c$. The following relations hold in every Witt complex.*

$$\begin{aligned}
&dF_n = nF_n d, \quad V_n d = ndV_n,\\
&F_m dV_n = idF_{m/c}V_{n/c} + jF_{m/c}V_{n/c}d + (c-1)d\log \eta_{S/m}([-1]_{S/m}) \cdot F_{m/c}V_{n/c},\\
&d\log \eta_S([-1]_S) = \textstyle\sum_{r\geqslant 1} 2^{r-1} dV_{2^r}\eta_{S/2^r}([1]_{S/2^r}),\\
&d\log \eta_S([-1]_S) \cdot d\log \eta_S([-1]_S) = 0, \quad dd\log \eta_S([-1]_S) = 0,\\
&F_n(d\log \eta_S([-1]_S)) = d\log \eta_{S/n}([-1]_{S/n}),
\end{aligned}$$

Proof. The following calculation verifies the first two relations.

$$\begin{aligned}
dF_n(x) &= F_n dV_n F_n(x) - (n-1)d\log\eta[-1]\cdot F_n(x)\\
&= F_n d(V_n\eta([1])\cdot x) - (n-1)d\log\eta([-1])\cdot F_n(x)\\
&= F_n(dV_n\eta([1])\cdot x + V_n\eta([1])\cdot dx) - (n-1)d\log\eta([-1])\cdot F_n(x)\\
&= F_n dV_n\eta([1])\cdot F_n(x) + F_nV_n\eta([1])\cdot F_n d(x) - (n-1)d\log\eta([-1])\cdot F_n(x)\\
&= (n-1)d\log\eta([-1])\cdot F_n(x) + nF_n d(x) - (n-1)d\log\eta([-1])\cdot F_n(x)\\
&= nF_n d(x)
\end{aligned}$$

$$\begin{aligned}
V_n d(x) &= V_n(F_n dV_n(x) - (n-1)d\log\eta([-1])\cdot x)\\
&= V_n\eta([1])dV_n(x) - (n-1)V_n(d\log\eta([-1])\cdot x)\\
&= d(V_n\eta([1])V_n(x)) - dV_n\eta([1])V_n(x) - (n-1)V_n(d\log\eta([-1])\cdot x)\\
&= dV_n(F_nV_n\eta([1])x) - V_n(F_n dV_n\eta([1])x) - (n-1)V_n(d\log\eta([-1])\cdot x)\\
&= ndV_n(x) - 2V_n(d\log\eta([-1])\cdot x)\\
&= ndV_n(x)
\end{aligned}$$

Next, the last formula follows from the calculation

$$\begin{aligned}
F_m(d\log\eta([-1]_S)) = F_m(\eta([-1]^{-1})d\eta([-1])) &= F_m\eta([-1]^{-1})F_m d\eta([-1])\\
&= \eta([-1]^{-m})\eta([-1]^{m-1})d\eta([-1])\\
&= \eta([-1]^{-1})d\eta([-1]) = d\log\eta([-1]).
\end{aligned}$$

Using the three relations proved thus far together with the projection formula, we find

$$\begin{aligned}
F_m dV_n(x) &= F_{m/c}F_c dV_c V_{n/c}(x)\\
&= F_{m/c}dV_{n/c}(x) + (c-1)d\log\eta([-1])\cdot F_{m/c}V_{n/c}(x)\\
&= ((m/c)i + (n/c)j)F_{m/c}dV_{n/c}(x) + (c-1)d\log\eta([-1])\cdot F_{m/c}V_{n/c}(x)\\
&= idF_{m/c}V_{n/c}(x) + jF_{m/c}V_{n/c}d(x) + (c-1)d\log\eta([-1])\cdot F_{m/c}V_{n/c}(x).
\end{aligned}$$

To prove the formula for $d\log\eta([-1]_S)$, we use the formula

$$[-1]_S = -[1]_S + V_2([1]_{S/2})$$

which is easily verified by evaluating ghost coordinates. We find

$$\begin{aligned}
d\log\eta([-1]_S) = \eta([-1]_S)d\eta([-1]_S) &= \eta(-[1]_S + V_2([1]_{S/2})d\eta(-[1]_S + V_2([1]_{S/2})\\
&= -dV_2\eta([1]_{S/2}) + V_2(F_2dV_2\eta([1]_{S/2}))\\
&= -dV_2\eta([1]_{S/2}) + V_2(d\log\eta([-1]_{S/2}))\\
&= dV_2\eta([1]_{S/2}) + V_2(d\log\eta([-1]_{S/2}))
\end{aligned}$$

where the last equality uses that $2dV_2\eta([1]_{S/2}) = Vd\eta([1]_{S/2}) = 0$. The stated formula follows by easy induction. Using this formula, we find

$$\begin{aligned}
dV_2 d\log\eta([-1]_{S/2}) &= \sum_{r\geqslant 1} 2^r ddV_{2^{r+1}}\eta([1]_{S/2^{r+1}})\\
&= \sum_{r\geqslant 1} 2^r d\log\eta([-1]_S)\cdot dV_{2^{r+1}}\eta([1]_{S/2^{r+1}})
\end{aligned}$$

which is zero, since $2d\log\eta([-1]_S) = 0$. The calculation

$$\begin{aligned}(d\log\eta([-1]_S))^2 &= (d\eta([-1]_S))^2 = (dV_2\eta([1]_{S/2}))^2\\ &= d(V_2\eta([1]_{S/2})dV_2\eta([1]_{S/2})) - V_2\eta([1]_{S/2})ddV_2\eta([1]_{S/2})\\ &= dV_2 d\log\eta([-1]_{S/2}) - V_2\eta([1]_{S/2})dV_2 d\log\eta([-1]_{S/2})\\ &= dV_2 d\log\eta([-1]_{S/2})\eta([1]_S - V_2([1]_{S/2}))\end{aligned}$$

then shows that $(d\log\eta([-1]_S))^2$ is zero as stated. This, in turn, shows that

$$\begin{aligned}dd\log\eta([-1]_S) &= d\eta([-1]_S d\eta([-1]_S))\\ &= d\eta([-1]_S)d\eta([-1]_S) + \eta([-1]_S)dd\eta([-1]_S)\\ &= \eta([-1]_S)dd\eta([-1]_S)\\ &= \eta([-1]_S)d\log\eta([-1]_S)d\eta([-1]_S)\\ &= d\eta([-1]_S)d\eta([-1]_S) = 0.\end{aligned}$$

This completes the proof. □

Proposition 4.4. *For every Witt complex and every positive integer m, the following diagram, where the horizontal maps take a_0da_1 to $\eta(a_0)d\eta(a_1)$, commutes.*

$$\begin{tikzcd}\Omega^1_{\mathbb{W}_S(A)} \arrow[r, "\eta_S"] \arrow[d, "F_m"] & E^1_S \arrow[d, "F_m"]\\ \Omega^1_{\mathbb{W}_{S/m}(A)} \arrow[r, "\eta_{S/m}"] & E^1_{S/m}\end{tikzcd}$$

Proof. We may assume that $m = p$ is a prime number and that $S = \mathbb{N}$. Moreover, by Lemma 1.5 (i), it will suffice to show that for every $n \in \mathbb{N}$ and $a \in A$,

$$F_p dV_n\eta_{\mathbb{N}}([a]_{\mathbb{N}}) = \eta_{\mathbb{N}}F_p dV_p([a]_{\mathbb{N}}).$$

To ease notation, we suppress the subscript $\mathbb{N}$. We first suppose n is prime to p and set $k = (1-n^{p-1})/p$ and $\ell = n^{p-2}$. Since $kp + \ell n = 1$, Lemma 4.3 shows that

$$\begin{aligned}F_p dV_n\eta([a]) &= k\cdot dV_nF_p\eta([a]) + \ell\cdot V_nF_pd\eta([a])\\ &= k\cdot dV_n\eta([a]^p) + \ell\cdot V_n(\eta([a])^{p-1}d\eta([a]))\\ &= k\cdot dV_n\eta([a]^p) + \ell\cdot V_n\eta([a]^{p-1}d[a])\end{aligned}$$

Moreover, by Theorem 2.13 and Remark 1.19, we have

$$\begin{aligned}\eta F_p dV_n([a]) &= \eta(V_n([a])^{p-1}dV_n([a]) + d\Delta_pV_n([a]))\\ &= \eta(\ell\cdot V_n([a]^{p-1})dV_n([a]) + k\cdot dV_n([a]^p))\\ &= \ell\cdot V_n\eta([a]^{p-1})dV_n\eta([a]) + k\cdot dV_n\eta([a]^p)\\ &= \ell\cdot V_n\eta([a]^{p-1}F_ndV_n\eta([a])) + k\cdot dV_n\eta([a]^p)\\ &= \ell\cdot V_n\eta([a]^{p-1}d[a]) + k\cdot dV_n\eta([a]^p)\end{aligned}$$

where the last equality uses that $n^{p-2}(n-1)d\log\eta([-1])$ is zero. This proves that the desired equality holds if p does not divide n. Suppose next that p divides n and write $n = pr$. We consider the cases $p = 2$ and p odd separately. First, for p odd,

$$\begin{aligned} F_p dV_n\eta([a]) &= dV_r\eta([a]) \\ \eta F_p dV_n([a]) &= \eta(V_n([a])^{p-1}dV_n([a]) + dV_r([a]) - p^{p-2}r^{p-1}dV_n([a]^p)) \\ &= V_n\eta([a])^{p-1}dV_n\eta([a]) + dV_r\eta([a]) - p^{p-2}r^{p-1}dV_n(\eta([a])^p) \end{aligned}$$

and the desired equality follows from the calculation

$$\begin{aligned} p^{p-2}r^{p-1}dV_n(\eta([a])^p) &= p^{p-3}r^{p-2}V_nd(\eta([a])^p) \\ &= p^{p-2}r^{p-2}V_n(\eta([a])^{p-1}d\eta([a])) \\ &= n^{p-2}V_n(\eta([a])^{p-1})dV_n\eta([a]) \\ &= V_n\eta([a])^{p-1}dV_n\eta([a]). \end{aligned}$$

Finally, if $p = 2$, then

$$\begin{aligned} F_2dV_n\eta([a]) &= dV_r\eta([a]) + d\log\eta([-1])\cdot V_r\eta([a]) \\ \eta F_2dV_n([a]) &= \eta(V_n([a])dV_n([a]) + dV_r([a]) - rdV_n([a]^2)) \\ &= V_n\eta([a])dV_n\eta([a]) + dV_r\eta([a]) - rdV_n(\eta([a])^2) \end{aligned}$$

and must show that

$$d\log\eta([-1])\cdot V_r\eta([a]) = V_n\eta([a])dV_n\eta([a]) - rdV_n(\eta([a])^2).$$

Suppose first that $r = 1$. Since $[-1] = -[1] + V_2([1])$, we have

$$d\log\eta([-1]) = V_2\eta([1])dV_2\eta([1]) - dV_2\eta([1]),$$

and hence,

$$\begin{aligned} &d\log\eta([-1])\cdot\eta([a]) = V_2(\eta([a])^2)dV_2\eta([1]) - \eta([a])dV_2\eta([1]) \\ &= V_2(\eta([a]^2)F_2dV_2\eta([1])) - dV_2(\eta([a])^2) + d\eta([a])\cdot V_2\eta([1]) \\ &= V_2(\eta([a])^2d\log\eta([-1])) - dV_2(\eta([a])^2) + V_2(\eta([a])d\eta([a])) \\ &= -dV_2(\eta([a])^2) + V_2(\eta([a]))dV_2\eta([a]) \end{aligned}$$

as desired. In general, we apply V_r to the equation just proved. This gives

$$\begin{aligned} &d\log\eta([-1])\cdot V_r\eta([a]) = -rdV_n(\eta([a])^2) + V_r(V_2\eta([a])dV_2\eta([a])) \\ &= -rdV_n(\eta([a])^2) + V_n(\eta([a])F_2dV_2\eta([a])) \\ &= -rdV_n(\eta([a])^2) + V_n(\eta([a])F_ndV_n\eta([a])) \\ &= -rdV_n(\eta([a])^2) + V_n\eta([a])dV_n\eta([a]) \end{aligned}$$

where $F_2dV_2 = F_ndV_n$ since n is even. This completes the proof. $\square$

Remark 4.5. For every Witt complex over the ring A, $E^1_{\mathbb{N}}$ is a $(\mathbb{W}(A),\Delta)$-module in the sense of Definition 2.4 with structure maps $\lambda_{E,n} = F_n \colon E^1_{\mathbb{N}} \to E^1_{\mathbb{N}}$. Now, Proposition 4.4 states that d is a derivation in the sense of Definition 2.11

$$d \colon (\mathbb{W}(A),\Delta_A) \to (E^1_{\mathbb{N}},\lambda_E).$$

Moreover, in Definition 4.1, we may substitute this statement for axiom (v).

Corollary 4.6. *Let $E^{\cdot}_S$ be a Witt complex over the ring A. There is a unique natural homomorphism of graded rings*

$$\eta_S \colon \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \to E^{\cdot}_S$$

that extends the natural ring homomorphism $\eta_S \colon \mathbb{W}_S(A) \to E^0_S$ and commutes with the derivations. In addition, for every positive integer m, the diagram

$$\begin{CD}
\check{\Omega}^{\cdot}_{\mathbb{W}_S(A)} @>{\eta_S}>> E^{\cdot}_S \\
@V{F_m}VV @VV{F_m}V \\
\check{\Omega}^{\cdot}_{\mathbb{W}_{S/m}(A)} @>{\eta_{S/m}}>> E^{\cdot}_{S/m}
\end{CD}$$

commutes.

Proof. The map η_S necessarily is given by

$$\eta_S(a_0da_1\dots da_q) = \eta_S(a_0)d\eta_S(a_1)\dots d\eta_S(a_q).$$

We show that this formula gives a well-defined map. First, from Proposition 4.4, we find that for all $a \in \mathbb{W}(A)$,

$$F_2d\eta_{\mathbb{N}}(a) = \eta_{\mathbb{N}}F_2d(a) = \eta_{\mathbb{N}}(ada + d\Delta_2(a)) = \eta_{\mathbb{N}}(a)d\eta_N(a) + d\eta_{\mathbb{N}}\Delta_2(a).$$

Applying d to this equation, the left-hand and right-hand sides becomes

$$dF_2d\eta_{\mathbb{N}}(a) = 2F_2dd\eta_N(a) = 0$$

and

$$\begin{aligned}
&d\eta_N(a)d\eta_{\mathbb{N}}(a) + d\log\eta_{\mathbb{N}}([-1])_{\mathbb{N}} \cdot (\eta_{\mathbb{N}}(a)d\eta_{\mathbb{N}}(a) + d\eta_{\mathbb{N}}\Delta_2(a)) \\
&= d\eta_N(a)d\eta_{\mathbb{N}}(a) + d\log\eta_{\mathbb{N}}([-1])_{\mathbb{N}} \cdot F_2d\eta_{\mathbb{N}}(a),
\end{aligned}$$

respectively. This shows that the formula above gives a well-defined map of graded rings from $\hat{\Omega}^{\cdot}_{\mathbb{W}_S(A)}$ to $E^{\cdot}_S$, and by axiom (iv) in Definition 4.1, this map factors through the projection onto $\check{\Omega}^{\cdot}_{\mathbb{W}_S(A)}$. Finally, Proposition 4.4 shows that the diagram in the statement commutes. $\square$

Proof of Theorem B. We define quotient maps of graded rings

$$\eta_S\colon \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \to \mathbb{W}_S\Omega^{\cdot}_A$$

together with maps R^S_T and F_n of graded rings and a graded derivation d that make the diagrams of the statement commute and a map of graded abelian groups

$$V_n\colon \mathbb{W}_{S/n}\Omega^{\cdot}_A \to \mathbb{W}_S\Omega^{\cdot}_A$$

that make the following diagrams commute.

$$\begin{tikzcd}
\mathbb{W}_{S/n}(A) \arrow[r, "\eta_{S/n}"] \arrow[d, "V_n"] & \mathbb{W}_{S/n}\Omega^0_A \arrow[d, "V_n"] \\
\mathbb{W}_S(A) \arrow[r, "\eta_S"] & \mathbb{W}_S\Omega^0_A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{W}_{S/n}\Omega^{\cdot}_A \arrow[r, "V_n"] \arrow[d, "R^{S/n}_{T/n}"] & \mathbb{W}_S\Omega^{\cdot}_A \arrow[d, "R^S_T"] \\
\mathbb{W}_{T/n}\Omega^{\cdot}_A \arrow[r, "V_n"] & \mathbb{W}_T\Omega^{\cdot}_A
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{W}_{S/n}\Omega^{\cdot}_A \otimes \mathbb{W}_{S/n}\Omega^{\cdot}_A \arrow[d, "\mu"] & \mathbb{W}_{S/n}\Omega^{\cdot}_A \otimes \mathbb{W}_S\Omega^{\cdot}_A \arrow[l, "\mathrm{id}\otimes F_n"'] \arrow[r, "V_n\otimes\mathrm{id}"] & \mathbb{W}_S\Omega^{\cdot}_A \otimes \mathbb{W}_S\Omega^{\cdot}_A \arrow[d, "\mu"] \\
\mathbb{W}_{S/n}\Omega^{\cdot}_A \arrow[rr, "V_n"] & & \mathbb{W}_S\Omega^{\cdot}_A
\end{tikzcd}$$

It will suffice to define the above structure for finite truncation sets S. We proceed recursively starting from $\mathbb{W}_{\emptyset}\Omega^{\cdot}_A$ that we define to be the zero graded ring. So let $S \neq \emptyset$ be a finite truncation set and assume, inductively, that the maps η_T, R^T_U, F_n, d, and V_n have been defined for all proper truncation sets $T \subset S$, all truncation sets $U \subset T$, and all positive integers n such that the diagrams in the statement and the diagrams above commute. We then define

$$\eta_S\colon \check{\Omega}^{\cdot}_{\mathbb{W}_S(A)} \to \mathbb{W}_S\Omega^{\cdot}_A$$

to be the quotient map that annihilates the graded ideal $N^{\cdot}_S$ generated by all sums

$$\sum_{\alpha} V_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}$$
$$d(\sum_{\alpha} V_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha})$$

for which the Witt vectors $x_\alpha \in \mathbb{W}_{S/n}(A)$ and $y_{1,\alpha},\dots,y_{q,\alpha} \in \mathbb{W}_S(A)$ and the integers $n \geqslant 2$ and $q \geqslant 1$ satisfy that the sum

$$\eta_{S/n}(\sum_{\alpha} x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha})$$

is zero in $\mathbb{W}_{S/n}\Omega^q_A$. For every positive integer n, we now define

$$V_n\colon \mathbb{W}_{S/n}\Omega^{\cdot}_A \to \mathbb{W}_S\Omega^{\cdot}_A$$

to be the map of graded abelian groups given by

$$V_n\eta_{S/n}(xF_n dy_1\dots F_n dy_q) = \eta_S(V_n(x)dy_1\dots dy_q).$$

Here we use that every element of $\mathbb{W}_{S/n}\Omega^q_A$ can be written as a sum of elements of the form $\eta_{S/n}(xF_n dy_1 \dots F_n dy_q)$ with $x \in \mathbb{W}_{S/n}(A)$ and $y_1, \dots, y_q \in \mathbb{W}_S(A)$. Indeed,

$$dx = F_n dV_n(x) - (n-1)d\log[-1]_{S/n} \cdot x = F_n dV_n(x) - (n-1)xF_n d([-1]_S).$$

By definition, the map V_1 is the identity map, and that

$$V_{mn} = V_m V_n \colon \mathbb{W}_{S/mn}\Omega^q_A \to \mathbb{W}_S\Omega^q_A$$

for all integers $m, n \geqslant 2$ and $q \geqslant 0$ follows from the calculation

$$\begin{aligned}
&V_{mn}\eta_{S/mn}(xF_{mn}dy_1 \dots F_{mn}dy_q) = \eta_S(V_{mn}(x)dy_1 \dots dy_q) \\
&= \eta_S(V_m(V_n(x))dy_1 \dots dy_q) = V_m\eta_{S/m}(V_n(x)F_m dy_1 \dots F_m dy_q) \\
&= V_m(V_n(\eta_{S/mn}(x))F_m d\eta_{S/m}y_1 \dots F_m d\eta_{S/m}y_q) \\
&= V_m V_n \eta_{S/mn}(xF_{mn}dy_1 \dots F_{mn}dy_q)
\end{aligned}$$

where the last equality uses the inductive hypothesis.

We proceed to construct the right-hand vertical maps in the square diagrams in the statement. To this end, it will suffice to show that the left-hand vertical maps in these diagrams satisfy $R^S_T(N^q_S) \subset N^q_T$, $d(N^q_S) \subset N^{q+1}_S$, and $F_n(N^q_S) \subset N^q_{S/n}$, respectively. We therefore fix a positive integer n and an element

$$\omega = \sum_\alpha V_n(x_\alpha)dy_{1,\alpha} \dots dy_{q,\alpha} \in \check{\Omega}^q_{\mathbb{W}_S(A)}$$

such that

$$\eta_{S/n}(\sum_\alpha x_\alpha F_n dy_{1,\alpha} \dots F_n dy_{q,\alpha}) \in \mathbb{W}_{S/n}\Omega^q_A$$

is zero and show that $\eta_T R^S_T(\omega)$, $\eta_S(dd\omega)$, $\eta_{S/m}F_m(\omega)$, and $\eta_{S/m}F_m(d\omega)$ are all zero. First, in order to show that

$$\eta_T R^S_T(\omega) = \eta_T(\sum_\alpha V_n R^{S/n}_{T/n}(x_\alpha)dR^S_T(y_{1,\alpha}) \dots dR^S_T(y_{q,\alpha}))$$

is zero, it suffices by the definition of the ideal $N^{\cdot}_T$ to show that

$$\eta_{T/n}(\sum_\alpha R^{S/n}_{T/n}(x_\alpha)F_n dR^S_T(y_{1,\alpha}) \dots F_n dR^S_T(y_{q,\alpha}))$$

is zero. But this element is equal to

$$\eta_{T/n}R^{S/n}_{T/n}(\sum_\alpha x_\alpha F_n dy_{1,\alpha} \dots F_n dy_{q,\alpha})$$

which, by the inductive hypothesis, is equal to

$$R^{S/n}_{T/n}\eta_{S/n}(\sum_\alpha x_\alpha F_n dy_{1,\alpha} \dots F_n dy_{q,\alpha})$$

which we assumed to be zero. Similarly, we have

$$\begin{aligned}\eta_S(dd\omega) &= \eta_S(d\log\eta_S([-1]_S)\cdot d\omega) = \eta_S d(d\log\eta_S([-1]_S)\cdot\omega)\\ &= \eta_S d(\sum_\alpha V_n(x_\alpha[-1]_{S/n}^n)d[-1]_S dy_{1,\alpha}\dots dy_{q,\alpha}),\end{aligned}$$

and by the definition of $N_S^{\cdot}$, this is zero since

$$\begin{aligned}&\eta_{S/n}(\sum_\alpha x_\alpha\eta_{S/n}([-1]_{S/n}^n)F_n d\eta_S([-1]_S)F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha})\\ &= d\log\eta_{S/n}([-1]_{S/n})\cdot\eta_{S/n}(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha})\end{aligned}$$

is zero. Next, to prove that $\eta_{S/m}F_m(\omega)$ and $\eta_{S/m}F_m(d\omega)$ are zero, it suffices to consider the three cases $n=km$, $m=kn$, and $(m,n)=1$. First, if $n=km$, then

$$\begin{aligned}\eta_{S/m}F_m(\omega) &= \eta_{S/m}(\sum_\alpha F_m V_n(x_\alpha)F_m dy_{1,\alpha}\dots F_m dy_{q,\alpha})\\ &= m\eta_{S/m}(\sum_\alpha V_k(x_\alpha)F_m dy_{1,\alpha}\dots F_m dy_{q,\alpha})\\ &= mV_k\eta_{S/n}(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha})\end{aligned}$$

which is zero as desired. Here the last equality uses the inductive hypothesis. Similarly, using Proposition 3.3, we find

$$\begin{aligned}\eta_{S/m}F_m(d\omega) &= \eta_{S/m}F_m(\sum_\alpha dV_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha})\\ &\quad + qd\log\eta_{S/m}([-1]_{S/m})\cdot\eta_{S/m}F_m(\omega).\end{aligned}$$

The second summand on the right-hand side is zero, by what was just proved, and by Lemma 3.8, we may rewrite the first summand as the sum

$$\begin{aligned}&\eta_{S/m}(\sum_\alpha dV_k(x_\alpha)F_m dy_{1,\alpha}\dots F_m dy_{q,\alpha})\\ &\quad + (m-1)d\log\eta_{S/m}([-1]_{S/m})\cdot\eta_{S/m}(\sum_\alpha V_k(x_\alpha)F_m dy_{1,\alpha}\dots F_m dy_{q,\alpha})\end{aligned}$$

where again the second summand is zero by what was proved above. We further rewrite the first summand as

$$\begin{aligned}&\eta_{S/m}d(\sum_\alpha V_k(x_\alpha)F_m dy_{1,\alpha}\dots F_m dy_{q,\alpha})\\ &\quad - qd\log\eta_{S/m}([-1]_{S/m})\cdot\eta_{S/m}(\sum_\alpha V_k(x_\alpha)F_m dy_{1,\alpha}\dots F_m dy_{q,\alpha})\end{aligned}$$

where the second summand is zero as before. Finally, using the inductive hypothesis, we rewrite the first summand as

$$d\eta_{S/m}(\sum_\alpha V_k(x_\alpha)F_m dy_{1,\alpha}\dots F_m dy_{q,\alpha}) = dV_k\eta_{S/n}(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha})$$

which is zero as desired. Next, suppose that $m = kn$. Then

$$\begin{aligned}\eta_{S/m}F_m(\omega) &= n\eta_{S/m}F_k(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha}) \\ &= nF_k\eta_{S/n}(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha})\end{aligned}$$

which is zero as desired. Here the last equality uses the inductive hypothesis. Using that $\eta_{S/m}F_m(\omega)$ is zero, we further find that

$$\begin{aligned}\eta_{S/m}F_m(d\omega) &= \eta_{S/m}F_m(\sum_\alpha dV_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}) \\ &= \eta_{S/m}F_k(\sum_\alpha dx_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha}) \\ &= \eta_{S/m}F_k d(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha}) \\ &= \eta_{S/k}F_k d\eta_{S/n}(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha})\end{aligned}$$

which is zero as desired. Here, again, the last equality uses the inductive hypothesis. Finally, we consider the case where m and n are relatively prime. We have

$$\begin{aligned}\eta_{S/m}F_m(\omega) &= \eta_{S/m}(\sum_\alpha V_nF_m(x_\alpha)F_m dy_{1,\alpha}\dots F_m dy_{q,\alpha}) \\ &= \sum_\alpha V_nF_m\eta_{S/n}(x_\alpha)F_m d\eta_S y_{1,\alpha}\dots F_m d\eta_S y_{q,\alpha} \\ &= \sum_\alpha V_n(F_m\eta_{S/n}(x_\alpha)F_nF_m d\eta_S y_{1,\alpha}\dots F_nF_m d\eta_S y_{q,\alpha}) \\ &= V_nF_m\eta_{S/n}(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha})\end{aligned}$$

which is zero. Here the second and third equality uses the inductive hypothesis. To prove that also $\eta_{S/m}F_m d\omega$ is zero, it will suffice to show that both $m\eta_{S/m}F_m d\omega$ and $n\eta_{S/n}F_m d\omega$ are zero. First, since $\eta_{S/m}F_m(\omega)$ is zero, so is

$$m\eta_{S/m}F_m d\omega = \eta_{S/m}dF_m\omega = d\eta_{S/m}F_m(\omega).$$

We next use the equality

$$\begin{aligned}ndV_n(x) &= dV_nF_nV_n(x) = d(V_n([1]_{S/n})\cdot V_n(x)) \\ &= V_n(x)dV_n([1]_{S/n}) + V_n([1]_{S/n})dV_n(x)\end{aligned}$$

together with the fact that $\eta_{S/m}F_m(\omega)$ is zero to see that

$$
\begin{aligned}
n\eta_{S/m}F_m(d\omega) &= \eta_{S/m}F_m(\sum_{\alpha} ndV_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}) \\
&= \eta_{S/m}F_m(\sum_{\alpha} V_n(x_\alpha)dV_n([1]_{S/n})dy_{1,\alpha}\dots dy_{q,\alpha}) \\
&\quad + \eta_{S/m}F_m(\sum_{\alpha} V_n([1]_{S/n})dV_n(x_\alpha)dy_{1,\alpha}\dots dy_{q,\alpha}) \\
&= \eta_{S/m}(\sum_{\alpha} V_nF_m(x_\alpha)F_mdV_n([1]_{S/n})F_mdy_{1,\alpha}\dots F_mdy_{q,\alpha}) \\
&\quad + \eta_{S/m}(\sum_{\alpha} V_nF_m([1]_{S/mn})F_mdV_n(x_\alpha)F_mdy_{1,\alpha}\dots F_mdy_{q,\alpha}) \\
&= V_n\eta_{S/mn}F_m(F_ndV_n([1]_{S/n})\cdot \sum_{\alpha} x_\alpha F_ndy_{1,\alpha}\dots F_ndy_{q,\alpha}) \\
&\quad + V_n\eta_{S/mn}F_m(\sum_{\alpha} F_ndV_n(x_\alpha)F_ndy_{1,\alpha}\dots F_ndy_{q,\alpha}) \\
&= V_n\eta_{S/mn}F_md(\sum_{\alpha} x_\alpha F_ndy_{1,\alpha}\dots F_ndy_{q,\alpha}) \\
&= V_nF_md\eta_{S/n}(\sum_{\alpha} x_\alpha F_ndy_{1,\alpha}\dots F_ndy_{q,\alpha})
\end{aligned}
$$

which is zero as desired. Here, again, the third and the last equalites use the inductive hypothesis.

To complete the recursive definition, it remains only to prove that for all positive integers n, $\omega \in \mathbb{W}_S\Omega_A^q$, and $\tau \in \mathbb{W}_{S/n}\Omega_A^r$, the projection formula

$$V_n(\omega)\tau = V_n(\omega F_n(\tau))$$

holds. It suffices to let $\omega = \eta_{S/n}(xF_ndy_1\dots F_ndy_q)$ and $\tau = \eta_S(zdw_1\dots dw_r)$. Then

$$
\begin{aligned}
V_n(\omega)\tau &= V_n\eta_{S/n}(xF_ndy_1\dots F_ndy_q)\cdot \eta_S(zdw_1\dots dw_r) \\
&= \eta_S(V_n(x)dy_1\dots dy_q\cdot zdw_1\dots dw_r) \\
&= \eta_S(V_n(xF_n(z))dy_1\dots dy_qdw_1\dots dw_r) \\
&= V_n(\eta_{S/n}(xF_ndy_1\dots F_ndy_q)\cdot \eta_{S/n}F_n(zdw_1\dots dw_r)) \\
&= V_n(\eta_{S/n}(xF_ndy_1\dots F_ndy_q)\cdot F_n\eta_S(zdw_1\dots dw_r)) \\
&= V_n(\omega F_n(z))
\end{aligned}
$$

as desired. This completes the proof. □

Definition 4.7. The initial Witt complex $\mathbb{W}_S\Omega_A^{\cdot}$ over the ring A is called the big de Rham-Witt complex of A.

Addendum 4.8. (i) *For all q, the map* $\eta_{\{1\}}\colon \Omega_A^q \to \mathbb{W}_{\{1\}}\Omega_A^q$ *is an isomorphism.*
(ii) *For all S, the map* $\eta_S\colon \mathbb{W}_S(A) \to \mathbb{W}_S\Omega_A^0$ *is an isomorphism.*

Proof. This follows immediately from the proof of Theorem B and Lemma 3.14. □

5 Étale morphisms

The functor that to the ring A associates the $\mathbb{W}_S(A)$-module $\mathbb{W}_S\Omega^q_A$ defines a presheaf of $\mathbb{W}_S(\mathscr{O})$-modules on the category of affine schemes. In this section, we use the theorem of Borger [3] and van der Kallen [22] which we recalled as Theorem 1.22 to show that for S finite, this presheaf is a quasi-coherent sheaf of $\mathbb{W}_S(\mathscr{O})$-modules for the étale topology. This is the statement of Theorem C which we now prove.

Proof of Theorem C. Let α denote the map of the statement. We define a structure of Witt complex over B on the domain of α. By Theorem 1.22, the map

$$\mathbb{W}_S(f)\colon \mathbb{W}_S(A) \to \mathbb{W}_S(B)$$

is étale. Hence, the derivation $d\colon \mathbb{W}_s\Omega^q_A \to \mathbb{W}_S\Omega^{q+1}_A$ extends uniquely to a derivation

$$d\colon \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^q_A \to \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^{q+1}_A$$

defined by the formula

$$d(b \otimes x) = (db)x + b \otimes dx$$

where db is the image of b by the composition

$$\mathbb{W}_S(B) \xrightarrow{d} \Omega^1_{\mathbb{W}_S(B)} \xleftarrow{\sim} \mathbb{W}_S\,(B) \otimes_{\mathbb{W}_S(A)} \Omega^1_{\mathbb{W}_S(A)} \to \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^1_A.$$

Here the middle map and the right-hand map are the canonical isomorphism and the canonical projection, respectively. We further define $R^S_T = R^S_T \otimes R^S_T$ and $F_n = F_n \otimes F_n$. To define the map

$$V_n\colon \mathbb{W}_{S/n}(B) \otimes_{\mathbb{W}_{S/n}(A)} \mathbb{W}_{S/n}\Omega^q_A \to\colon \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^q_A$$

we use that the square in the statement of Theorem 1.22 is cocartesian. It follows that the map

$$F_n \otimes \mathrm{id}\colon \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_{S/n}\Omega^q_A \to \mathbb{W}_{S/n}(B) \otimes_{\mathbb{W}_{S/n}(A)} \mathbb{W}_{S/n}\Omega^q_A$$

is an isomorphism, and we then define V_n to be the composition of the inverse of this isomorphism and the map

$$\mathrm{id} \otimes V_n\colon \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_{S/n}\Omega^q_A \to \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^q_A.$$

Finally, we define the map η_S to be the composition

$$\mathbb{W}_S(B) \to \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S(A) \xrightarrow{\mathrm{id}\otimes\eta} \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^0_A$$

of the canonical isomorphism and the map $\mathrm{id} \otimes \eta_S$. We proceed to show that this defines a Witt complex over B. The axioms (i)–(iii) of Definition 4.1 follow immediately from the definitions. For example, the identity $dd(x) = d\log\eta_S([-1]_S)d(x)$ holds as the two sides are derivations which agree on $\mathbb{W}_S\Omega^q_A$. It remains to verify axioms (iv)–(v) of Definition 4.1.

To prove axiom (iv), we first show that for all $b \in \mathbb{W}_{S/n}(B)$,

$$F_n dV_n(b) + (n-1) d\log \eta_{S/n}([-1]_{S/n}) \cdot b = d(b).$$

The right-hand side is the unique extension to $\mathbb{W}_{S/n}(B)$ of the derivation

$$d\colon \mathbb{W}_{S/n}(A) \to \mathbb{W}_{S/n}\Omega_A^1.$$

The left-hand side, $D_n(b)$, is also an extension of this map since $\mathbb{W}_S\Omega_A^q$ is a Witt complex over A. Hence, it will suffice to show that D_n is a derivation. Moreover, since D_n is an additive function of b, and since the square of rings in Theorem 1.22 is cocartesion, it is enough to consider elements of the form $F_n(b)a$ with $a \in \mathbb{W}_{S/n}(A)$ and $b \in \mathbb{W}_S(B)$. Now,

$$\begin{aligned}
&D_n(F_n(b)aF_n(b')a') = D_n(F_n(bb')aa') \\
&= F_n d(bb'V_n(aa')) + (n-1) d\log \eta([-1]) F_n(bb')aa' \\
&= F_n d(bb') F_n V_n(aa') + F_n(bb') F_n dV_n(aa') + (n-1) d\log \eta([-1]) F_n(bb')aa' \\
&= F_n d(bb') paa' + F_n(bb') d(aa') \\
&= (F_n d(b) F_n(b') + F_n(b) F_n d(b')) paa' + F_n(bb')(d(a)a' + ad(a')) \\
&= (F_n d(b) pa + F_n(b) d(a)) F_n(b')a' + F_n(b)a(F_n d(b') pa + F_n(b) d(a')) \\
&= D_n(F_n(b)a) F_n(b')a' + F_n(b) a D_n(F_n(b')a')
\end{aligned}$$

which shows that D_n is a derivation, and hence, equal to the derivation d. We note that the special case of axiom (iv) proved thus far implies that

$$dF_n = nF_n d\colon \mathbb{W}_S(B) \to \mathbb{W}_{S/n}(B) \otimes_{\mathbb{W}_{S/n}(A)} \mathbb{W}_{S/n}\Omega_A^1.$$

To prove axiom (iv) in general, we must show that

$$F_n dV_n(b \otimes x) + (n-1) d\log \eta_{S/n}([-1]_{S/n}) \cdot b \otimes x = d(b \otimes x)$$

for all $b \in \mathbb{W}_{S/n}(B)$ and $x \in \mathbb{W}_{S/n}\Omega_A^q$. As before, this follows once we prove that the left-hand side, $D_n(b \otimes x)$, is a derivation. Moreover, since D_n is additive and since the square of rings in Theorem 1.22 is cocartesian, it will suffice to consider elements of the form $F_n(b) \otimes x$ with $b \in \mathbb{W}_S(B)$ and $x \in \mathbb{W}_{S/n}\Omega_A^q$. Finally, to prove that D_n is a derivation on elements of this form, it suffices to show that

$$D_n(F_n(b) \otimes x) = D_n(F_n(b))x + F_n(b) \otimes D_n(x).$$

Now, by the definition of V_n and by the case of axiom (iv) proved above, we find

$$\begin{aligned}
&D_n(F_n(b) \otimes x) = F_n dV_n(F_n(b) \otimes x) + (n-1) d\log \eta([-1]) \cdot F_n(b) \otimes x \\
&= F_n d(b \otimes V_n(x)) + (n-1) d\log \eta([-1]) \cdot F_n(b) \otimes x \\
&= F_n d(b) \cdot F_n V_n(x) + F_n(b) \otimes F_n dV_n(x) + (n-1) d\log \eta([-1]) \cdot F_n(b) \otimes x \\
&= nF_n d(b) \cdot x + F_n(b) \otimes D_n(x) = dF_n(b) \cdot x + F_n(b) \otimes D_n(x) \\
&= D_n(F_n(b)) \cdot x + F_n(b) \otimes D_n(x)
\end{aligned}$$

as desired. This completes the proof of axiom (iv).

To prove axiom (v), we consider the following diagram where the left-hand horizontal maps are the canonical isomorphisms.

$$\begin{tikzcd}
\Omega^1_{\mathbb{W}_S(B)} \arrow[d, "F_n"] & \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \Omega^1_{\mathbb{W}_S(A)} \arrow[l] \arrow[r, "\mathrm{id} \otimes \eta_S"] \arrow[d, "F_n \otimes F_n"] & \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^1_A \arrow[d, "F_n \otimes F_n"] \\
\Omega^1_{\mathbb{W}_{S/n}(B)} & \mathbb{W}_{S/n}(B) \otimes_{\mathbb{W}_{S/n}(A)} \Omega^1_{\mathbb{W}_{S/n}(A)} \arrow[l] \arrow[r, "\mathrm{id} \otimes \eta_{S/n}"] & \mathbb{W}_{S/n}(B) \otimes_{\mathbb{W}_{S/n}(A)} \mathbb{W}_{S/n}\Omega^1_A
\end{tikzcd}$$

The left-hand square commutes since $F_n \colon \Omega^1_{\mathbb{W}(B)} \to \Omega^1_{\mathbb{W}(B)}$ is F_n-linear, and the right-hand square commutes since $\mathbb{W}_S\Omega^q_A$ is a Witt complex. Hence, also the outer square commutes and this immediately implies axiom (v); compare Remark 4.5.

We have proved that the domains of the canonical map α at the beginning of the proof form a Witt complex over B. Therefore, there exists a unique map

$$\beta \colon \mathbb{W}_S\Omega^q_B \to \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^q_A$$

of Witt complexes over B. The composition $\alpha \circ \beta$ is a selfmap of the initial object $\mathbb{W}_S\Omega^q_B$, and therefore, is the identity map. The composition $\beta \circ \alpha$ is a map of Witt complexes over B. In particular, it is a map of $\mathbb{W}_S(B)$-modules, and therefore, is determined by the composition with the map

$$\iota \colon \mathbb{W}_S\Omega^q_A \to \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^q_A$$

that takes x to $[1]_S \otimes x$. But ι and $\beta \circ \alpha \circ \iota$ both are maps of Witt complexes over A with domain the initial Witt complex over A. Therefore, the two maps are equal, and hence, also $\beta \circ \alpha$ is the identity map. This completes the proof. □

6 The big de Rham-Witt complex of the ring of integers

We finally evaluate the absolute de Rham-Witt complex of the ring of integers. If m and n are positive integers, we write (m,n) and $[m,n]$ for the greatest common divisor and least common multiple of m and n, respectively. We define $(m,n]$ to be the unique integer modulo $[m,n]$ such that $(m,n] \equiv 0$ modulo m and $(m,n] \equiv (m,n)$ modulo n, and define $\{m,n\}$ to the unique integer modulo 2 that is non-zero if and only if both m and n are even. We note that $(m,n] + (n,m] \equiv (m,n)$ modulo $[m,n]$.

Theorem 6.1. *The big de Rham-Witt complex of $\mathbb{Z}$ is given as follows.*

$$\mathbb{W}_S\Omega^0_{\mathbb{Z}} = \prod_{n \in S} \mathbb{Z} \cdot V_n\eta([1]_{S/n})$$
$$\mathbb{W}_S\Omega^1_{\mathbb{Z}} = \prod_{n \in S} \mathbb{Z}/n\mathbb{Z} \cdot dV_n\eta([1]_{S/n})$$

and the groups in degrees $q \geqslant 2$ are zero. The multiplication is given by

$$V_m\eta([1]_{S/m}) \cdot V_n\eta([1]_{S/n}) = (m,n) \cdot V_{[m,n]}\eta([1]_{S/[m,n]})$$
$$V_m\eta([1]_{S/m}) \cdot dV_n\eta([1]_{S/n}) = (m,n] \cdot dV_{[m,n]}\eta([1]_{S/[m,n]}) + \{m,n\} \sum_{r \geqslant 1} 2^{r-1}[m,n] \cdot dV_{2^r[m,n]}\eta([1]_{S/2^r[m,n]})$$

and the mth Frobenius and Verschiebung maps are given by

$$\begin{aligned}
F_m V_n \eta([1]_{S/n}) &= (m,n) \cdot V_{n/(m,n)} \eta([1]_{S/(m,n)}) \\
F_m dV_n \eta([1]_{S/n}) &= (m,n]/m \cdot dV_{n/(m,n)} \eta([1]_{S/[m,n]}) \\
&\quad + \{m,n\} \sum_{r \geqslant 1} (2^{r-1} n/(m,n)) \cdot dV_{2^r n/(m,n)} \eta([1]_{S/2^r[m,n]}) \\
V_m(V_n \eta([1]_{S/mn})) &= V_{mn} \eta([1]_{S/mn}) \\
V_m(dV_n \eta([1]_{S/mn})) &= m \cdot dV_{mn} \eta([1]_{S/mn}).
\end{aligned}$$

Proof. We begin by showing that the groups E_S^q and structure maps listed in the statement form a Witt complex over $\mathbb{Z}$. For notational convenience we suppress the subscript S. We first show that the product is associative. The formula for the product shows in particular that

$$d\log \eta([-1]) = (-\eta([1]) + V_2 \eta([1])) dV_2 \eta([1]) = \sum_{r \geqslant 1} 2^{r-1} dV_{2^r} \eta([1]).$$

We claim that

$$V_m \eta([1]) \cdot dV_n \eta([1]) = (m,n] \cdot dV_{[m,n]} \eta([1]) + \{m,n\} \cdot V_{[m,n]} \eta([1]) d\log \eta([-1]).$$

Indeed, by the formula for $d\log\eta([-1])$ which we just proved,

$$\begin{aligned}
&V_e \eta([1]) \cdot d\log \eta([-1]) = \sum_{r \geqslant 1} 2^{r-1} V_e \eta([1]) dV_{2^r} \eta([1]) \\
&= \sum_{r \geqslant 1} 2^{r-1} (e, 2^r] dV_{[e,2^r]} \eta([1]) + \{e,2\} \sum_{s \geqslant 1} 2^{s-1} [e,2] dV_{2^s[e,2]} \eta([1])
\end{aligned}$$

But $2^{r-1}(m,2^r]$ is congruent to $2^{r-1}m$ modulo $[m,2^r]$, and therefore, if e is odd, the lower left-hand summand is equal to $\sum_{r \geqslant 1} 2^{r-1} e dV_{2^r e} \eta([1])$ and the lower right-hand summand is zero, and if e is even, the lower left-hand summand is zero and the lower right-hand summand is equal to $\sum_{s \geqslant 1} 2^{s-1} e dV_{2^s e} \eta([1])$. The claim follows. A similar calculation shows that for all positive integers a and b,

$$V_a \eta([1]) \cdot (V_b \eta([1]) \cdot d\log \eta([-1])) = (V_a \eta([1]) \cdot V_b \eta([1])) \cdot d\log \eta([-1]).$$

We now show that the product is associative. On the one hand,

$$\begin{aligned}
&V_l \eta([1]) \cdot (V_m([1]) \cdot dV_n \eta([1])) = (l,[m,n]](m,n] \cdot dV_{[l,[m,n]]} \eta([1]) \\
&\quad + (\{l,[m,n]\}(m,n] + (l,[m,n])\{m,n\}) \cdot V_{[l,[m,n]]} \eta([1]) d\log \eta([-1])
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
&(V_l \eta([1]) \cdot V_m([1])) \cdot dV_n \eta([1]) = (l,m)([l,m],n] \cdot dV_{[[l,m],n]} \eta([1]) \\
&\quad + (l,m)\{[l,m],n\} \cdot V_{[[l,m],n]} \eta([1]) d\log \eta([1]).
\end{aligned}$$

Here $[l,[m,n]] = [[l,m],n]$ and to prove that $(l,[m,n]](m,n]$ and $(l,m)([l,m],n]$ are congruent modulo $[l,[m,n]]$, we use that $[l,[m,n]]\mathbb{Z}$ is the kernel of the map

$$\mathbb{Z} \to \mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$$

that takes a to $(a+l\mathbb{Z}, a+m\mathbb{Z}, a+n\mathbb{Z})$. So it will suffice to verify that the desired congruence holds modulo l, m, and n, respectively. By definition, both numbers are zero modulo l and m, and the congruence modulo n follows from the identity

$$(l,[m,n])\cdot(m,n) = (l,m)\cdot([l,m],n)$$

which is readily verified by multiplying by $[l,[m,n]] = [[l,m],n]$ on both sides. We also note that $\{l,[m,n]\}(m,n] + (l,[m,n])\{m,n\}$ and $(l,m)\{[l,m],n\}$ are well-defined integers modulo 2 which are non-zero if and only if n and exactly one of l and m are even. This shows that the product is associative.

We proceed to verify the axioms (i)–(v) of Definition 4.1. First, we note that since the sum $(m,n]+(n,m]$ is congruent to (m,n) modulo $[m,n]=[n,m]$, we find

$$\begin{aligned}
&dV_m\eta([1])\cdot V_n\eta([1]) + V_m\eta([1])\cdot dV_n\eta([1]) \\
&= (n,m]dV_{[n,m]}\eta([1]) + \{n,m\}V_{[n,m]}\eta([1])d\log\eta([-1]) \\
&\quad + (m,n]dV_{[m,n]}\eta([1]) + \{m,n\}V_{[m,n]}\eta([1])d\log\eta([-1]) \\
&= (m,n)dV_{[m,n]}\eta([1])
\end{aligned}$$

which verifies axiom (i). Next, to verify axiom (iv) we note that

$$\begin{aligned}
F_m(d\log\eta([-1])) &= \sum_{r\geqslant 1} 2^{r-1}F_m dV_{2^r}\eta([1]) \\
&= \sum_{r\geqslant 1} 2^{r-1}(m,2^r]/m\cdot dV_{[m,2^r]/m}\eta([1]) \\
&\quad + \{m,2\}V_{[m,2]/m}\eta([1])d\log\eta([-1]) \\
&= d\log\eta([-1]),
\end{aligned}$$

where the last equality uses that $2^{r-1}(m,2^r] \equiv 2^{r-1}m$ modulo $[m,2^r]$. Therefore, we may rewrite the stated formula for the Frobenius in the form

$$F_m dV_n\eta([1]) = (m,n]/m\cdot dV_{n/(m,n)}\eta([1]) + \{m,n\}\cdot V_{n/(m,n)}\eta([1])d\log\eta([-1])$$

which verifies axiom (iv). Similarly, to prove axiom (iii), we first note that

$$\begin{aligned}
V_m(d\log\eta([1])) &= V_m(\sum_{r\geqslant 1} 2^{r-1}dV_{2^r}\eta([1])) = \sum_{r\geqslant 1} 2^{r-1}mdV_{2^r m}\eta([1]) \\
&= V_m\eta([1])d\log\eta([1]).
\end{aligned}$$

Therefore, we find that

$$\begin{aligned}
V_mF_mdV_n\eta([1]) &= V_m((m,n]/m\cdot dV_{[m,n]/m}\eta([1]) \\
&\quad + \{m,n\}\cdot V_{[m,n]/m}\eta([1])d\log\eta([1])) \\
&= (m,n]\cdot dV_{[m,n]}\eta([1]) + \{m,n\}\cdot V_{[m,n]}\eta([1])d\log\eta([-1]) \\
&= V_m\eta([1])dV_n\eta([1])
\end{aligned}$$

where we have used that $[m,n]/m = n/(m,n)$. This proves that axiom (iii) holds. We next consider axiom (ii) and begin by proving that the identity $F_lF_m = F_{lm}$ holds. Since $(lm,n) = (l,n/(m,n))(m,n)$ we find on the one hand that

$$\begin{aligned}F_l(F_m dV_n\eta([1])) &= (l,n/(m,n)]/l \cdot (m,n]/m \cdot dV_{n/(lm,n)}\eta([1]) \\ &\quad + (\{l,n/(m,n)\}(m,n]/m + (l,n/(m,n))\{m,n\}) \cdot V_{n/(lm,n)}\eta([1])d\log\eta([-1])\end{aligned}$$

and on the other hand that

$$\begin{aligned}F_{lm}dV_n\eta([1]) &= (lm,n]/lm \cdot dV_{n/(lm,n)}\eta([1]) \\ &\quad + \{lm,n\} \cdot V_{n/(lm,n)}\eta([1])d\log\eta([-1]).\end{aligned}$$

The classes $(lm,n]$ and $(l,n/(m,n)](m,n]$ are congruent modulo $[lm,n]$. For both are congruent to 0 modulo lm and congruent to $(lm,n) = (l,n/(m,n))(m,n)$ modulo n. It remains only to note that $\{lm,n\}$ and $\{l,n/(m,n)\}(m,n]/m + (l,n/(m,n))\{m,n\}$ are well-defined integers modulo 2 which are non-zero if and only if n and at least one of l and m are even. This shows that $F_lF_m = F_{lm}$. The formulas $V_lV_m = V_{lm}$ and $F_mV_m = m \cdot \mathrm{id}$ are readily verified, so we next show that $F_lV_m = V_mF_l$ provided that l and m are relatively prime. Using what was proved earlier, we find that

$$\begin{aligned}F_lV_m dV_n\eta([1]) &= m(l,mn]/l \cdot dV_{mn/(l,mn)}\eta([1]) \\ &\quad + m\{l,mn\} \cdot V_{mn/(l,mn)}\eta([1])d\log\eta([-1]) \\ V_mF_l dV_n\eta([1]) &= m(l,n]/l \cdot dV_{mn/(l,n)\eta([1])} \\ &\quad + \{l,n\} \cdot V_{mn/(l,n)}\eta([1])d\log\eta([-1])\end{aligned}$$

Since $(l,m) = 1$, we have $(l,mn) = (l,n)$. Moreover, modulo $n/(l,n)$, the congruence classes of $(l,n]/l$ and $l/(l,n)$ are inverse to each other, and modulo $mn/(l,mn)$, the congruence classes of $(l,mn]/l$ and $l/(l,mn)$ are inverse to each other. It follows that the congruence classes of $m(l,mn]/l$ and $m(l,n]/l$ modulo $mn/(l,n)$ are equal. Since also $m\{l,mn\} = \{l,n\}$, we find that $F_lV_m = V_mF_l$ as desired. This proves axiom (ii). Finally, we consider axiom (v). Since the formula for $[a]_S$ in Addendum 1.7 is quite complicated, it is an onerous task to verify this axiom directly. However, as noted in Remark 4.5, we may instead show that for every prime number p, the diagram

$$\begin{array}{ccc}\Omega^1_{\mathbb{W}(A)} & \xrightarrow{\eta} & E^1_{\mathbb{N}} \\ \downarrow{\scriptstyle F_p} & & \downarrow{\scriptstyle F_p} \\ \Omega^1_{\mathbb{W}(A)} & \xrightarrow{\eta} & E^1_{\mathbb{N}}\end{array}$$

commutes. It further suffices to show that for every positive integer n, the image of the element $dV_n([1])$ by the two composites in the diagram are equal. We consider three cases separately. First, if p is odd and $n = ps$ is divisible by p, then

$$F_p\eta dV_n([1]) = F_p dV_n\eta([1]) = dV_s\eta([1])$$

which is equal to

$$\eta F_p dV_n([1]) = \eta(dV_s([1]) + n^{p-2}(V_n([1])dV_n([1]) - sdV_n([1]))) = dV_s\eta([1])$$

as desired. Second, if $p = 2$ and $n = 2s$ is even, then

$$F_2\eta dV_n([1]) = F_2 dV_n\eta([1]) = dV_s\eta([1]) + \sum_{r\geqslant 1} 2^{r-1} sdV_{2^r s}\eta([1])$$

while

$$\begin{aligned}\eta F_2 dV_n([1]) &= \eta(dV_s([1]) + V_n([1])dV_n([1]) - sdV_n([1])) \\ &= dV_s\eta([1]) + \sum_{t\geqslant 1} 2^{t-1} ndV_{2^t n}\eta([1]) - sdV_n\eta([1]),\end{aligned}$$

so the desired equality holds in this case, too. Third, if n is not divisible by p, we have $((1-n^{p-1})/p)\cdot p + n^{p-2}\cdot n = 1$, and hence,

$$F_p\eta dV_n([1]) = F_p dV_n\eta([1]) = \frac{1-n^{p-1}}{p} dV_n\eta([1]).$$

We wish to prove that this equal to

$$\eta F_p dV_n([1]) = \eta(n^{p-2}V_n([1]) + dV_n([1]) + \frac{1-n^{p-1}}{p} dV_n([1])),$$

or equivalently, that $n^{p-2}V_n\eta([1])dV_n\eta([1])$ is zero. This is true for p odd, since $ndV_n\eta([1])$ is zero, and for $p = 2$, since $V_n\eta([1])dV_n\eta([1])$ is zero, for n odd.

We have now proved that the groups E_S^q and structure maps listed in the statement form a Witt complex over $\mathbb{Z}$. It follows that there is unique map

$$\mathbb{W}_S\Omega_{\mathbb{Z}}^q \to E_S^q$$

of Witt complexes over $\mathbb{Z}$ which one readily verifies to be an isomorphism for $q \leqslant 1$. Therefore, it remains only to prove that $\mathbb{W}_S\Omega_{\mathbb{Z}}^q$ is zero for $q \geqslant 2$. It will suffice to show that for S finite, the elements $ddV_n\eta([1]_{S/n})$ are zero for all $n \in S$. Now,

$$\begin{aligned}ddV_n\eta([1]_{S/n}) &= d\log\eta([-1]_S)\cdot dV_n\eta([1]_{S/n}) = d(d\log\eta([-1]_S)\cdot V_n\eta([1]_{S/n})) \\ &= dV_n(d\log\eta([-1]_{S/n})) = \sum_{r\geqslant 1} 2^{r-1} dV_n dV_{2^r}\eta([1]_{S/2^r n}) \\ &= nddV_{2n}\eta([1]_{S/2n}).\end{aligned}$$

Since S is assumed to be finite, this furnishes an induction argument which shows that $ddV_n\eta([1]_{S/n})$ is zero. We conclude that $\mathbb{W}_S\Omega_{\mathbb{Z}}^q$ is zero for $q \geqslant 2$ as desired. This completes the proof. □

Addendum 6.2. *Let S be a finite truncation set. The kernel of the canonical map*

$$\eta_S\colon \hat{\Omega}^{\cdot}_{\mathbb{W}_S(Z)} \to \mathbb{W}_S\Omega^{\cdot}_Z$$

is equal to the graded ideal generated by the following elements (i)–(ii) *together with their images by the derivation d.*

(i) *For all* $m,n \in S$, *the element*

$$V_m([1]_{S/m})dV_n([1]_{S/n}) - (m,n]dV_{[m,n]}([1]_{S/[m,n]}) \\ - \{m,n\}\sum_{r\geqslant 1} 2^{r-1}[m,n]dV_{2^r[m,n]}([1]_{S/2^r[m,n]})$$

(ii) *For all* $n \in S$, *the element* $ndV_n([1]_{S/n})$.

Proof. This follows from the proof of Theorem 6.1. ☐

We remark that in Addendum 6.2, the graded ring $\hat{\Omega}^{\cdot}_{\mathbb{W}_S(\mathbb{Z})}$ may be replaced by the graded ring $\Omega^{\cdot}_{\mathbb{W}_S(\mathbb{Z})}$.

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