On the K-theory of complete regular local $\mathbb{F}_p$-algebras

Thomas Geisser [a,1] Lars Hesselholt [b,2]

[a] *University of Southern California, Los Angeles, California*

[b] *Massachusetts Institute of Technology, Cambridge, Massachusetts*

Abstract

Let A be a noetherian $\mathbb{F}_p$-algebra that is finitely generated as a module over the subring A^p of pth powers. We give an explicit formula for the de Rham-Witt complex of the power series ring $A[\![t]\!]$ in terms of that of the ring A. We use this formula to show that, for every complete regular local $\mathbb{F}_p$-algebra whose residue field is a finite extension of the subfield of pth powers, the canonical map from the algebraic K-theory with $\mathbb{Z}/p^v$-coefficients to the topological K-theory with $\mathbb{Z}/p^v$-coefficients is an isomorphism.

Key words: De Rham-Witt complex, K-theory, complete regular local ring.

Introduction

Let $(R, \mathfrak{m}, k)$ be a complete regular local $\mathbb{F}_p$-algebra and assume that the residue field k is a finite extension of the subfield k^p of pth powers. We show that the canonical map

$$K_q(R, \mathbb{Z}/p^v) \to K_q^{\mathrm{top}}(R, \mathbb{Z}/p^v)$$

from the algebraic K-theory with $\mathbb{Z}/p^v$-coefficients to the topological K-theory with $\mathbb{Z}/p^v$-coefficients is an isomorphism. The proof gives an inductive procedure for evaluating the common group in terms of the de Rham-Witt groups of k. This extends previous results of Dundas [2] and the authors [4, Prop. 5.3.1],

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where the case of a one-dimensional complete regular local $\mathbb{F}_p$-algebra with perfect residue field was considered. The topological K-groups, which take the $\mathfrak{m}$-adic topology on R into account, are defined to be the homotopy groups with $\mathbb{Z}/p^v$-coefficients of the homotopy limit spectrum

$$K^{\mathrm{top}}(R) = \operatorname*{holim}_n K(R/\mathfrak{m}^n).$$

Hence there is natural exact sequence

$$0 \to R^1 \lim_s K_{q+1}(R/\mathfrak{m}^s, \mathbb{Z}/p^v) \to K_q^{\mathrm{top}}(R, \mathbb{Z}/p^v) \to \lim_s K_q(R/\mathfrak{m}^s, \mathbb{Z}/p^v) \to 0$$

compare Wagoner [19]. By the structure theorem for complete regular local $\mathbb{F}_p$-algebras, the ring R is non-canonically isomorphic to the power series ring $k[\![t_1, \dots, t_d]\!]$, where d is the Krull dimension of R. We shall allow for a slightly more general ring of coefficients:

Theorem A *Let A be a regular local $\mathbb{F}_p$-algebra and assume that A is finitely generated as a module over the subring A^p. Let $R = A[\![t_1, \dots, t_d]\!]$ and let $I \subset R$ be the ideal generated by $t_1, \dots, t_d$. Then the canonical map*

$$K_q(R, \mathbb{Z}/p^v) \to K_q^{\mathrm{top}}(R, \mathbb{Z}/p^v)$$

is an isomorphism, for all integers q and $v \geqslant 1$.

We note that the Gabber-Suslin rigidity theorem [17,3] implies that for m prime to p and for all $s \geqslant 1$, the natural projection induces an isomorphism

$$K_q(R, \mathbb{Z}/m) \xrightarrow{\sim} K_q(R/I^s, \mathbb{Z}/m).$$

Hence, in this case, the limit system is constant. On the other hand, continuity fails rationally, and hence integrally.

We briefly discuss the method of proof. The K-theory spectra of the rings R and R/I^s naturally decompose as wedge sums

$$K(A) \vee K(R, I) \xrightarrow{\sim} K(R)$$
$$K(A) \vee K(R/I^s, I) \xrightarrow{\sim} K(R/I^s)$$

and the natural projection induces the identity map of the first summand on the left. Thus we may as well show that the induced map of relative theories

$$K(R, I) \to \operatorname*{holim}_s K(R/I^s, I)$$

induces an isomorphism of homotopy groups with $\mathbb{Z}/p^v$-coefficients. To this end, we compare the K-theory spectra to the corresponding topological cyclic homology spectra via the cyclotomic trace in the following diagram.

$$\begin{array}{ccc} K(R,I) & \xrightarrow{\quad\text{tr}\quad} & \mathrm{TC}(R,I;p) \\ \downarrow & & \downarrow \\ \mathrm{holim}_s\, K(R/I^s,I) & \xrightarrow{\text{tr}} & \mathrm{holim}_s\, \mathrm{TC}(R/I^s,I;p). \end{array}$$

By a theorem of McCarthy [15], the lower horizontal map induces an isomorphism of homotopy groups with $\mathbb{Z}/p^v$-coefficients. A theorem of Popescu states that every regular $\mathbb{F}_p$-algebra is a filtered colimit of smooth $\mathbb{F}_p$-algebras [16]. (See also [18].) This implies that the results of [5] and [6], which were proved originally for smooth $\mathbb{F}_p$-algebras, are valid, more generally, for regular $\mathbb{F}_p$-algebras. Hence, there is a natural long-exact sequence

$$\cdots \to \mathrm{TC}_q(R,I;p) \to W\Omega^q_{(R,I)} \xrightarrow{1-F} W\Omega^q_{(R,I)} \to \mathrm{TC}_{q-1}(R,I;p) \to \cdots$$

where $W\Omega^q_{(R,I)}$ is the de Rham-Witt complex of [12]. Moreover, the composite

$$K_q(R,I;\mathbb{Z}_p) \to \mathrm{TC}_q(R,I;p) \to W\Omega^q_R$$

maps the relative K-group with $\mathbb{Z}_p$-coefficients isomorphically onto the kernel of $1-F$. We prove the following result which is also interesting in its own right. The proof uses the corresponding result, proved in [11], for polynomial algebras, and the extension to power series rings was inspired by Kato's paper [13].

Theorem B *Let A be a noetherian $\mathbb{F}_p$-algebra and suppose that A is finitely generated as an A^p-module. Then every element $\omega^{(n)} \in W_n\Omega^q_{A[\![t]\!]}$ can be written uniquely as an infinite series*

$$\begin{aligned} \omega^{(n)} = & \sum_{i\in\mathbb{N}_0} a^{(n)}_{0,i}[t]^i_n + \sum_{i\in\mathbb{N}} b^{(n)}_{0,i}[t]^{i-1}_n d[t]_n \\ & + \sum_{s\in\mathbb{N}}\sum_{j\in I_p} \Big(V^s(a^{(n-s)}_{s,j}[t]^j_{n-s}) + dV^s(b^{(n-s)}_{s,j}[t]^j_{n-s}) \Big) \end{aligned}$$

with the components $a^{(m)}_{s,i} \in W_m\Omega^q_A$ and $b^{(m)}_{s,i} \in W_m\Omega^{q-1}_A$. Here I_p is the set of positive integers that are not divisible by p and $[t]_n$ is the multiplicative representative of t in $W_n(A[\![t]\!])$.

Using this result we prove, by induction on the number of variables, that the map $1-F$ in the sequence above is surjective. It follows that the cyclotomic trace induces an isomorphism

$$K_q(R,I;\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{TC}_q(R,I;p).$$

Finally, we show that the canonical map

$$\mathrm{TC}(R,I;p) \to \operatorname*{holim}_s \mathrm{TC}(R/I^s,I;p)$$

is a weak equivalence. This completes the outline of the proof of Thm. A.

We mention that Thm. B gives a canonical isomorphism

$$W\Omega^q_{A,\log} \oplus \mathbf{W}\Omega^{q-1}_A \xrightarrow{\sim} W\Omega^q_{A[\![t]\!],\log}$$

where the second summand on the left is the big de Rham-Witt group considered in [9]. Previously, the graded pieces of a complete and separated filtration of the left-hand side were known [13].

In this paper, A will always denote an $\mathbb{F}_p$-algebra, R the power series algebra $A[\![t_1, \dots, t_d]\!]$, and $I \subset R$ the ideal generated by $t_1, \dots, t_d$. A pro-object of a category $\mathcal{C}$ is a functor to $\mathcal{C}$ from the set of positive integers viewed as a category with one morphism from m to n, if $m \geqslant n$. A strict map from a pro-object X to a pro-object Y is a natural transformation and a general map from X to Y is an element of the set

$$\mathrm{Hom}_{\mathrm{pro}-\mathcal{C}}(X, Y) = \lim_n \operatorname*{colim}_m \mathrm{Hom}_{\mathcal{C}}(X_m, Y_n).$$

1 Continuity and topological cyclic homology

Let A be a regular $\mathbb{F}_p$-algebra, finitely generated as a module over the subring A^p, and let R and $I \subset R$ be as in the introduction. In this paragraph, we show that the canonical map

$$\mathrm{TC}(R; p) \to \operatorname*{holim}_s \mathrm{TC}(R/I^s; p)$$

is a weak equivalence. We briefly recall the definition of $\mathrm{TC}(R; p)$, refering the reader to [10, Sect. 1] for a fuller discussion.

For any ring S, the spectrum $\mathrm{TC}(S; p)$ is defined as the homotopy fixed points of an operator called Frobenius on another spectrum $\mathrm{TR}(S; p)$. Hence, there is a natural cofibration sequence

$$\mathrm{TC}(S; p) \to \mathrm{TR}(S; p) \xrightarrow{1-F} \mathrm{TR}(S; p) \xrightarrow{\partial} \Sigma\, \mathrm{TC}(S; p).$$

The spectrum $\mathrm{TR}(S; p)$, in turn, is the homotopy limit of a pro-spectrum $\mathrm{TR}^{\boldsymbol{\cdot}}(S; p)$ and there are strict maps of pro-spectra

$$F\colon\ \mathrm{TR}^n(S; p) \to \mathrm{TR}^{n-1}(S; p)$$
$$V\colon\ \mathrm{TR}^{n-1}(S; p) \to \mathrm{TR}^n(S; p).$$

The spectrum $\mathrm{TR}^1(S; p)$ is the topological Hochschild spectrum $\mathrm{TH}(S)$. It has an action by the circle group $\mathbb{T}$ and the higher levels in the pro-system by definition are the fixed sets of the cyclic subgroups of $\mathbb{T}$ of p-power order

$$\mathrm{TR}^n(S;p) = \mathrm{TH}(S)^{C_{p^{n-1}}}.$$

The map F is the obvious inclusion and V is the accompanying transfer. The structure map R in the pro-system is harder to define and uses the so-called cyclotomic structure of $\mathrm{TH}(S)$. There is a natural cofibration sequence

$$\mathbb{H}.(C_{p^{n-1}}, \mathrm{TH}(S)) \xrightarrow{N} \mathrm{TR}^n(S;p) \xrightarrow{R} \mathrm{TR}^{n-1}(S;p) \xrightarrow{\partial} \Sigma\mathbb{H}.(C_{p^{n-1}}, \mathrm{TH}(S))$$

which gives the "layers" in the tower $\mathrm{TR}^{\cdot}(S;p)$. Here the left-hand term is the group homology spectrum (or borel spectrum) of the group $C_{p^{n-1}}$ acting on $\mathrm{TH}(S)$. Its homotopy groups are given by a strongly convergent first quadrant homology type spectral sequence

$$E^2_{s,t} = H_s(C_{p^v}, \mathrm{TH}_t(S)) \Rightarrow \mathbb{H}_{s+t}(C_{p^v}, \mathrm{TH}(S)).$$

We first show that, quite generally, continuity for topological Hochschild homology implies continuity for topological cyclic homology.

Lemma 1.1 *Let G be a finite group and let X_n, $n \geqslant 1$, be a limit system of G-spectra. Suppose that there exists $m \in \mathbb{Z}$ such that $\pi_t X_n$ vanishes for all $n \geqslant 1$ if $t < m$. Then the canonical map*

$$\mathbb{H}.(G, \operatorname*{holim}_n X_n) \to \operatorname*{holim}_n \mathbb{H}.(G, X_n)$$

is a weak equivalence.

Proof. Let EG be a free contractible G-CW complex. Then by definition

$$\mathbb{H}.(G, X) = (X \wedge EG_+)^G.$$

The filtration of EG by the skeleta defines an exact couple

$$\begin{aligned} D_{s,t} &= \pi_{s+t}\Big((X \wedge E_sG_+)^G\Big) \\ E_{s,t} &= \pi_{s+t}\Big((X \wedge (E_sG/E_{s-1}G))^G\Big) \end{aligned}$$

and this in turn gives rise to the spectral sequence

$$E^2_{s,t} = H_s(G, \pi_t X) \Rightarrow \mathbb{H}_{s+t}(G, X).$$

In particular, if $\pi_t X$ vanishes for $t < m$, then the map induced from the canonical inclusion

$$\pi_n\Big((X \wedge E_sG_+)^G\Big) \to \pi_n\Big((X \wedge EG_+)^G\Big)$$

is an isomorphism for $n \leqslant s+m$. Now consider the following diagram

$$\begin{array}{ccc} \Big((\operatorname{holim}_n X_n) \wedge E_sG_+\Big)^G & \longrightarrow & \Big((\operatorname{holim}_n X_n) \wedge EG_+\Big)^G \\ \downarrow & & \downarrow \\ \operatorname{holim}_n \Big((X_n \wedge E_sG_+)^G\Big) & \longrightarrow & \operatorname{holim}_n \Big((X_n \wedge EG_+)^G\Big). \end{array}$$

Since G is finite, we can choose EG such that E_sG is a finite CW complex, for all $s \geqslant 0$. This implies that the left-hand vertical map is a weak equivalence. Indeed, E_sG_+ has a dual. Moreover, the horizontal maps induce isomorphism of homotopy groups in degrees less than m. (The assumption that $\pi_t X_n$ vanishes for for all $n \geqslant 1$ if $t < m$ implies that $\pi_t \operatorname{holim}_n X_n$ vanishes for $t < m-1$.) Hence the right-hand vertical map will induce an isomorphism of homotopy groups in degrees less than $s+m$. But this is true for all $s \geqslant 0$, so the right-hand vertical map is a weak equivalence. □

Proposition 1.2 *Let S be a ring and let $J \subset S$ be an ideal. If the canonical map* $\mathrm{TH}(S) \to \operatorname{holim}_s \mathrm{TH}(S/J^s)$ *is a weak equivalence, then so is*

$$\mathrm{TC}(S;p) \to \operatorname*{holim}_s \mathrm{TC}(S/J^s;p).$$

Proof. We show inductively that the canonical map

$$\mathrm{TR}^n(S;p) \to \operatorname*{holim}_s \mathrm{TR}^n(S/J^s;p)$$

is a weak equivalence. The case $n = 1$ is the assumption. In the induction step, we must show that the canonical map

$$\mathbb{H}.(C_{p^v}, \mathrm{TH}(S)) \to \operatorname*{holim}_s \mathbb{H}.(C_{p^v}, \mathrm{TH}(S/J^s))$$

is a weak equivalence. This map is equal to the composite map

$$\mathbb{H}.(C_{p^v}, \mathrm{TH}(S)) \to \mathbb{H}.(C_{p^v}, \operatorname*{holim}_s \mathrm{TH}(S/J^s)) \to \operatorname*{holim}_s \mathbb{H}.(C_{p^v}, \mathrm{TH}(S/J^s)).$$

The right-hand map is an equivalence by Lemma 1.1, and the left-hand map is an equivalence since forming the group homology spectrum preserves weak equivalences. Finally, the result for topological cyclic homology follows since homotopy limits commute. □

We will show in Prop. 1.7 below that if A is a regular $\mathbb{F}_p$-algebra which is finitely generated as an A^p-module, then the canonical map

$$\mathrm{TH}(R) \to \operatorname*{holim}_s \mathrm{TH}(R/I^s)$$

is a weak equivalence. The proof is based on a series of lemmas.

Lemma 1.3 *Let S be a ring and $J \subset S$ an ideal. Then the canonical map*

$$S/J^{\cdot} \otimes_S \Omega^q_S \to \Omega^q_{S/J^{\cdot}}$$

is an isomorphism of pro-abelian groups.

Proof. Let $\Omega^*_{(S,J)} \subset \Omega^*_S$ be the differential graded ideal generated by J. Then the projection induces an isomorphism $\Omega^*_S/\Omega^*_{(S,J^{\cdot})} \xrightarrow{\sim} \Omega^*_{S/J}$, and by the Leibniz rule, $\Omega^*_{(S,J^{2s})} \subset J^s \otimes_S \Omega^*_{(S,J^s)}$. It follows that in the diagram

$$\begin{array}{ccccccc} S/J^{2s} \otimes_S \Omega^q_{(S,J^{2s})} & \longrightarrow & S/J^{2s} \otimes_S \Omega^q_S & \longrightarrow & \Omega^q_{S/J^{2s}} & \longrightarrow & 0 \\ \downarrow{\scriptstyle 0} & & \downarrow & & \downarrow & & \\ S/J^{s} \otimes_S \Omega^q_{(S,J^{s})} & \longrightarrow & S/J^{s} \otimes_S \Omega^q_S & \longrightarrow & \Omega^q_{S/J^{s}} & \longrightarrow & 0 \end{array}$$

the left-hand vertical map is zero, and this, in turn, shows that the the map of the statement is an isomorphism of pro-abelian groups. $\square$

Lemma 1.4 *Let A be a ring, let $S = A[t_1, \ldots, t_d]$, and let $J \subset S$ be the ideal generated by $t_1, \ldots, t_d$. Then the canonical map*

$$S/J^{\cdot} \otimes_S \mathrm{TH}_q(S) \to \mathrm{TH}_q(S/J^{\cdot})$$

is an isomorphism of pro-abelian groups.

Proof. Let $J_s \subset S$ be the ideal generated by $t_1^s, \ldots, t_d^s$. We note that

$$J_s \subset J^s \subset J_{[s/d]}$$

where $[m]$ is the largest integer less than or equal to m. This shows that in the diagram

$$\begin{array}{ccc} S/J. \otimes_S \mathrm{TH}_q(S) & \longrightarrow & \mathrm{TH}_q(S/J.) \\ \downarrow & & \downarrow \\ S/J^{\cdot} \otimes_S \mathrm{TH}_q(S) & \longrightarrow & \mathrm{TH}_q(S/J^{\cdot}) \end{array}$$

the vertical maps are isomorphisms of pro-abelian groups. Hence, we may instead show that the lower horizontal map is an isomorphism of pro-abelian groups. The rings S and S/J_s both are pointed monoid algebras in the sense of [8, Sect. 7] and their topological Hochschild homology groups therefore are given by *loc. cit.* Thm. 7.1. We use this to show that in the diagram

$$\begin{array}{ccc} S/J. \otimes_S \mathrm{TH}_*(S) & \longrightarrow & \mathrm{TH}_*(S/J.) \\ \uparrow & & \uparrow \\ S/J. \otimes_S \Omega^*_S \otimes_{\Omega^*_A} \mathrm{TH}_*(A) & \longrightarrow & \Omega^*_{S/J.} \otimes_{\Omega^*_A} \mathrm{TH}_*(A) \end{array}$$

the left-hand vertical map is an isomorphism, and the right-hand vertical map an isomorphism of pro-abelian groups. Since Lemma 1.3 implies that also the lower horizontal map is an isomorphism of pro-abelian groups, the lemma will follow.

For the ring S in question, we get a weak equivalence

$$\mathrm{TH}(A) \wedge N^{\mathrm{cy}}(\Pi_\infty^{\wedge d}) \to \mathrm{TH}(S)$$

where the second factor on the left is the cyclic bar-construction of the d-fold smash product of the pointed monoid $\Pi_\infty = \{0, 1, t, t^2, \dots\}$. Moreover, the cyclic bar-construction decomposes as wedge sum

$$\bigvee_{i_1,\dots,i_d \in \mathbb{N}_0} (N^{\mathrm{cy}}(\Pi_\infty, i_1) \wedge \cdots \wedge N^{\mathrm{cy}}(\Pi_\infty; i_d)) \xrightarrow{\sim} N^{\mathrm{cy}}(\Pi_\infty^{\wedge d}).$$

It is proved in [6, Lemma 3.1.6] that $N^{\mathrm{cy}}(\Pi_\infty, 0) = S^0$ and that for $i > 0$, there is a $\mathbb{T}$-equivariant deformation retract

$$\mathbb{T}/C_{i+} \xrightarrow{\sim} N^{\mathrm{cy}}(\Pi_\infty; i)$$

where $\mathbb{T}$ and C_i denote the circle group and the cyclic subgroup of order i. On homotopy groups in degree q, this shows that for $i > 0$,

$$\pi_q(\mathrm{TH}(A) \wedge N^{\mathrm{cy}}(\Pi_\infty, i)) = \mathrm{TH}_q(A) \cdot t^i \oplus \mathrm{TH}_{q-1}(A) \cdot t^{i-1} dt,$$

and hence the canonical map

$$\Omega^*_S \otimes_{\Omega^*_A} \mathrm{TH}_*(A) \to \mathrm{TH}_*(S)$$

is an isomorphism.

The description of the topological Hochschild homology of the ring S/J_s is completely parallel. There is a natural equivalence

$$\bigvee_{i_1,\dots,i_d \in \mathbb{N}_0} \mathrm{TH}(A) \wedge (N^{\mathrm{cy}}(\Pi_s, i_1) \wedge \cdots \wedge N^{\mathrm{cy}}(\Pi_s; i_d)) \xrightarrow{\sim} \mathrm{TH}(S/J_s)$$

where Π_s is the pointed monoid $\{0, 1, t, \dots, t^{s-1}\}$ with base-point 0 and with $t^s = 0$. The $\mathbb{T}$-equivariant homotopy type of the spaces $N^{\mathrm{cy}}(\Pi_s, i)$ was determined in [7, Thm. B]. In particular, the natural projection

$$N^{\mathrm{cy}}(\Pi_\infty, i) \to N^{\mathrm{cy}}(\Pi_s, i)$$

is a homeomorphism, for $i < s$, and for $i = s$, there is a cofibration sequence

$$N^{\mathrm{cy}}(\Pi_\infty, 1) \to N^{\mathrm{cy}}(\Pi_\infty, s) \to N^{\mathrm{cy}}(\Pi_s, s) \to \Sigma N^{\mathrm{cy}}(\Pi_\infty, 1).$$

On homotopy groups in degree q, the latter gives rise to a short exact sequence

$$0 \to (\mathrm{TH}_{q-1}(A)/s) \cdot t^{s-1}dt \to \pi_q(\mathrm{TH}(A) \wedge N^{\mathrm{cy}}(\Pi_s, s)) \to \mathrm{TH}_{q-2}(A)[s] \cdot t \to 0.$$

It follows that the map

$$\Omega^*_{S/J_s} \otimes_{\Omega^*_A} \mathrm{TH}_*(A) \to \mathrm{TH}_*(S/J_s)$$

is injective and that the cokernel C^*_s only involves the summands $(i_1, \dots, i_d)$ where $i_j \geqslant s$ for some $j = 1, \dots, d$. Finally, since $N^{\mathrm{cy}}(\Pi_s, i)$ is $[(i-1)/s]$-connected, the canonical inclusion

$$\bigvee_{0 \leqslant i_1, \dots, i_d < qs+1} \mathrm{TH}(A) \wedge (N^{\mathrm{cy}}(\Pi_s, i_1) \wedge \cdots \wedge N^{\mathrm{cy}}(\Pi_s; i_d)) \to \mathrm{TH}(S/J_s)$$

induces an isomorphism on homotopy groups in degrees $\leqslant q$. It follows that in degree q, the map of cokernels $C^q_{qs+1} \to C^q_s$ is zero. □

Remark 1.5 It would be interesting to determine if, generally, given a ring S and an ideal $J \subset S$, the canonical map

$$S/J^{\cdot} \otimes_S \mathrm{TH}_q(S)) \to \mathrm{TH}_q(S/J^{\cdot})$$

is an isomorphism of pro-abelian groups. As far as we know the analogous question for ordinary Hochschild homology also has not been settled.

Lemma 1.6 *Let A be a regular $\mathbb{F}_p$-algebra. Then the canonical map*

$$\Omega^*_{R/I^{\cdot}} \otimes \mathrm{TH}_*(\mathbb{F}_p) \to \mathrm{TH}_*(R/I^{\cdot})$$

is an isomorphism of pro-abelian groups.

Proof. Let S be the polynomial algebra $A[t_1, \dots, t_d]$ and let J be the ideal generated by $t_1, \dots, t_d$. Then, since $S/J^s \xrightarrow{\sim} R/I^s$, the map of the statement may be identified with the lower horizontal map in the diagram

$$\begin{array}{ccc} S/J^{\cdot} \otimes_S \Omega^*_S \otimes \mathrm{TH}_*(\mathbb{F}_p) & \longrightarrow & S/J^{\cdot} \otimes_S \mathrm{TH}_*(S) \\ \downarrow & & \downarrow \\ \Omega^*_{S/J^{\cdot}} \otimes \mathrm{TH}_*(\mathbb{F}_p) & \longrightarrow & \mathrm{TH}_*(S/J^{\cdot}). \end{array}$$

In this diagram, the left-hand vertical map is an isomorphism of pro-abelian groups by Lemma 1.3 and the right-hand vertical map is an isomorphism of pro-abelian groups by Lemma 1.4. Finally, the upper horizontal map is an isomorphism by [6, Thm. B]. □

Proposition 1.7 *Suppose that A is regular and finitely generated as a module over the subring A^p. Then the canonical map*

$$\mathrm{TH}_q(R) \to \lim_s \mathrm{TH}_q(R/I^s)$$

is an isomorphism and the derived limit $R^1 \lim_s \mathrm{TH}_q(R/I^s)$ vanishes.

Proof. We consider the following diagram

$$\begin{array}{ccc} \Omega^*_R \otimes \mathrm{TH}_*(\mathbb{F}_p) & \longrightarrow & \mathrm{TH}_*(R) \\ \downarrow & & \downarrow \\ \lim_s \Omega^*_{R/I^s} \otimes \mathrm{TH}_*(\mathbb{F}_p) & \longrightarrow & \lim_s \mathrm{TH}_*(R/I^s) \end{array}$$

where the upper and lower horizontal maps are isomorphisms by [6, Thm. B] and Lemma 1.6. We recall that $\mathrm{TH}_q(\mathbb{F}_p)$ is isomorphic to $\mathbb{F}_p$, if q is a non-negative even integer, and is zero, otherwise. We show that the canonical map

$$\Omega^q_R \to \lim_s \Omega^q_{R/I^s}$$

is an isomorphism and that the derived limit $R^1 \lim_s \Omega^q_{R/I^s}$ vanishes. The canonical map is equal to the composite map

$$\Omega^q_R \to \lim_s (R/I^s \otimes_R \Omega^q_R) \to \lim_s \Omega^q_{R/I^s}.$$

It follows from Lemma 1.3 that the right-hand map and the corresponding map of derived limits are isomorphisms. Hence, it suffices to show that the R-module Ω^q_R is I-adically complete. We recall from [14, Thm. 8.7] that since R is noetherian and I-adically complete, every finitely generated R-module, too, is I-adically complete. So it will suffice to prove that the R-module Ω^q_R is finitely generated. But if x_r^s, $r = 1, \dots, m$, $0 \leqslant s < p$, generate A as an A^p-module, then $dx_1, \dots, dx_m, dt_1, \dots, dt_d$ generates Ω^1_R as an R-module. $\square$

2 De Rham-Witt complexes

In this section, we show that if the $\mathbb{F}_p$-algebra A is finitely generated as a module over the subring A^p, then the map

$$1 - F \colon W\Omega^q_{(R,I)} \to W\Omega^q_{(R,I)}$$

is surjective. We begin with some lemmas on Witt vectors and completion. We note that the definition of the ring $W_n(J)$ of Witt vectors associated with a ring J does not require the ring to be unital.

Lemma 2.1 *Let S be a ring and $J \subset S$ an ideal. Then $W_n(J) \subset W_n(S)$ is an ideal and the canonical projection induces an isomorphism*

$$W_n(S)/W_n(J) \xrightarrow{\sim} W_n(S/J).$$

Proof. It is clear that $W_n(J) \subset W_n(S)$ is an ideal and that the natural projection factors as stated. Indeed, as a set $W_n(A)$ is the n-fold product of copies of A. We show inductively that the sequence

$$0 \to W_n(J) \to W_n(S) \to W_n(S/J) \to 0$$

is exact. The case $n = 1$ is trivial. It is also clear that the left-hand map is injective, that the right-hand map is surjective, and that the composite of the two maps is zero. The proof of the induction step now follows from the natural exact sequence

$$0 \to A \xrightarrow{V^{n-1}} W_n(A) \xrightarrow{R} W_{n-1}(A) \to 0$$

by a diagram chase. □

Lemma 2.2 *Let S be a ring and let $x_1, \dots, x_d \in S$. Then*

$$([x_1]_n^{p^{n-1}ds}, \dots, [x_d]_n^{p^{n-1}ds}) \subset W_n((x_1, \dots, x_d)^{p^{n-1}ds}) \subset ([x_1]_n^s, \dots, [x_d]_n^s),$$

where $[x]_n \in W_n(S)$ is the Teichmüller representative.

Proof. The inclusion on the left is trivial. To prove the right-hand inclusion, we first note that since

$$(x_1, \dots, x_d)^{p^{n-1}ds} \subset (x_1^{p^{n-1}s}, \dots, x_d^{p^{n-1}s}),$$

it will suffice to prove that for all $y_1, \dots, y_d \in S$,

$$W_n((y_1^{p^{n-1}}, \dots, y_d^{p^{n-1}})) \subset ([y_1]_n, \dots, [y_d]_n).$$

To this end, we show by descending induction on $0 \leqslant s \leqslant n$ that

$$V^s W_n((y_1^{p^{n-1}}, \dots, y_d^{p^{n-1}})) \subset ([y_1]_n, \dots, [y_d]_n).$$

The case $s = n$ is trivial. We write $J = (y_1^{p^{n-1}}, \dots, y_d^{p^{n-1}})$ and assume the statement is true for $s + 1$. The elements $V^s([x]_{n-s})$ with $x \in J$ form a set of coset representatives of $V^{s+1}W_n(J)$ as a subgroup of $V^s W_n(J)$. Now if

$$x = a_1 y_1^{p^{n-1}} + \dots a_d y_d^{p^{n-1}}$$

then, modulo $VW_{n-s}(J)$,

$$[x]_{n-s} \equiv [a_1]_{n-s}[y_1]_{n-s}^{p^{n-1}} + \cdots + [a_d]_{n-s}[y_d]_{n-s}^{p^{n-1}}$$

which shows that, modulo $V^{s+1}W_n(J)$,

$$V^s([x]_{n-s}) \equiv V^s([a_1]_{n-s})[y_1]_n^{p^{n-1-s}} + \cdots + V^s([a_d]_{n-s})[y_d]_n^{p^{n-1-s}}.$$

This proves the induction step. □

Corollary 2.3 *Let S be a ring and $J \subset S$ a finitely generated ideal. Then for all $s \geqslant 1$, there exists $r \geqslant s$ such that*

$$W_n(J^r) \subset W_n(J)^s \subset W_n(J^s).$$

Proof. Indeed, if $x_1, \dots, x_d$ generates J, then by Lemma 2.2,

$$W_n(J^{p^{n-1}ds}) \subset ([x_1]_n^s, \dots, [x_d]_n^s) \subset W_n(J)^s,$$

which proves the left-hand inclusion of the statement with $r = p^{n-1}ds$. The right-hand inclusion follows from the more general fact that if $I, J \subset S$ are two ideals, then $W_n(I)W_n(J) \subset W_n(IJ)$. □

The de Rham-Witt complex $W.\Omega^*_S$ is defined, for any ring S, to be the initial example of an algebraic structure called a Witt complex over S [12,11]. By definition a Witt complex over S consists of the following data (i)–(iii).

(i) A pro-differential graded ring $E^*_{\cdot}$ and a strict map of pro-rings

$$\lambda\colon W.(S) \to E^0_{\cdot}.$$

(ii) A strict map of pro-graded rings

$$F\colon E^*_n \to E^*_{n-1}$$

such that $F\lambda = \lambda F$ and such that

$$Fd\lambda([a]_n) = \lambda([a]_{n-1})^{p-1}d\lambda([a]_{n-1}), \quad \text{for all } a \in S.$$

(iii) A strict map of pro-graded modules over the pro-graded ring $E^*_{\cdot}$,

$$V\colon F_*E^*_{n-1} \to E^*_n,$$

such that $V\lambda = \lambda V$ and such that $FV = p$ and $FdV = d$.

A map of Witt functors is a strict map of pro-differential graded rings that commutes with the maps λ, F and V. The structure map in the de Rham-Witt complex $W.\Omega^*_S$ is called the restriction and written R.

Lemma 2.4 *Let S be a ring, let $J \subset S$ be an ideal, and let $W_n\Omega^*_{(S,J)}$ be the differential graded ideal in $W_n\Omega^*_S$ generated by $W_n(J)$. Then the canonical projection induces an isomorphism*

$$W_n\Omega^q_S/W_n\Omega^q_{(S,J)} \xrightarrow{\sim} W_n\Omega^q_{S/J}.$$

Proof. We claim that $W.\Omega^*_S/W.\Omega^*_{(S,J)}$ is a Witt complex over S/J. Indeed, by definition and by Lemma 2.1, it is a pro-differential graded ring with underlying pro-ring $W.(S/J)$. Hence, we need only show that the operators F, R and V on $W.\Omega^*_S$ descend to operators on $W.\Omega^*_S/W.\Omega^*_{(S,J)}$, or equivalently, that

$$\begin{aligned} RW_n\Omega^q_{(S,J)} &\subset W_{n-1}\Omega^q_{(S,J)} \\ FW_n\Omega^q_{(S,J)} &\subset W_{n-1}\Omega^q_{(S,J)} \\ VW_n\Omega^q_{(S,J)} &\subset W_{n+1}\Omega^q_{(S,J)}. \end{aligned}$$

The elements of $W_n\Omega^q_{(S,J)}$ are sums of elements of the form $\omega = a_0 da_1 \dots da_q$ with $a_i \in W_n(S)$, for all i, and $a_i \in W_n(J)$, for some i. We find that

$$V(\omega) = V(a_0\, FdV(a_1) \dots FdV(a_q)) = V(a_0)dV(a_1)\dots dV(a_q),$$

which is therefore in $W_{n+1}\Omega^q_{(S,J)}$. Since the Frobenius is multiplicative,

$$F(\omega) = Fa_0 \cdot Fda_1 \cdot \ldots \cdot Fda_q.$$

If $a_0 \in W_n(J)$ then $F(a_0) \in W_{n-1}(J)$, and hence, $F(\omega) \in W_{n-1}\Omega^q_{(S,J)}$. Suppose that $a_i \in W_n(J)$, for some $1 \leqslant i \leqslant q$. We write out a_i in Witt coordinates

$$a_i = [a_{i,0}]_n + V[a_{i,1}]_{n-1} + \dots + V^{n-1}[a_{i,n-1}]_1$$

and find that

$$Fda_i = [a_{i,0}]^{p-1}_{n-1} d[a_{i,0}]_{n-1} + d[a_{i,1}]_{n-1} + dV[a_{i,2}]_{n-2} + \dots + dV^{n-2}[a_{i,n-1}]_1.$$

This shows that $Fda_i \in W_{n-1}\Omega^1_{(S,J)}$ and hence $F(\omega) \in W_{n-1}\Omega^q_{(S,J)}$. Finally, the statement for R is clear.

It remains to show that the quotient $W.\Omega^*_S/W.\Omega^*_{(S,J)}$ is the universal Witt complex over S/J. Let $E^*_.$ be a Witt complex over S/J. By regarding $E^*_.$ as a Witt complex over S, we have the unique map $W.\Omega^*_S \to E^*_.$ of Witt complexes of S. But this factors through the canonical projection to give a map

$$W.\Omega^*_S/W.\Omega^*_{(S,J)} \to E^*_.$$

of Witt complexes over S/J. Finally, this map is unique because the canonical projection is surjective. This completes the proof. $\square$

Proposition 2.5 *Let S be a ring and let $J \subset S$ be a finitely generated ideal. Then for all $n \geqslant 1$ and $q \geqslant 0$, the natural map*

$$W_n(S/J^{\cdot}) \otimes_{W_n(S)} W_n\Omega^q_S \to W_n\Omega^q_{S/J^{\cdot}}$$

is an isomorphism of pro-abelian groups.

Proof. It follows from Lemma 2.4 that we have a natural exact sequence of limit systems

$$W_n(S/J^{\cdot}) \otimes_{W_n(S)} W_n\Omega^q_{(S,J^{\cdot})} \to W_n(S/J^{\cdot}) \otimes_{W_n(S)} W_n\Omega^q_S \to W_n\Omega^q_{S/J^{\cdot}} \to 0.$$

Hence, it suffices to show that the limit system on the left-hand side is zero as a pro-abelian group. This means that given $s \geqslant 1$, we must find $r \geqslant s$ such that the map

$$W_n(S/J^r) \otimes_{W_n(S)} W_n\Omega^q_{(S,J^r)} \to W_n(S/J^s) \otimes_{W_n(S)} W_n\Omega^q_{(S,J^s)}$$

is zero, or equivalently, such that

$$W_n\Omega^*_{(S,J^r)} \subset W_n(J^s) \otimes_{W_n(S)} W_n\Omega^*_S.$$

To this end, we choose $r \geqslant s$ such that $W_n(J^r) \subset W_n(J^s)^2$. This is possible by Cor. 2.3 since the ideal J and hence also J^s is finitely generated. The desired inclusion follows from the Leibniz rule. $\square$

Lemma 2.6 *Let A be a noetherian $\mathbb{F}_p$-algebra and suppose that A is finitely generated as a module over A^p. Then $W_n(A)$ is a noetherian $\mathbb{Z}/p^n$-algebra.*

Proof. It is enough to show that every prime ideal of $W_n(A)$ is finitely generated [14, Thm. 3.4]. We prove this by induction on n. The case $n = 1$ is by assumption. In the induction step, we use that $V^{n-1}(A) \subset W_n(A)$ is a nilpotent ideal and that the restriction map induces an isomorphism

$$R \colon W_n(A)/V^{n-1}(A) \xrightarrow{\sim} W_{n-1}(A).$$

It follows that every prime ideal $\mathfrak{p} \subset W_n(A)$ is of the form $\mathfrak{p} = R^{-1}(\bar{\mathfrak{p}})$ for some prime ideal $\bar{\mathfrak{p}} \subset W_{n-1}(A)$. Hence, we have a short-exact sequence of $W_n(A)$-modules

$$0 \to V^{n-1}(A) \to \mathfrak{p} \to R_*\bar{\mathfrak{p}} \to 0$$

and it will suffice to show that the right and left-hand modules are finitely generated. By the induction hypothesis, $W_{n-1}(A)$ is noetherian, and hence

$\bar{\mathfrak{p}}$ is a finitely generated over $W_{n-1}(A)$. This implies, since the restriction is surjective, that $R_*\bar{\mathfrak{p}}$ is a finitely generated $W_n(A)$-module. Finally, the iterated Verschiebung gives an isomorphism of $W_n(A)$-modules

$$V^{n-1}\colon F_*^{n-1}A \xrightarrow{\sim} V^{n-1}(A),$$

and the iterated Frobenius factors as the composite map

$$F^{n-1}\colon W_n(A) \xrightarrow{R^{n-1}} A \xrightarrow{\varphi^{n-1}} A$$

where φ is the Frobenius endomorphism of A. The left-hand map is always finite, and the right-hand map is finite by assumption. Hence $V^{n-1}(A)$ is a finitely generated $W_n(A)$-module. □

Lemma 2.7 *Suppose that A is finitely generated as an module over the subring A^p. Then $W_n\Omega^q_R$ is a finitely generated $W_n(R)$-module.*

Proof. The proof is by induction on n. The case $n = 1$ was treated in the proof of Prop. 1.7. In the induction step we use the following exact sequence of $W_n(R)$-module from [10, Prop. 3.2.6]

$${}_hW_n\Omega^q_R \xrightarrow{N} W_n\Omega^q_R \xrightarrow{R} W_{n-1}\Omega^q_R \to 0.$$

The left-hand term, as an abelian group, is the quotient of $\Omega^{q-1}_R \oplus \Omega^q_R$ by the subgroup consisting of the elements $(p^{n-1}\omega, -d\omega)$ with $\omega \in \Omega^{q-1}_R$. We recall from *op. cit.*, Lemma 3.2.5, that it has a natural $W_n(R)$-module structure such that there is an exact sequence of $W_n(R)$-modules

$$F_*^{n-1}\Omega^q_R \to {}_hW_n\Omega^q_R \to F_*^{n-1}(\Omega^{q-1}_R/p^{n-1}\Omega^{q-1}_R) \to 0.$$

Since A/A^p is finitely generated, Ω^q_R is a finitely generated R-module, and $F_*^{n-1}R$ is a finitely generated $W_n(R)$-module. This completes the proof of the induction step. □

Proof of Thm. B Let $R = A[\![t]\!]$, let $S = A[t]$, and let $I \subset R$ and $J \subset S$ be the ideals generated by t. Then in the diagram

$$\begin{array}{ccc}
\lim_s(W_n(S/J^s) \otimes_{W_n(S)} W_n\Omega^q_S) & \longrightarrow & \lim_s(W_n(R/I^s) \otimes_{W_n(R)} W_n\Omega^q_R) \\
\downarrow & & \downarrow \\
\lim_s W_n\Omega^q_{S/J^s} & \longrightarrow & \lim_s W_n\Omega^q_{R/I^s}
\end{array}$$

the vertical maps are isomorphisms by Prop. 2.5 and the lower horizontal map is an isomorphism since $S/J^s \xrightarrow{\sim} R/I^s$. Hence the top horizontal map is an isomorphism. Moreover, we claim that the map

$$W_n\Omega^q_R \to \lim_s (W_n(R/I^s) \otimes_{W_n(R)} W_n\Omega^q_R)$$

is an isomorphism. According to Cor. 2.3 this is equivalent to the statement that the $W_n(R)$-module $W_n\Omega^q_R$ is $W_n(I)$-adically complete. We know from Lemma 2.7 that $W_n\Omega^q_R$ is a finitely generated $W_n(R)$-module, from Lemma 2.6 that $W_n(R)$ is a noetherian ring, and from Cor. 2.3 that the ring $W_n(R)$ is $W_n(I)$-adically complete. The claim then follows from [14, Thm. 8.7]. Concluding, we have isomorphisms

$$W_n\Omega^q_R \xrightarrow{\sim} \lim_s (W_n(R/I^s) \otimes_{W_n(R)} W_n\Omega^q_R) \xleftarrow{\sim} \lim_s (W_n(S/J^s) \otimes_{W_n(S)} W_n\Omega^q_S)$$

and the right-hand term is equal to the completion of the $W_n(S)$-module $W_n\Omega^q_S$ with respect to the topology given by the ideals $W_n(J^s)$, $s \geqslant 0$. We recall from [11, Thm. B] that every element $\omega^{(n)} \in W_n\Omega^q_S$ can be written uniquely

$$\begin{aligned}\omega^{(n)} = & \sum_{i\in\mathbb{N}_0} a^{(n)}_{0,i}[t]^i_n + \sum_{i\in\mathbb{N}} b^{(n)}_{0,i}[t]^{i-1}_n d[t]_n \\ & + \sum_{s\in\mathbb{N}}\sum_{j\in I_p} \left(V^s(a^{(n-s)}_{s,j}[t]^j_{n-s}) + dV^s(b^{(n-s)}_{s,j}[t]^j_{n-s})\right)\end{aligned}$$

where $a^{(m)}_{s,i} \in W_m\Omega^q_A$, $b^{(m)}_{s,i} \in W_m\Omega^{q-1}_A$, and where only finitely many $a^{(r)}_{s,i}$ and $b^{(r)}_{s,i}$ are non-zero. The effect of the completion is to remove the latter requirement. Indeed, according to Cor. 2.3, the ideals $W_n(J^s)$ and $([t]^s_n)$, where $s \geqslant 0$, define the same topology on $W_n\Omega^q_S$. □

Proposition 2.8 *Let A be a noetherian $\mathbb{F}_p$-algebra and suppose that A is a finitely generated A^p-module. Then the map*

$$1 - F \colon W\Omega^q_{(R,I)} \to W\Omega^q_{(R,I)}$$

is surjective.

Proof. We first reduce to the one variable case. Let $R = A[\![t_1, \dots, t_d]\!]$ and let I and J be the ideals generated by $t_1, \dots, t_d$ and by t_d, respectively. Then Lemma 2.4 and the snake lemma give rise to an exact sequence

$$0 \to W\Omega^q_{(R,J)} \to W\Omega^q_{(R,I)} \to W\Omega^q_{(R/J,I/J)} \to 0,$$

and by induction, the endomorphisms $1 - F$ of the right and left-hand terms are surjective. Here, for the left-hand term, we have used that if A is a finitely generated A^p-module, then R is a finitely generated R^p-module.

So suppose that $R = A[\![t]\!]$ and $I = (t)$. Then $W_n\Omega^q_{(R,I)}$ is given by Thm. B, and the value of the Frobenius endomorphism is given by

$$F(w^{(n)}) = \sum_{i\in\mathbb{N}} \Big(Fa^{(n)}_{0,i}[t]^{pi}_{n-1} + Fb^{(n)}_{0,i}[t]^{pi-1}_{n-1}d[t]_{n-1}\Big) + \sum_{s\in\mathbb{N}}\sum_{j\in I_p} \Big(V^{s-1}(pa^{(n-s)}_{s,j}[t]^j_{n-s}) + dV^{s-1}(b^{(n-s)}_{s,j}[t]^j_{n-s})\Big).$$

To see that $1-F$ is onto, let $\omega = (\omega^{(n)})_{n\in\mathbb{N}}$ be an element of

$$W\Omega^q_{(R,I)} = \lim_R W_n\Omega^q_{(R,I)}$$

and write

$$\omega^{(n)} = \sum_{i\in\mathbb{N}} \Big(a^{(n)}_{0,i}[t]^i_n + b^{(n)}_{0,i}[t]^{i-1}_n d[t]_n\Big) + \sum_{s\in\mathbb{N}}\sum_{j\in I_p} \Big(V^s(a^{(n-s)}_{s,j}[t]^j_{n-s}) + dV^s(b^{(n-s)}_{s,j}[t]^j_{n-s})\Big)$$

with $a^{(m)}_{s,i} \in W_m\Omega^d_A$ and $b^{(m)}_{s,i} \in W_m\Omega^{q-1}_A$. We consider the case where $b^{(n)}_{s,j} = 0$, for all $n, s \in \mathbb{N}$ and $j \in I_p$, and the case where $a^{(n)}_{0,i} = b^{(n)}_{0,i} = 0$, for all $n, i \in \mathbb{N}$, and $a^{(n)}_{s,j} = 0$, for all $s, n \in \mathbb{N}$ and $j \in I_p$, separately. In the first case, the geometric series

$$\psi^{(n)} = \sum_{m\geqslant 0} F^m\omega^{(m+n)} \in W_n\Omega^q_{(R,I)}$$

converges and defines an element $\psi = (\psi^{(n)})_{n\in\mathbb{N}}$ with $(1-F)\psi = \omega$. In the second case, we define

$$\varphi^{(n)} = -\sum_{m\in\mathbb{N}}\sum_{s\in\mathbb{N}}\sum_{j\in I_p} dV^{m+s}(b^{(n-m-s)}_{s,j}[t]^j_{n-m-s}).$$

This makes sense since, for each $n \in \mathbb{N}$, the sum over $m \in \mathbb{N}$ is finite. The resulting element $\varphi = (\varphi^{(n)})_{n\in\mathbb{N}}$ satisfies $(1-F)\varphi = \omega$. We note that, formally, we may write $\varphi = \sum_{s\geqslant 1} F^{-s}\omega$. This completes the proof. □

Remark 2.9 It is interesting to use Thm. B to evaluate the kernel of $1-F$. Let $\omega = (\omega^{(n)})_{n\in\mathbb{N}}$ be an element of $W\Omega^q_{(A[\![t]\!],(t))}$. Then

$$F(w^{(n)}) = \sum_{i\in\mathbb{N}} \Big(Fa^{(n)}_{0,i}[t]^{pi}_{n-1} + Fb^{(n)}_{0,i}[t]^{pi-1}_{n-1}d[t]_{n-1}\Big) + \sum_{s\in\mathbb{N}}\sum_{j\in I_p} \Big(V^{s-1}(pa^{(n-s)}_{s,j}[t]^j_{n-s}) + dV^{s-1}(b^{(n-s)}_{s,j}[t]^j_{n-s})\Big),$$

which we compare to

$$\omega^{(n-1)} = \sum_{i\in\mathbb{N}} \Big(a^{(n-1)}_{0,i}[t]^i_{n-1} + b^{(n-1)}_{0,i}[t]^{i-1}_{n-1}d[t]_{n-1}\Big) + \sum_{s\in\mathbb{N}}\sum_{j\in I_p} \Big(V^s(a^{(n-1-s)}_{s,j}[t]^j_{n-1-s}) + dV^s(b^{(n-1-s)}_{s,j}[t]^j_{n-1-s})\Big).$$

We find that for all $n, i \in \mathbb{N}$, $s \in \mathbb{N}_0$ and $j \in I_p$, the coefficients $a_{s,j}^{(m)}$ and $b_{s,j}^{(m)}$ must satisfy the equations

$$\begin{aligned} a_{0,pi}^{(n-1)} &= F a_{0,i}^{(n)}, \quad a_{s-1,j}^{(n-s)} = p a_{s,j}^{(n-s)}, \\ b_{0,pi}^{(n-1)} &= F b_{0,i}^{(n)}, \quad b_{s-1,j}^{(n-s)} = b_{s,j}^{(n-s)}. \end{aligned}$$

It follows that for all $j \in I_p$, there exist unique elements

$$\begin{aligned} a_j &= (a_j^{(n)}) \in \lim_R \operatorname{Hom}(\mathbb{Z}[\tfrac{1}{p}], W_n\Omega_A^q), \\ b_j &= (b_j^{(n)}) \in W\Omega_A^{q-1} = \lim_R W_n\Omega_A^{q-1}, \end{aligned}$$

such that for all $s \in \mathbb{N}_0$ and all $i \in \mathbb{N}$,

$$\begin{aligned} a_{s,j}^{(n)} &= a_j^{(n)}(p^{-s}), \quad a_{0,i}^{(n)} = F^v a_j^{(n+v)}(1), \\ b_{s,j}^{(n)} &= b_j^n, \quad b_{0,i}^{(n)} = F^v b_j^{(n+v)}, \end{aligned}$$

where we write $i = p^v d$ with $j \in I_p$. But $W_n\Omega_A^q$ is p^n-torsion, and hence, the coefficients a_j must all be zero. It follows that the kernel of $1 - F$ is isomorphic to a product indexed by I_p of copies of $W\Omega_A^{q-1}$. This group is canonically isomorphic to the big de Rham Witt group $\mathbf{W}\Omega_A^{q-1}$ introduced in [9]. Hence, we can write our findings as a short exact sequence

$$0 \to \mathbf{W}\Omega_A^{q-1} \to W\Omega^q_{(A[\![t]\!],(t))} \xrightarrow{1-F} W\Omega^q_{(A[\![t]\!],(t))} \to 0$$

which is valid whenever A is a noetherian $\mathbb{F}_p$-algebra that is finitely generated as a module over A^p. This also implies a canonical isomorphism

$$K_q(A[[t]], (t), \mathbb{Z}_p) \xrightarrow{\sim} \mathbf{W}\Omega_A^{q-1}$$

with the relative p-adic K-group on the left.

3 Proof of Theorem A

Let A be a regular $\mathbb{F}_p$-algebra. It follows from [6, Thm. B] and [16] that the canonical map

$$W.\Omega_A^* \to \mathrm{TR}^{\cdot}_*(A;p)$$

is an isomorphism of pro-abelian groups, and hence, there is a natural long-exact sequence

$$\cdots \to \mathrm{TC}_q(A;p) \to W\Omega_A^q \xrightarrow{1-F} W\Omega_A^q \to \mathrm{TC}_{q-1}(A;p) \to \cdots$$

The cyclotomic trace induces a map

$$K_q(A;\mathbb{Z}_p) \to \mathrm{TC}_q(A;p)$$

that is defined to be the composite

$$\pi_q(K(A), \mathbb{Z}_p) \to \pi_q(\mathrm{TC}(A;p), \mathbb{Z}_p) \xleftarrow{\sim} \pi_q(\mathrm{TC}(A;p)).$$

The right-hand map is an isomorphism since the topological cyclic homology spectrum of an $\mathbb{F}_p$-algebra is p-complete. We recall the following result from [5] and [4].

Theorem 3.1 *Let A be a regular local $\mathbb{F}_p$-algebra. Then the composite map*

$$K_q(A, \mathbb{Z}_p) \to \mathrm{TC}_q(A;p) \to W\Omega^q_A$$

is an isomorphism onto the kernel of $1 - F$.

Proof. Suppose first that A is an essentially smooth local $\mathbb{F}_p$-algebra. Then it was proved in [5] that $K_q(A)$ is p-torsion free and that $K_q(A)/p^n$ is generated by symbols. It follows that the composite

$$K_q(A) \to \mathrm{TC}^n_q(A;p) \to \mathrm{TR}^n_q(A;p)$$

factors through the canonical map from $W_n\Omega^q_A$ to $\mathrm{TR}^n_q(A;p)$. Indeed, in the following diagram, the upper horizontal map is an isomorphism.

$$\begin{array}{ccc} (W_n\Omega^1_A)^{\otimes q} & \longrightarrow & \mathrm{TR}^n_1(A;p)^{\otimes q} \\ \downarrow & & \downarrow \\ W_n\Omega^q_A & \longrightarrow & \mathrm{TR}^n_q(A;p) \end{array}$$

It follows further from [5], [1], and [4] that the induced map

$$K_q(A)/p^n \to W_n\Omega^q_A$$

is injective. Moreover, according to [12, I.5.7.4], its image $W_n\Omega^q_{A,\log}$ is related to the kernel K^q_n of

$$R - F \colon W_n\Omega^q_A \to W_{n-1}\Omega^{q-1}_A$$

by

$$W_n\Omega^q_{A,\log} \subset K^q_n \subset W_n\Omega^q_{A,\log} + \mathrm{Fil}^{n-1} W_n\Omega^q_A.$$

The above statements are all stable under filtered colimits, and hence they remain valid for any regular local $\mathbb{F}_p$-algebra. Indeed, by [16] a regular local $\mathbb{F}_p$-algebra is a filtered colimit of essentially smooth local $\mathbb{F}_p$-algebras. The result follows by forming the limit over n. □

Proof of Thm. A Let A be a regular local $\mathbb{F}_p$-algebra which is finitely generated as a module over A^p. We consider the square from the introduction

$$\begin{array}{ccc} K(R,I) & \longrightarrow & \mathrm{TC}(R,I;p) \\ \downarrow & & \downarrow \\ \mathrm{holim}_s\, K(R/I^s,I) & \longrightarrow & \mathrm{holim}_s\, \mathrm{TC}(R/I^s,I;p). \end{array}$$

Since R is regular, the topological cyclic group in the upper right-hand corner is given by the long exact sequence

$$\cdots \to \mathrm{TC}_q(R,I;p) \to W\Omega^q_{(R,I)} \xrightarrow{1-F} W\Omega^q_{(R,I)} \xrightarrow{\partial} \mathrm{TC}_{q-1}(R,I;p) \to \dots$$

and we proved in Prop. 2.8 above that the map $1-F$ is surjective. Hence Thm. 3.1 shows that the top horizontal map in the diagram above becomes a weak equivalence after p-completion. Moreover, Props. 1.2 and 1.7 show that the right-hand vertical map is weak equivalence. Finally, the lower horizontal map becomes a weak equivalence after p-completion by the main theorem of [15]. Hence the left-hand vertical map becomes a weak equivalence after p-completion. This is the statement of Thm. A. □

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