THE S^1-TATE SPECTRUM FOR J

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0. Introduction

0.1. For any compact Lie group G and any G-equivariant spectrum X_G, Greenlees and May have introduced the Tate spectrum $\hat{\mathbb{H}}(G;X)$, the G-fixed set of a G-equivariant spectrum denoted $t(X_G)$ in [6]. In this paper we evaluate the homotopy groups of this construction when G is the circle group and $X_G = J_G$, the S^1-equivariant non-connective image of J-spectrum completed at p. When there is no danger of confusion we will omit the subscribt G and just write J instead of J_G.

Our interest in the Tate spectrum for J comes in part from the connection of this construction with the topological cyclic homology theory $\mathrm{TC}(R,p)$, demonstrated in [4] when the ring R is equal to the integers. While we at the time of writing do not know of any direct interplay between $\hat{\mathbb{H}}(S^1;J)$ and the topological cyclic homology theory we do believe that the calculations below will aid the calculation of $\mathrm{TC}(R,p)$ for $R=\mathbb{Z}[v,v^{-1}]$. In any event the Tate spectrum is an interesting construction in its own right, and $\hat{\mathbb{H}}(S^1;J)$ is a non-trivial example. From this point of view one would like to evaluate $\hat{\mathbb{H}}(G;J)$ for any compact Lie group G. This seems to be a non-trivial task. We remark that $\hat{\mathbb{H}}(G;J)$ is a rational spectrum when G is a finite group, e.g. $\hat{\mathbb{H}}(C_{p^n};J) = H(\mathbb{Q}_p[C_{p^n}],0)\vee H(\mathbb{Q}_p[C_{p^n}],-1)$.

Let K_G be p-completion in the sense of [3, ch.IV] of the G-equivariant non-connected spectrum which represents equivariant K-theory. We choose an integer g which generates $(\mathbb{Z}/p^2)^\times$ and let J_G be the homotopy fiber of $\psi^g-1\colon K_G\to K_G$ where ψ^g is the stable Adams operation. There is a map from the equivariant sphere spectrum S^0_G into J_G which is a kind of an equivariant K-theoretic localization (all fixed sets are K-local). In the rest of this paper p is a fixed odd prime.

We emphasize that K_G denotes the p-completion of the non-connected G-spectrum which represents G-equivariant K-theory. Thus K_G and J_G as well as their underlying non-equivariant spectra are all p-complete.

0.2. The Tate spectrum fits into the norm cofibration, cf. [6], which in our case

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takes the form

$$\Sigma J_{hS^1} \xrightarrow{N} J^{hS^1} \xrightarrow{\Psi} \hat{\mathbb{H}}(S^1; J).$$

Since K_G and hence also J_G are split G-spectra in the sense that the fixed sets contain the non-equivariant J as a direct factor,

$$J_{hS^1} \simeq J \wedge \mathbb{C}\mathrm{P}^\infty_+, \qquad J^{hS^1} \simeq F(\mathbb{C}\mathrm{P}^\infty_+, J)$$

so the norm cofibration takes a particular simple form. The homotopy types of these two spectra are given in terms of K-theory, cf. §1, 2 below, so our calculation thus amounts to evaluating N.

Our main result is the following formula

$$\pi_n \hat{\mathbb{H}}(S^1; J) = \begin{cases} \mathbb{Q}/\mathbb{Z}_{(p)} \oplus \mathcal{S}(\mathbb{Q}_p) & \text{if } n \equiv -1 \pmod{2p-2} \text{ and } n \neq -1 \\ \mathcal{S}(\mathbb{Q}_p) & \text{otherwise when } n \text{ is odd} \\ \mathbb{Z}_p & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \text{ is even.} \end{cases}$$

Here $\mathcal{S}(\mathbb{Q}_p) = \varinjlim(\prod_{s\in S} \mathbb{Q}_p \times \prod_{s\notin S} \mathbb{Z}_p)$, the colimit taken over finite subsets $S \subset \mathbb{N}$.

0.3. The homotopy groups of the p-completion $\hat{\mathbb{H}}(S^1; J)^\wedge_p$ can be evaluated from the formula of [3], recalled in 1.1 below. Let $\mathcal{N}(\mathbb{Z}_p) \subset \prod_{s=0}^\infty \mathbb{Z}_p$ be the sequences which converges to 0 in the p-adic topology, i.e. the profinite completion of $\bigoplus_{s=0}^\infty \mathbb{Z}_p$. Then

$$\pi_n \hat{\mathbb{H}}(S^1; J)^\wedge_p = \begin{cases} \mathbb{Z}_p \oplus \prod \mathbb{Z}_p/\mathcal{N}(\mathbb{Z}_p) & \text{if } n \equiv 1 \pmod{2p-2} \\ \prod \mathbb{Z}_p/\mathcal{N}(\mathbb{Z}_p) & \text{otherwise when } n \text{ is odd} \\ \mathbb{Z}_p & \text{if } n = 0 \pmod{2p-2} \\ 0 & \text{otherwise if } n \text{ is even.} \end{cases}$$

0.4. The term $\prod \mathbb{Z}_p/\mathcal{N}(\mathbb{Z}_p)$ is not surprising in view of the norm cofibration since one has (cf. §1, 2 below)

$$(J_{hS^1})^\wedge_p \simeq J \vee (\bigvee_{s=0}^\infty K)^\wedge_p$$

$$J^{hS^1} \simeq J \vee \prod_{s=0}^\infty \Sigma K.$$

The surprise in 0.3 is the lack of torsion. The homotopy groups indicate that one might have

$$\hat{\mathbb{H}}(S^1; J)^\wedge_p \simeq K^{[1]} \vee \Sigma K^{[1]} \vee (\prod_{r=0}^\infty \Sigma K)/(\bigvee_{r=0}^\infty \Sigma K)^\wedge_p.$$

We have not so far been able to prove this so we leave it as a conjecture (see also §5 below). There is a similar conjecture before p-completion which we leave for the reader to formulate.

0.5. Our main tool for proving 0.2 is a spectral sequence

$$\hat{E}^2_{s,t} = \begin{cases} H^{-s}(BS^1; \pi_t J) & \text{when } s \le 0 \\ H_{s-2}(BS^1; \pi_t J) & \text{when } s \ge 2 \end{cases}$$

which approximates $\pi_* \hat{\mathbb{H}}(S^1; J)$. The spectral sequence has non-zero terms in all quadrants, and in general there are convergence problems in such a situation. In §3 we examine the convergence. The results are based on an unpublished paper by J.M.Boardman, [1]. We show that the spectral sequence above converges, but oddly enough that the corresponding spectral sequence with $\mathbb{F}_p$-coefficients diverges very badly. Nevertheless it is the spectral sequence with $\mathbb{F}_p$-coefficients, along with the norm cofibration which supply the necessary information for determining N_* and for solving the extension problems to obtain the results above. We are much indebted to Peter May for drawing our attension to [1], and for warning us of convergence problems.

1. The homotopy fixed points of J

1.1. We begin with a short review of localization, cf. [2].

A spectrum Z is said to be E-local if $F(V, Z) \simeq *$ whenever $V \wedge E \simeq *$. Homotopy limits of E-local spectra are again E-local. In particular any product of E-local spectra is E-local and if two of the spectra in a cofibration $Z \to V \to W$ are E-local so is the third. If E is a ring spectrum and Z is an E-module spectrum then Z is E-local.

The E-localization functor L_E associates to any spectrum V its E-localization $L_E V$. It comes with a natural transformation $V \to L_E V$ such that $E \wedge V \to E \wedge L_E V$ is an equivalence and such that $F(L_E V, Z) \to F(V, Z)$ is an equivalence whenever Z is E-local. Furthermore if W is an E-local spectrum which satisfies these two conditions then $W \simeq L_E V$. The functor L_E preserves cofibrations and $L_E(F \wedge V) \simeq F \wedge L_E(V)$ for any finite complex F. In particular L_E commutes with finite wedge sums.

Consider first the arithmetic localizations $L_{S^0 R}$ where $S^0 R$ is a Moore spectrum for the ring R. If $R \subset \mathbb{Q}$ is a subring of the rationals then

$$L_{S^0 R} V = V \wedge S^0 R = VR,$$

the spectrum with coefficients in R. The homotopy groups are $\pi_*(VR) = \pi_*(V) \otimes R$. When $R = \mathbb{Z}/p$ we prefer to write $Z/p = ZR$. Then,

$$L_{S^0/p} V = V_p^\wedge = F(S^{-1}\mathbb{Q}/\mathbb{Z}_{(p)}, V)$$

is the p-completion of V in the sense of [3]. The homotopy groups are given by the split short exact sequence

$$0 \to \operatorname{Ext}(\mathbb{Q}/\mathbb{Z}_{(p)}, \pi_n V) \to \pi_n(V_p^\wedge) \to \operatorname{Hom}(\mathbb{Q}/\mathbb{Z}_{(p)}, \pi_{n-1} V) \to 0.$$

For any spectrum E we have the homotopy cartesian square

$$\begin{array}{ccc} L_{E\mathbb{Z}_{(p)}}V & \longrightarrow & L_{E/p}V \\ \downarrow & & \downarrow \\ L_{E\mathbb{Q}}V & \longrightarrow & L_{E\mathbb{Q}}(L_{E/p}V). \end{array}$$

When $E = S^0$ this is the usual arithmetic square for V. One may use this to prove that if either R is torsion or $E\mathbb{Q} \not\simeq *$ then $L_{ER}V = L_{S^0R}(L_E V)$. In particular $L_{E/p}V = (L_E V)^{\wedge}_p$. Also one may see that $L_{E\mathbb{Z}_{(p)}}V = L_{E^{\wedge}_p}V$ provided $(E^{\wedge}_p)\mathbb{Q} \not\simeq *$.

The K-localization of the sphere spectrum is related to the J-spectrum. More precisely we have ([2], [8])

$$J = L_{K/p}S^0 = (L_K S^0)^{\wedge}_p.$$

It has homotopy groups

$$\pi_n J = \begin{cases} \mathbb{Z}_p & \text{if } n = 0, -1 \\ \mathbb{Z}/p^{v_p(n+1)+1} & \text{if } n \equiv -1 \pmod{2p-2} \\ 0 & \text{otherwise} \end{cases}$$

with $v_p(-)$ denoting the p-adic valuation.

Finally we recall that K is 'smashing', i.e. that $L_K V = V \wedge L_K S^0$. This is not the case for K/p, since $L_{K/p}V = (L_K S^0 \wedge V)^{\wedge}_p \neq (L_K S^0)^{\wedge}_p \wedge V$.

1.2. The functor $F(X, -)$ preserves cofibrations so the definition of J in 0.1 gives us a cofibration for computing $F(\mathbb{C}\mathrm{P}^\infty, J)$.

We recall that $\pi_* F(\mathbb{C}\mathrm{P}^\infty_+, K) = \mathbb{Z}[u, u^{-1}] \otimes \mathbb{Z}_p[\![\lambda]\!]$ where u is the Bott class and $\lambda = H - 1$ is the reduced Hopf bundle; $\deg u = -2$ and $\deg \lambda = 0$. The ideal generated by λ corresponds to $\pi_* F(\mathbb{C}\mathrm{P}^\infty, K)$, and ψ^g is the ring homomorphism determined by

$$\psi^g(u) = g^{-1}u, \qquad \psi^g(\lambda) = (1+\lambda)^g - 1.$$

Note that all elements are concentrated in even degrees so that the homotopy groups of $F(\mathbb{C}\mathrm{P}^\infty, J)$ are given by short exact sequences

$$0 \to \pi_{-2i}F(\mathbb{C}\mathrm{P}^\infty, J) \to \tilde{\mathbb{Z}}_p[\![\lambda]\!] \xrightarrow{\psi^g - g^i} \tilde{\mathbb{Z}}_p[\![\lambda]\!] \to \pi_{-2i-1}F(\mathbb{C}\mathrm{P}^\infty, J) \to 0.$$

Here we have written $\tilde{\mathbb{Z}}_p[\![\lambda]\!]$ for the ideal in $\mathbb{Z}_p[\![\lambda]\!]$ generated by λ.

1.3. Modulo p we have $(T-1)^{p^n} = T^{p^n} - 1$ and therefore the power series ring $\mathbb{F}_p[\![\lambda]\!]$ is isomorphic to the inverse limit of group rings

$$\varprojlim_n \mathbb{F}_p[T]/(T^{p^n}-1) = \varprojlim_n \mathbb{F}_p[C_{p^n}],$$

with the isomorphism given by $1+\lambda \mapsto T$. We thank Steffen Bentzen for reminding us that there is the following p-adic analog of this.

Proposition. *There are ring-homomorphisms $\mathbb{Z}_p[\![\lambda]\!] \to \mathbb{Z}_p[C_{p^n}]$ which maps $1+\lambda$ to the generator $T \in C_{p^n}$ and the induced map*

$$\mathbb{Z}_p[\![\lambda]\!] \to \varprojlim_n \mathbb{Z}_p[C_{p^n}]$$

is an isomorphism of rings.

Proof. By the Atiyah-Segal completion theorem $K(BG) = K_G(EG) = RG^\wedge_{IG}$. The map induced by the inclusion of groups

$$\mathbb{Z}[T, T^{-1}] = RS^1 \to RC_{p^n} = \mathbb{Z}[C_{p^n}]$$

maps T to the generator $T \in C_{p^n}$. We may use p-complete theories instead to get a commutative diagram of rings

$$\begin{array}{ccc} \mathbb{Z}_p[T,T^{-1}] & \longrightarrow & \mathbb{Z}_p[C_{p^n}] \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c'} \\ \mathbb{Z}_p[\![T-1]\!] & \longrightarrow & \mathbb{Z}_p[C_{p^n}]^\wedge_{IC_{p^n}} \end{array}$$

where the vertical maps are completion in the augmentation ideals. To compute c' we apply the exact sequence of limit-systems (in m)

$$0 \to (T-1)^m \to \mathbb{Z}_p[T]/(T^{p^n}-1) \to \mathbb{Z}_p[T]/(T^{p^n}-1, (T-1)^m) \to 0.$$

Since all the groups involved are compact we may take the limit and maintain exactness. Thus c' is an isomorphism and we obtain maps $\mathbb{Z}_p[\![\lambda]\!] \to \mathbb{Z}_p[C_{p^n}]$ and consequently

$$\mathbb{Z}_p[\![\lambda]\!] \to \varprojlim_n \mathbb{Z}_p[C_{p^n}].$$

Finally this is an isomorphism since $K(\varinjlim BC_{p^n}) = \varprojlim K(BC_{p^n})$ and because the map

$$\varinjlim_n BC_{p^n} = B\mathbb{Q}/\mathbb{Z}_{(p)} \to BS^1$$

induced by group inclusions becomes a homotopy equivalence upon p-completion. $\square$

Of course the proposition may also be proved by purely algebraic means; cf. [9]. The inverse of the map in the proposition can be defined as follows. Consider $\mathbb{Z}_p[\lambda]$ in the (p, λ)-adic topology. Since $\lambda \in (p, \lambda)$ and since

$$\frac{(1+\lambda)^{p^{n+1}}-1}{(1+\lambda)^{p^n}-1} = (1+\lambda)^{p^n(p-1)} + (1+\lambda)^{p^n(p-2)} + \cdots + (1+\lambda)^{p^n} + 1 \in (p, \lambda)$$

induction gives $(1+\lambda)^{p^n} - 1 \in (p, \lambda)^{n+1}$. Thus the map $\mathbb{Z}_p[T] \to \mathbb{Z}_p[\lambda]$ which sends T to $1+\lambda$ is continuous if we give $\mathbb{Z}_p[T]$ the linear topology defined by the descending chain of ideals $(T-1) \supset (T^p - 1) \supset \cdots \supset (T^{p^n} - 1) \supset \cdots$. The induced map on completions is inverse to the map in the proposition,

$$\varprojlim \mathbb{Z}_p[T]/(T^{p^n} - 1) \to \mathbb{Z}_p[\![\lambda]\!].$$

1.4. Let $C_{p^n} = \langle T \rangle$ denote the cyclic group of order p^n and let $G_n = \operatorname{Aut}(C_{p^n})$ denote its group of automorphisms. We recall that $G_n = \langle f \rangle$ is cyclic of order $\phi(p^n) = p^{n-1}(p-1)$ with the action of the generator f given by $T \cdot f = T^g$.

We may extend the action of G_n to a representation on the group algebra $\mathbb{Z}_p[C_{p^n}]$ by linearity. The decomposition of C_{p^n} into orbits under G_n gives a splitting as G_n-modules

$$\mathbb{Z}_p[C_{p^n}] = \prod_{k=0}^{n} \mathbb{Z}_p[T^{p^k} G_n] = \prod_{k=0}^{n} \mathbb{Z}_p[G_{n-k}].$$

The G_n-module structure on $\mathbb{Z}_p[G_{n-k}]$ is as follows. The projection $C_{p^n} \twoheadrightarrow C_{p^{n-k}}$ induces a group homomorphism $G_n \to G_{n-k}$, the induced map on group algebras turns $\mathbb{Z}_p[G_{n-k}]$ into a G_n-module. In this setup ψ^g corresponds to the action of the generator f.

Lemma. *Let $H = \langle h \rangle$ be a finite cyclic group and $u \in \mathbb{Z}_p^\times$ a unit of infinite order. Then there is a short exact sequence*

$$0 \to \mathbb{Z}_p[H] \xrightarrow{h-u} \mathbb{Z}_p[H] \to \mathbb{Z}_p/(u^{|H|} - 1)\mathbb{Z}_p \to 0$$

and this sequence is natural with respect to projections $H \twoheadrightarrow \bar{H}$.

Proof. Suppose $h - u$ had a kernel. Then this would be an H-module on which H acted by the rule $x \cdot h = ux$. But u has infinite order and H is finite, hence the kernel is zero.

In the cokernel we have the identities $h^s = uh^{s-1}$. Thus the cokernel has only one generator as a $\mathbb{Z}_p$-module and is subject to the relation $u^{|H|} = 1$. □

1.5. We return to the calculation of the homotopy groups of $F(\mathbb{C}\mathrm{P}^\infty, J)$ by the short exact sequences of 1.2.

Proposition. *There is an exact sequence*

$$0 \to \tilde{\mathbb{Z}}_p[\![\lambda]\!] \xrightarrow{\psi^g - g^i} \tilde{\mathbb{Z}}_p[\![\lambda]\!] \xrightarrow{\pi} \prod_{k=0}^{\infty} \mathbb{Z}_p\langle x_k \rangle \to 0$$

Here $x_k = \pi((1+\lambda)^{p^k} - 1)$ *and* $\pi((1+\lambda)^s - 1) = \gamma_s x_{v_p(s)}$, *where* $\gamma_s \in \mathbb{Z}_p^\times$ *is a unit.*

Proof. We only deal with the case $i \neq 0$ leaving the simpler case $i = 0$ to the reader. Then g^i is a unit in $\mathbb{Z}_p$ of infinite order and we can use lemma 1.4. In proposition 1.3 we may restrict ourselves to augmentation ideals,

$$\tilde{\mathbb{Z}}_p[\![\lambda]\!] \cong \varprojlim_n \tilde{\mathbb{Z}}_p[C_{p^n}].$$

We let G_n be as in 1.4 and let $\pi_n \colon G_n \to G_{n-1}$ be the map on automorphism groups induced by the reduction map $\mathrm{red}_n \colon C_{p^n} \twoheadrightarrow C_{p^{n-1}}$. The reduction map also induces a map $\mathbb{Z}_p[\mathrm{red}_n] \colon \mathbb{Z}_p[C_{p^n}] \to \mathbb{Z}_p[C_{p^{n-1}}]$ and this decomposes as

$$\prod_{k=0}^{n} \mathbb{Z}_p[\pi_{n-k}] \colon \prod_{k=0}^{n} \mathbb{Z}_p[G_{n-k}] \to \prod_{k=0}^{n-1} \mathbb{Z}_p[G_{n-1-k}].$$

Since $\lim^{(1)}$ vanishes for compact groups lemma 1.4 gives us an exact sequence

$$0 \to \tilde{\mathbb{Z}}_p[\![\lambda]\!] \xrightarrow{\psi^g - g^i} \tilde{\mathbb{Z}}_p[\![\lambda]\!] \to \varprojlim_n \prod_{k=0}^{n-1} \mathbb{Z}_p/(g^{i\phi(p^{n-1-k})} - 1)\mathbb{Z}_p \to 0.$$

Finally $v_p(g^{i\phi(p^s)} - 1) = s + 1 + v_p(i)$ when $s > 0$, in particular $v_p(g^{i\phi(p^s)} - 1)$ tends to infinity with s. $\square$

It follows that $\pi_n F(\mathbb{C}\mathrm{P}^\infty, J)$ is an infinite product of $\mathbb{Z}_p$'s in odd degrees and zero in even degrees. We end this paragraph with a determination of the homtopy type.

1.6. We define $a_s(x) \in \mathbb{Z}[x]$ recursively by the formulae $a_0(x) = x - 1$, $a_{s+1}(x) = a_s(x^p) - p^s a_s(x)$. If $\psi^p \colon \mathbb{Z}[x] \to \mathbb{Z}[x]$ is the ring homomorphism given by $\psi^p(x) = x^p$ then $a_{s+1}(x) = (\psi^p - p^s)(\psi^p - p^{s-1}) \cdots (\psi^p - 1)a_0(x)$. This shows that $a_s(x)$ belongs to the augmentation ideal in $\mathbb{Z}[x-1]$. Therefore we can define maps $a_s \colon \mathbb{C}\mathrm{P}^\infty \to K$ by the formula $a_s = a_s(H)$ where H is the Hopf bundle. We let $b \colon \Sigma^2 K \to K$ represent the Bott isomorphism $K^2(-) \cong K^0(-)$ and define a map $f_s\colon \Sigma K \to F(\mathbb{C}\mathrm{P}^\infty, J)$ as the adjoint of a desuspension of the composition

$$\mathbb{C}\mathrm{P}^\infty \wedge \Sigma^2 K \xrightarrow{a_s \wedge b} K \wedge K \xrightarrow{\mu} K \xrightarrow{\partial} \Sigma J.$$

We can take the wedge sum of all the maps f_s to get

$$f\colon \bigvee_{s=0}^{\infty} \Sigma K \to F(\mathbb{C}\mathrm{P}^\infty, J).$$

We would like to extend f over the product. We shall need a duality result.

Since $\mathbb{Q}/\mathbb{Z}_{(p)}$ is divisible and hence injective

$$D^n(X) = \mathrm{Hom}(\pi_{n-2}(L_K S^0 \wedge X), \mathbb{Q}/\mathbb{Z}_{(p)}) = \mathrm{Hom}(\pi_{n-2}(L_K X), \mathbb{Q}/\mathbb{Z}_{(p)})$$

is a cohomology theory. Now Hom takes sums in the first variable to products, so D^* is completely additive and is represented by a spectrum D. The composition

$$\begin{aligned}(L_K S^0)^n(X) &= \pi_{-n} F(X, L_K S^0) = \pi_{-n}(L_K X, L_K S^0) = [\Sigma^{-n+2} L_K X, \Sigma^2 L_K S^0] \\ &\xrightarrow{\pi_0} \mathrm{Hom}(\pi_{n-2}(L_K X), \mathbb{Q}/\mathbb{Z}_{(p)}) = D^n(X)\end{aligned}$$

is induced by a map $\delta\colon L_K S^0 \to D$ and comparing homotopy groups we get

Lemma. *There is a cofibration of spectra* $L_K S^0 \xrightarrow{\delta} D \to H(\mathbb{Q}_p/\mathbb{Q}, 0)$. $\square$

We can now prove our main result in this paragraph.

Theorem. *Up to homotopy there exists a unique extension $\hat{f}$ of the map f to the product*

$$\hat{f}\colon \prod_{s=0}^{\infty} \Sigma K \xrightarrow{\simeq} F(\mathbb{C}\mathrm{P}^\infty, J),$$

and this is a homotopy equivalence

Proof. The n'th derivative $a_s^{(n)}(1) = 0$ when $s > n$. Indeed this is true when $s = 0$ and we may differentiate the recursion formula to get

$$a_{s+1}^{(n)}(x) = a_s^{(n)}(x^p) p^n x^{n(p-1)} - p^s a_s^{(n)}(x) + \sum_{k=1}^{n-1} g_k(x) a_s^{(k)}(x^p)$$

which gives the induction step. Thus $(x-1)^s | a_s(x)$ such that $a_s|_{\mathbb{C}\mathrm{P}^n} \simeq 0$ when $s > n$ and consequently the same holds for the composition

$$\Sigma K \xrightarrow{f_s} F(\mathbb{C}\mathrm{P}^\infty, J) \xrightarrow{\iota^*} F(\mathbb{C}\mathrm{P}^n, J).$$

Now as $F(\mathbb{C}\mathrm{P}^\infty, J) = \operatorname{holim}_{\leftarrow} F(\mathbb{C}\mathrm{P}^n, J)$ a choice of null-homotopies gives the desired extension of f. Furthermore the homotopy class of this extension is uniquely determined if and only if

$$\varprojlim_n{}^{(1)}[S^1 \wedge \prod_{s=0}^{\infty} \Sigma K, F(\mathbb{C}\mathrm{P}^n, J)] = 0.$$

We proceed to show that each group in the limit system is zero. We have by Bott periodicity,

$$[S^1 \wedge \prod_{s=0}^{\infty} \Sigma K, F(\mathbb{C}\mathrm{P}^n, J)] = [\mathbb{C}\mathrm{P}^n \wedge \prod_{s=0}^{\infty} K, J] = [\prod_{s=0}^{\infty}(\mathbb{C}\mathrm{P}^n \wedge K), J].$$

where the last equality holds becauce $\mathbb{C}\mathrm{P}^n$ is a finite complex (so has an S-dual). To calculate the J-cohomology we first notice that $Z = \prod(\mathbb{C}\mathrm{P}^n \wedge K)$ is K-local. Indeed $\mathbb{C}\mathrm{P}^n \wedge K$ is K-local and a product of K-local spectra is again K-local. Thus $D^n(Z) = \mathrm{Hom}(\pi_{n-2}(Z), \mathbb{Q}/\mathbb{Z}_{(p)})$. For abelian groups A, B with A torsion free and B divisible, $\mathrm{Hom}(A, B)$ is divisible as we have a short exact sequence

$$0 \to \mathrm{Hom}(A/nA, B) \to \mathrm{Hom}(A, B) \xrightarrow{n} \mathrm{Hom}(A, B) \to 0.$$

Now $\pi_{\mathrm{ev}}(Z)$ is torsion free and $\pi_{\mathrm{od}}(Z) = 0$, so $D^{\mathrm{ev}}(Z)$ is divisible and $D^{\mathrm{od}}(Z) = 0$. The groups $H^*(Z; \mathbb{Q}_p/\mathbb{Q})$ are $\mathbb{Q}$-vectorspaces, and by the lemma $(L_K S^0)^{\mathrm{ev}}(Z)$ is divisible and $(L_K S^0)^{\mathrm{od}}(Z)$ is a $\mathbb{Q}$-vectorspace. We can calculate

$$[Z, J] = \pi_0 F(Z, J) = \pi_0(F(Z, L_K S^0)^\wedge_p)$$

from the short exact sequence (cf. 1.1)

$$0 \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}_{(p)}, (L_K S^0)^0(Z)) \to [Z, J] \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}_{(p)}, (L_K S^0)^1(Z)) \to 0.$$

The Ext-group is zero as a divisible abelian group is injective. The Hom-group vanishes since there are no maps from a torsion group into a torsion free group. We conclude that $[Z, J] = 0$.

In view of proposition 1.5 we only have left to show that $x^{p^s} - 1$ is a linear combination of the polynomials $a_t(x)$, $t \le s$. Since $x - 1 = a_0(x)$ the case $s = 0$ is trivial. Now suppose $x^{p^s} - 1 = \sum_{t=0}^{s} n_t a_t(x)$ then

$$x^{p^{s+1}} - 1 = \sum_{t=0}^{s} n_t a_t(x^p) = \sum_{t=0}^{s} n_t(a_{t+1}(x) + p^t a_t(x))$$

and we are done by induction. $\square$

2. The homotopy orbits of J

2.1. Since J_{S^1} is a split S^1-spectrum the homotopy orbits are

$$J_{hS^1} = J \wedge BS^1_+ \simeq J \vee (J \wedge \mathbb{C}\mathrm{P}^\infty).$$

The 'arithmetic' square of 1.1 gives a cartesian square

$$\begin{array}{ccc} L_K S^0 \wedge \mathbb{C}\mathrm{P}^\infty & \longrightarrow & L_{K/p} S^0 \wedge \mathbb{C}\mathrm{P}^\infty \\ \downarrow & & \downarrow \\ L_K S^0 \mathbb{Q} \wedge \mathbb{C}\mathrm{P}^\infty & \longrightarrow & (L_K S^0)^\wedge_p \mathbb{Q} \wedge \mathbb{C}\mathrm{P}^\infty . \end{array}$$

which relates $J \wedge \mathbb{C}\mathrm{P}^\infty$ to the K-localization of the suspension spectrum $S^0 \wedge \mathbb{C}\mathrm{P}^\infty$. We recall form 1.1 that p-completed and p-localized K-theory gives the same localization functors L_E. The spectrum $L_K S^0 \wedge \mathbb{C}\mathrm{P}^\infty$ is calculated in [8], [7], we give a short review.

2.2. The maps $a_s \colon \mathbb{C}\mathrm{P}^\infty \to K$ of 1.5 induce a map $a \colon \mathbb{C}\mathrm{P}^\infty \to \prod_{s=0}^\infty K$ to the product. Now as $a_s|_{\mathbb{C}\mathrm{P}^n} \simeq 0$ when $s > n$ and $\mathbb{C}\mathrm{P}^\infty = \varinjlim \mathbb{C}\mathrm{P}^n$ this map factors through the inclusion of the sum and we obtain

$$a \colon \mathbb{C}\mathrm{P}^\infty \to \bigvee_{s=0}^\infty K.$$

It is shown in [7] that this map induces an isomorphism on $K_*(-;\mathbb{F}_p)$ and thus that $a \colon S^0 \wedge \mathbb{C}\mathrm{P}^\infty \to (\bigvee K)^\wedge_p$ is a K/p-localization. Alternatively

$$(L_K S^0 \wedge \mathbb{C}\mathrm{P}^\infty)^\wedge_p \xrightarrow{\simeq} (\bigvee_{s=0}^\infty K)^\wedge_p .$$

However as $(L_K S^0 \wedge \mathbb{C}\mathrm{P}^\infty)\mathbb{Q} = H(\mathbb{Q}, 0) \wedge \mathbb{C}\mathrm{P}^\infty$ this is far from a rational equivalence. In order to state the integral result we need a new spectrum. We let bu denote the connected cover of K, i.e. $\mathrm{bu} = K[2,\infty)$. Its K-localization is the homotopy fiber of the composition $K \to K\mathbb{Q} \to K\mathbb{Q}(-\infty, 0]$, cf. [7], [8]. The homotopy groups $\pi_* L_K(\mathrm{bu})$ are $\mathbb{Z}_{(p)}$ in even degrees ≥ 2 and $\mathbb{Q}/\mathbb{Z}_{(p)}$ in odd degrees ≤ -1.

Theorem. ([7], [8]) *The K-localization of the suspension spectrum $\Sigma^\infty \mathbb{C}\mathrm{P}^\infty$ is*

$$L_K S^0 \wedge \mathbb{C}\mathrm{P}^\infty = L_K(\mathrm{bu}) \vee \bigvee_{s=1}^\infty \Sigma^{-1} K\mathbb{Q}/\mathbb{Z}.$$

2.3. In the square 2.1 the rational term is $L_K S^0\mathbb{Q} \wedge \mathbb{C}\mathrm{P}^\infty = \mathrm{bu}\,\mathbb{Q}$ and the left vertical map is projection onto $L_K(\mathrm{bu})$ followed by rationalisation. Thus the mutual cofibers of the vertical maps are $K\mathbb{Q}/\mathbb{Z} \vee \bigvee_{s=1}^{\infty} K\mathbb{Q}/\mathbb{Z}$. The cartesian square gives the following diagram of homotopy groups when $n \geq 1$.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathbb{Z}_{(p)} \arrow[r] \arrow[d] & \pi_{2n} J \wedge \mathbb{C}\mathrm{P}^\infty \arrow[d] \\
\mathbb{Q} \arrow[r, "c"] \arrow[d, "\pi\times 0"'] & \mathbb{Q}_p \arrow[d, "\rho"] \\
\mathbb{Q}/\mathbb{Z}_{(p)} \times \bigoplus_{s=1}^{\infty} \mathbb{Q}/\mathbb{Z}_{(p)} \arrow[r, "\mathrm{id}"] \arrow[d] & \mathbb{Q}_p/\mathbb{Z}_p \times \bigoplus_{s=1}^{\infty} \mathbb{Q}/\mathbb{Z}_{(p)} \arrow[d] \\
\bigoplus_{s=1}^{\infty} \mathbb{Q}/\mathbb{Z}_{(p)} \arrow[r] \arrow[d] & \pi_{2n-1} J \wedge \mathbb{C}\mathrm{P}^\infty \arrow[d] \\
0 & \mathbb{Q}_p \arrow[d] \\
 & 0
\end{tikzcd}$$

We claim that $\rho = \pi \times 0$ where $\pi\colon \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p$ is the canonical projection. Indeed if we restrict ρ to $\mathbb{Q} \subset \mathbb{Q}_p$ this is the case and since c is the completion map and ρ is $\mathbb{Z}_p$-linear the claim follows. Finally since $\bigoplus \mathbb{Q}/\mathbb{Z}_{(p)}$ is divisible there are no extension problems.

Corollary. *The homotopy groups of the smash product $J \wedge \mathbb{C}\mathrm{P}^\infty$ are*

$$\pi_n(J \wedge \mathbb{C}\mathrm{P}^\infty) = \begin{cases} \mathbb{Z}_p & \text{if } n \geq 2 \text{ and even} \\ \mathbb{Q}_p \times \bigoplus_{s=1}^{\infty} \mathbb{Q}/\mathbb{Z}_{(p)} & \text{if } n \geq 1 \text{ and odd} \\ \bigoplus_{s=0}^{\infty} \mathbb{Q}/\mathbb{Z}_{(p)} & \text{if } n \leq -1 \text{ and odd} \\ 0 & \text{else.} \end{cases}$$

2.4. We have determined the homotopy groups of the first two terms in the norm cofibration, cf. 0.2. Thus in order to calculate $\pi_*\hat{\mathbb{H}}(S^1; J)$ we must evaluate N_* and solve extension problems.

For degree reasons only N_0 and N_{2i-1}, $i \geq 1$ may be non-zero. To settle this we use the spectral sequence $\hat{E}^*(S^1; J)$ constructed below and we prove the following result in 4.5.

Proposition. *The norm map induces a map on homotopy groups*

$$N_s \colon \pi_{s-1} J_{hS^1} \to \pi_s J^{hS^1}$$

which is injective if $s = 2i - 1$, $i \geq 1$ and zero otherwise.

The basic extension problem is now the following. Let $n \leq -1$ be an odd integer and suppose $n \not\equiv \pm 1 \pmod{2p-2}$. Then the homotopy group of the Tate spectrum is an extension

$$0 \to \prod_{k=0}^{\infty} \mathbb{Z}_p \to \pi_n \hat{\mathbb{H}}(S^1; J) \to \bigoplus_{k=0}^{\infty} \mathbb{Q}/\mathbb{Z}_{(p)} \to 0.$$

We show in 4.4, using the spectral sequence with $\mathbb{F}_p$-coefficients, $\hat{E}^*(S^1; J/p)$ that $\pi_n \hat{\mathbb{H}}(S^1; J)$ is torsion free, and in 5.2 that this in turn determines the group completely.

3. The Tate spectral sequence

3.1. Let G be a compact Lie group of dimension d and V_G a G-spectrum. The Tate spectral sequence $\hat{E}^*(G; V)$ is a whole plane spectral sequence which one might expect to approximates the homotopy groups of $\hat{\mathbb{H}}(G; V)$, cf. [4], [6]. In this paragraph we study convergence properties.

Let us recall how the spectral sequence is constructed. The G-space $\tilde{E}G$ is defined by the cofibration

$$EG_+ \to S^0 \to \tilde{E}G.$$

It is a G-CW-complex whose k-skeleton is $\tilde{E}_k G = \mathrm{cof}(E_{k-1}G_+ \to S^0)$. The spectral sequence comes from Greenlees 'filtration'

$$\cdots \longrightarrow \mathrm{F}_{-2} \longrightarrow \mathrm{F}_{-1} \longrightarrow \mathrm{F}_0 \longrightarrow \mathrm{F}_1 \longrightarrow \mathrm{F}_2 \longrightarrow \cdots$$

of the suspension spectrum of $\tilde{E}G$. Here F_s is the suspension spectrum of $\tilde{E}_{s-d}G$ when $s \geq d$, the functional dual of $\tilde{E}_{-s}G$ when $s \leq 0$ and S^0 when $0 \leq s \leq d$. The exact couple associated with the spectral sequence is

$$D_{s,t} = \pi_{s+t}([\mathrm{F}_s \wedge F(EG_+, V)]^G)$$
$$E_{s,t} = \pi_{s+t}([\mathrm{F}_s / \mathrm{F}_{s-1} \wedge F(EG_+, V)]^G).$$

One expects that the spectral sequence approximate the colimit $D_n = \varinjlim D_{s,n-s}$ which is equal to $\pi_n \hat{\mathbb{H}}(G;V)$. It has an induced filtration

$$\mathrm{F}_s\, D_n = \mathrm{im}(D_{s,n-s} \to D_n).$$

We recall some notions of convergence from [1].

Definition. We say that the spectral sequence

(1) *converges to D_n conditionally* if $\varprojlim_s D_{s,n-s} = \varprojlim_s{}^{(1)} D_{s,n-s} = 0$.

(2) *converges to D_n weakly* if $\hat{E}^{\infty}_{s,n-s} = \mathrm{F}_s\, D_n / \mathrm{F}_{s-1}\, D_n$.

(3) *converges to D_n strongly* if it converges to D_n weakly and in addition

$$\varprojlim_s F_s D_n = \varprojlim_s{}^{(1)} F_s D_n = 0.$$

A word of explanation is in place. Condition (1) is rather weak but has the effect that the question of strong convergence can be settled by studying the internal structure of the spectral sequence. If for example the lenght of the non-zero differentials is bounded conditional convergence implies stong convergence. To understand strong convergence note that a filtered group can be viewed as a topological group by taking the stages of the filtration as a base for the neighborhoods of zero. The vanishing of the two groups in (3) is then equivalent to the statements that this topology is Hausdorff and complete respectively. Here complete means that Cauchy sequences converge. Let us finally note that if the topology is complete but not Hausdorff, then the spectral sequence converges strongly to

$$\bar{D}_n = D_n / \varprojlim \mathrm{F}_s\, D_n.$$

3.2. Our first convergence result for the Tate spectral sequence imposes no further conditions on the G-spectrum V.

Lemma. *The spectral sequence $\hat{E}^*(G;V)$ is conditionally convergent to $\pi_* \hat{\mathbb{H}}(G;V)$.*

Proof. We have a diagram of G-spaces in which the rows are cofibrations

$$\begin{tikzcd}
(E_{k-1}G \times EG)_+ \arrow[r] & EG_+ \arrow[r] & \tilde{E}_k G \wedge EG_+ \\
E_{k-1}G_+ \arrow[r] \arrow[u, "\Delta"', "\simeq_G"] & EG_+ \arrow[r] \arrow[u, equal] & EG/E_{k-1}G. \arrow[u]
\end{tikzcd}$$

It shows that $D_{s,n-s} = V_G^{-n}(EG, E_{-s-1})$ when $s \leq 0$. Thus the lemma follows from Milnors short exact sequence

$$0 \to \varprojlim_s{}^{(1)} D_{s,n+1-s} \to V_G^{-n}(EG, EG) \to \varprojlim_s D_{s,n-s} \to 0. \qquad \square$$

3.3. In the present general setting the norm cofibration has the form

$$\Sigma^{\mathrm{Ad}(G)} V_{hG} \xrightarrow{N} V^{hG} \xrightarrow{\Psi} \hat{\mathbb{H}}(G; V).$$

It induces a long exact sequence in homotopy groups which we compare to the long exact cohomology sequence of the pair $(EG, E_k G)$

$$\cdots \xrightarrow{\partial} \pi_{n-d} V_{hG} \xrightarrow{N_*} \pi_n V^{hG} \xrightarrow{\Psi_*} \pi_n \hat{\mathbb{H}}(G; V) \xrightarrow{\partial} \cdots$$
$$\cdots \xrightarrow{\delta} V_G^{-n}(EG, E_k G) \xrightarrow{j^*} V_G^{-n}(EG) \xrightarrow{i^*} V_G^{-n}(E_k G) \xrightarrow{\delta} \cdots .$$

They have the term $\pi_n(V^{hG}) = V_G^{-n}(EG)$ in common, and we can define subgroups

$$\mathcal{O}_k = \mathrm{im}(j^*\colon V_G^{-n}(EG, E_k G) \to V_G^{-n}(EG)) \subset \pi_n V^{hG}.$$

Taking these as a base for the neighborhoods of zero, $\pi_n(V^{hG})$ becomes a topological group.

Lemma. *Suppose V is a p-complete spectrum with $\pi_n(V)$ finitely generated as a $\mathbb{Z}_p$-module for all n. Then the group $\pi_n(V^{hG})$ is complete Hausdorff in the $\mathcal{O}$-topology.*

Proof. The statement is equivalent to the vanishing og $\varprojlim \mathcal{O}_k$ and $\varprojlim^{(1)} \mathcal{O}_k$. We let N_k be the kernel of $j^*\colon V_G^{-n}(EG, E_k G) \to V_G^{-n}(EG)$ and obtain a six term exact sequence

$$0 \to \varprojlim_k N_k \to \varprojlim_k V_G^{-n}(EG, E_k G) \to \varprojlim_k \mathcal{O}_k$$
$$\to \varprojlim_k{}^{(1)} N_k \to \varprojlim_k{}^{(1)} V_G^{-n}(EG, E_k G) \to \varprojlim_k{}^{(1)} \mathcal{O}_k \to 0.$$

By lemma 3.2 it suffices to show that $\varprojlim^{(1)} N_k = 0$. To this end note that N_k is also the image of $\delta\colon V_G^{-n-1}(E_k G) \to V_G^{-n}(EG, E_k G)$ so is compact in its p-adic topology as $E_k G$ is a finite free CW-complex. Finally $\varprojlim^{(1)}$ vanishes for compact groups. $\square$

Proposition. *Let V be a p-complete spectrum with $\pi_n(V)$ finitely generated as a $\mathbb{Z}_p$-module for all n. Then the spectral sequence $\hat{E}^*(G; V)$ converges strongly to the cokernel of Ψ_* in the exact sequence*

$$0 \longrightarrow \mathrm{im}\, N_* \longrightarrow \overline{\mathrm{im}\, N_*} \xrightarrow{\Psi_*} \pi_n \hat{\mathbb{H}}(G; V).$$

Here $\overline{\mathrm{im}\, N_*}$ denotes the closure of $\mathrm{im}\, N_*$ in the $\mathcal{O}$-topology. In particular the spectral sequence converges strongly to $\pi_n(\hat{\mathbb{H}}(G; V))$ if $\mathrm{im}\, N_*$ is closed.

Proof. Recall from 3.1 that $D_n = \pi_n(\hat{\mathbb{H}}(G;V))$ is filtered by

$$\mathrm{F}_s\, D_n = \operatorname{im}(D_{s,n-s} \to D_n) = \operatorname{im}(\mathcal{O}_{-s-1} \xrightarrow{\Psi_*} D_n).$$

It follows from (the proof of) [1; 10.1] that the spectral sequence converges weakly to D_n. Indeed the criterion is that the group $RE^{\infty}_{s,t}$ defined as $\varprojlim^{(1)}$ of a certain system of $\mathbb{Z}_p$-submodules of $E_{s,t}$ vanishes. In our case these submodules are compact, so the criterion is satisfied.

We next claim that the filtration topology on D_n is complete, i.e. $\varprojlim^{(1)} \mathrm{F}_s\, D_n = 0$. To see this notice that

$$\mathrm{F}_{-k-1}\, D_n = \Psi_*(\mathcal{O}_k) \cong \mathcal{O}_k/(\mathcal{O}_k \cap \operatorname{im} N_*) \cong (\mathcal{O}_k + \operatorname{im} N_*)/\operatorname{im} N_*$$

so that

$$0 \to \operatorname{im} N_* \to \mathcal{O}_k + \operatorname{im} N_* \to \mathrm{F}_{-k-1}\, D_n \to 0$$

is exact. Now $\varprojlim(\mathcal{O}_k + \operatorname{im} N_*)$ is precisely the closure of $\operatorname{im} N_*$ in the $\mathcal{O}$-topology on $\pi_n(V^{hG})$, and the six term exaxt sequence used in the above lemma reduces to

$$0 \to \operatorname{im} N_* \to \overline{\operatorname{im} N_*} \to \varprojlim_s \mathrm{F}_s\, D_n \to 0$$

since $\operatorname{im} N_*$ does not depend on s. Thus we are done by the discussion following definition 3.1. □

Corollary. *The Tate spectral sequence $\hat{E}^*(S^1;J)$ converges strongly to the homotopy groups $\pi_*\hat{\mathbb{H}}(S^1;J)$.*

Proof. Our calculations in §1 and §2 shows that for dimension reasons $\operatorname{im} N_*$ is always compact. But then $\overline{\operatorname{im} N_*} = \operatorname{im} N_*$ as the $\mathcal{O}$-topology is Hausdorff. □

3.4. We conclude this paragraph with a list of the E^2-term of the Tate spectral sequence for various groups G. The cases are

(1) G finite, $\hat{E}^2_{s,t}(G;V) = \hat{H}^{-s}(G;\pi_t V)$ - the Tate cohomology of G with coefficients in the G-module $\pi_t V$.

(2) $G = S^1$ the circle group, $\hat{E}^2(S^1;V) = \mathbb{Z}[t,t^{-1}] \otimes \pi_*(V)$, and $t \in V^2_G(S^3, S^1)$ has bidegree $(-2,0)$.

(3) G of positive dimension and connected

$$\hat{E}^2_{s,t}(G;V) = \begin{cases} H^{-s}(BG;\pi_t V) & , s \le 0 \\ H_{s-d-1}(BG;\pi_t V) & , s \ge d+1 \\ 0 & , 1 \le s \le d. \end{cases}$$

If V_G is a ring G-spectrum and G is either finite or S^1 the Tate spectral sequence is multiplicative in the sense that $\hat{E}^r(G;V)$ is a bigraded algebra and $\hat{d}^r$ a derivation for the multiplication. For such V the formulas (1) and (2) are algebra isomorphism.

4. The Tate spectral sequence for J/p

4.1. We write $X/p = X \wedge S^0/p$ for the spectrum X reduced mod p. Since the Moore spectrum S^0/p is finite all the functors in the norm cofibration commute with reduction mod p. In particular $\pi_*(\hat{\mathbb{H}}(S^1; J); \mathbb{F}_p) = \pi_*\hat{\mathbb{H}}(S^1; J/p)$.

The E^2-term of the mod p Tate spectral sequence is

$$\hat{E}^2(S^1; J/p) = P[t, t^{-1}] \otimes E\{a\} \otimes P[v_1, v_1^{-1}]$$

where the bidegrees are $\deg t = (-2, 0)$, $\deg a = (0, 2p-3)$ and $\deg v_1 = (0, 2p-2)$. There are also spectral sequences approximating $\pi_*(J^{hS^1}; \mathbb{F}_p)$ and $\pi_*(J_{hS^1}; \mathbb{F}_p)$. They are left and right half plane spectral sequences respectively and have E^2-terms

$$\begin{aligned} E^2_{s,t}(J^{hS^1}; \mathbb{F}_p) &= \hat{E}^2_{s,t} = P[t] \otimes E\{a\} \otimes P[v_1, v_1^{-1}] && , s \leq 0 \\ E^2_{s,t}(J_{hS^1}; \mathbb{F}_p) &= \hat{E}^2_{s+2,t} = \sigma^{-2} IP[t^{-1}] \otimes E\{a\} \otimes P[v_1, v_1^{-1}] && , s \geq 0. \end{aligned}$$

Here σ is the shift operator $(\sigma M)_{s,t} = M_{s-1,t}$ and $IP[x] \subset P[x]$ is the augmentation ideal. They are both strongly convergent, cf. 3.1.

There is a fruitful interplay between these spectral sequences and the Tate spectral sequence. We sketch the interplay here and refer the reader to [4] where it is treated thoroughly. For any S^1-spectrum X there is a map of spectral sequences

$$\Psi^r : E^r_{s,t}(X^{hS^1}) \twoheadrightarrow \hat{E}^r_{s,t}(S^1; X)$$

which is iso for $r = 2$ and $s \leq 0$ and epi for $r > 2$ and $s \leq 0$. This implies that $x \in E^2_{s,t}(X^{hS^1})$ survives to E^∞ precisely when one of two things happens. Either $\Psi^2 x$ survive to $\hat{E}^\infty$ or $\Psi^r x = \hat{d}^r y$ for some y in the right half plane, i.e. $s + r > 0$. Similarly there is a map of degree -2

$$\partial^r : \hat{E}^r_{s+2,t}(S^1; X) \hookrightarrow E^r_{s,t}(X_{hS^1})$$

which is iso for $r = 2$ and $s \geq 0$ and mono for $r > 2$ and $s \geq 0$. Let $x \in \hat{E}^2_{s,t}(S^1; X)$. Then $\partial^2 x$ survives to infinity if and only if x does so or $\hat{d}^r x \neq 0$ and $s - r \leq 0$. We note that what we said above fit in well with the exactness of

$$\pi_n X^{hS^1} \xrightarrow{\Psi_*} \pi_n \hat{\mathbb{H}}(S^1; X) \xrightarrow{\partial_*} \pi_{n-2} X_{hS^1}.$$

From this point of view the norm map N_* should correspond to those differentials in the Tate spectral sequence which cross the line $s = 0$. Proposition 4.3 below is an instance of such a relation. Our calculations in this paragraph for $X = J/p$ are based on the following result.

Theorem. ([4]) *The non-trivial differentials in $\hat{E}^*(S^1; J/p)$ are multiplicatively induced from*

$$d^{2(p^{k+1}-1)} t^{p^k} = \gamma_k \, t^{p^{k+1}+p^k-1} a \, {v_1}^{p(\frac{p^k-1}{p-1})}.$$

Here $\gamma_k \in \mathbb{F}_p^\times$ is a unit. $\square$

4.2. Before we can write down the E^∞-terms we need some notation. We write $r \in \mathbb{Z}$ as

$$r = r_1(p-1) + r_2; \ 1 \leq r_2 \leq p-1$$

Next we define periodicy elements

$$\pi = (t^{p-1} v_1)^{-r_2}, \qquad \Pi = (t^{p-1} v_1)^{p-r_2}.$$

of total degree zero, and elements $w_r(k)$ and $z_r(k)$ for $k \geq 0$ by

$$\begin{aligned} w_r(0) &= t^{-r_2} {v_1}^{r_1}; \qquad & w_r(k) &= w_r(k-1) \cdot \pi^{p^{k-1}} = w_r(0) \cdot \pi^{\frac{p^k-1}{p-1}} \\ z_r(0) &= t^{p-1-r_2} a \, {v_1}^{r_1}; \qquad & z_r(k) &= z_r(k-1) \cdot \Pi^{p^{k-1}} = z_r(0) \cdot \Pi^{\frac{p^k-1}{p-1}}. \end{aligned}$$

They have total degrees $2r$ and $2r-1$ respectively.

Lemma. *The E^∞-terms of the mod p spectral sequences are*

$$\begin{aligned} &\hat{E}^\infty(S^1; J/p) = E\{t^{-1} a\} \otimes P[v_1, v_1^{-1}] \\ &E^\infty(J^{hS^1}; \mathbb{F}_p) = \mathbb{F}_p \langle z_r(k) | k \in \mathbb{N}, \ r \in \mathbb{Z} \rangle \oplus P[v_1, v_1^{-1}] \\ &E^\infty(J_{hS^1}; \mathbb{F}_p) = \sigma^{-2} \mathbb{F}_p \langle w_r(k) | k \in \mathbb{N}, \ r \in \mathbb{Z} \rangle \oplus \sigma^{-2} \mathbb{F}_p \langle t^{-1} a \, {v_1}^j | j \in \mathbb{Z} \rangle, \end{aligned}$$

where σ^{-2} indicates that degrees are shifted down by 2. Furthermore the differential of the Tate spectral sequence takes $w_r(k)$ to $\gamma_r(k) z_r(k)$ with $\gamma_r(k) \in \mathbb{F}_p^\times$ a unit.

Proof. We first derive $\hat{E}^\infty(S^1; J/p)$. If we write $n = m \, p^k$ with $(m, p) = 1$ the first non-zero differential on t^n gives

$$d^* t^n = t^{(m-1)p^k} d^* t^{p^k} = t^{p^{k+1}+mp^k-1} a \, {v_1}^{p(\frac{p^k-1}{p-1})}.$$

Here we have neglected to list the units $\gamma \in \mathbb{F}_p^\times$ and we shall do so throughout the proof. Now for $l \in \mathbb{N}$ to be of the form

$$l = p^{k+1} + mp^k - 1$$

we must have

$$k = v_p(l+1), \qquad m = \frac{l+1}{p^k} - p.$$

It follows that all powers of t are hit by exactly one differential save $l = -1$. This gives the claim for $\hat{E}^\infty(S^1; J/p)$

To evaluate $E^\infty(J_{hS^1}, \mathbb{F}_p)$ we seek $x \in \hat{E}^2_{s,t}(S^1; J/p)$, $s \geq 2$ such that $\hat{d}^r x \neq 0$ for some $r \geq s$, cf. 4.1. So let $x = t^n {v_1}^i$ with $n < 0$ and suppose $\hat{d}^r x = t^l a\, {v_1}^j$ with $l \geq 0$ for some r. We write $n = -mp^k$ with $(m, p) = 1$ and have from above

$$l = p^{k+1} - mp^k - 1.$$

Thus $l \geq 0$ if and only if $m \leq p-1$ and since also $m \geq 1$ (because $n < 0$) we obtain $x = t^{-mp^k} {v_1}^j$ with $1 \leq m \leq p-1$, $k \in \mathbb{N}$ and $j \in \mathbb{Z}$. Ordering these elements after total degree we get the elements $w_r(k)$ defined above.

We can similarly evaluate $E^\infty(J^{hS^1}; \mathbb{F}_p)$. Indeed the generators are exacly the $t^l a\, {v_1}^{j'} = d^r t^n {v_1}^j$ we have just found. Again ordering after total degree gives us the elements $z_r(k)$. $\square$

We shall see in 4.8 that $\hat{E}^\infty(S^1; J/p)$ is very far from $\pi_*(\hat{\mathbb{H}}(S^1; J); \mathbb{F}_p)$, so the mod p Tate spectral sequence diverges.

4.3. We next evaluate the norm map. We begin with a lemma.

Lemma. *Suppose that in the commutive diagram*

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "N'"', "\cong"] & A \arrow[r, "\pi"] \arrow[d, "N"'] & A'' \arrow[r] & 0 \\
0 & B' \arrow[l] & B \arrow[l] & B'' \arrow[l, "\iota"'] & 0 \arrow[l]
\end{tikzcd}$$

the rows are exact and the map N' is an isomorphism. Then the additive relation

$$N'' \colon A'' \xleftarrow{\pi} A \xrightarrow{N} B \xleftarrow{\iota} B''$$

is a homomorphism. $\square$

Proposition. *There are elements $\bar{w}_r(k)$ and $\bar{z}_r(k)$ which reduce to $\sigma^{-2} w_r(k)$ and $z_r(k)$ respectively such that*

$$\pi_{2s-2}(J_{hS^1}; \mathbb{F}_p) = \bigoplus_{k=0}^{\infty} \mathbb{F}_p \langle \bar{w}_r(k) \rangle, \qquad \pi_{2s-1}(J^{hS^1}; \mathbb{F}_p) = \prod_{k=0}^{\infty} \mathbb{F}_p \langle \bar{z}_r(k) \rangle.$$

Furthermore these elements may be chosen such that $N_*(\bar{w}_r(k)) = \gamma_r(k) \bar{z}_r(k)$ where $\gamma_r(k) \in \mathbb{F}_p^\times$ as in 4.5

Proof. We write $A = \pi_{2s-2}(J_{hS^1}; \mathbb{F}_p)$ and $B = \pi_{2s-1}(J^{hS^1}; \mathbb{F}_p)$. For notational convenience we pick subfiltrations of the filtration associated with the spectral sequence such that

$$\mathrm{F}_k A/\mathrm{F}_{k-1} A = \mathbb{F}_p\langle \sigma^{-2} w_r(k)\rangle, \qquad \mathrm{F}^k B/\mathrm{F}^{k+1} B = \mathbb{F}_p\langle z_r(k)\rangle.$$

We shall prove the proporsition by induction. So assume that we have chosen elements $\bar{w}_r(k)$ for all k and $\bar{z}_r(k)$ for $k < n$ which represents $\sigma^{-2}w_r(k)$ and $z_r(k)$ such that $N\bar{w}_r(k) = \gamma_r(k)\bar{z}_r(k)$ when $k < n$ and such that $N\bar{w}_r(k) \in \mathrm{F}^n B$ when $k \geq n$. Repeated use of the lemma gives us a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{F}_n A/\mathrm{F}_{n-1} A \arrow[r] \arrow[d, "\cong"'] & A/\mathrm{F}_{n-1} A \arrow[r] \arrow[d, "N^{(n)}"'] & A/\mathrm{F}_n A \arrow[r] & 0 \\
0 & \mathrm{F}^n B/\mathrm{F}^{n+1} B \arrow[l] & \mathrm{F}^n B \arrow[l] & \mathrm{F}^{n+1} B \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

We may define $\bar{z}_r(k)$ by the equation $N^{(n)}\sigma^{-2}w_r(k) = \gamma_r(k)\bar{z}_r(k)$ and can change our choices of $\bar{w}_r(k)$ for $k > n$ by adding an appropriate multiple of $\bar{w}_r(n)$ so that we obtain $N\bar{w}_r(k) \in \mathrm{F}^{n+1} B$ when $k > n$. Now the proposition follows by induction with $n = 0$ as a trivial case. $\square$

4.4. In the Tate spectral sequence $\hat{E}^*(S^1; K)$ all elements are concentrated in even degrees. Thus the spectral sequence collapses and by proposition 3.3 it converges strongly to $\pi_*\hat{\mathbb{H}}(S^1; K)$. It follows that the odd degree homotopy groups of K_{hS^1}, K^{hS^1} and $\hat{\mathbb{H}}(S^1, K)$ all are zero.

We may apply the snake lemma to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_{2n}K^{hS^1} \arrow[r] \arrow[d, "(\psi^g)^{hS^1}-1"'] & \pi_{2n}\hat{\mathbb{H}}(S^1; K) \arrow[r] \arrow[d, "\hat{\mathbb{H}}(S^1;\psi^g)-1"'] & \pi_{2n-2}K_{hS^1} \arrow[r] \arrow[d, "(\psi^g)_{hS^1}-1"'] & 0 \\
0 \arrow[r] & \pi_{2n}K^{hS^1} \arrow[r] & \pi_{2n}\hat{\mathbb{H}}(S^1; K) \arrow[r] & \pi_{2n-2}K_{hS^1} \arrow[r] & 0
\end{tikzcd}$$

to obtain a six term exact sequence

$$\begin{aligned}
0 \longrightarrow \pi_{2n}J^{hS^1} &\longrightarrow \pi_{2n}\hat{\mathbb{H}}(S^1; J) \longrightarrow \pi_{2n-2}J_{hS^1} \\
&\xrightarrow{N_*} \pi_{2n-1}J^{hS^1} \longrightarrow \pi_{2n-1}\hat{\mathbb{H}}(S^1; J) \longrightarrow \pi_{2n-3}J_{hS^1} \to 0.
\end{aligned}$$

These statements are true with any coefficients; we take $\pi_* = \pi_*(-; \mathbb{F}_p)$. Since the $\mathcal{O}$-topology on $\pi_*(J^{hS^1}; \mathbb{F}_p)$ is Hausdorff so that $\{0\} \subset \pi_{2n}(J^{hS^1}; \mathbb{F}_p)$ is a closed subset proposition 3.3 shows that in even degrees the Tate spectral sequence

$\hat{E}^*(S^1; J/p)$ converges strongly to $\pi_*(\hat{\mathbb{H}}(S^1; J); \mathbb{F}_p)$. The calculations of the E^∞-terms in 4.2 therefore show that $\Psi_*\colon \pi_*(J^{hS^1}; \mathbb{F}_p) \to \pi_*(\hat{\mathbb{H}}(S^1; J); \mathbb{F}_p)$ is an isomorphism in even degrees. For the odd degree homotopy groups we get an exact sequence

$$0 \longrightarrow \pi_{2n-2}(J_{hS^1}; \mathbb{F}_p) \xrightarrow{N_*} \pi_{2n-1}(J^{hS^1}; \mathbb{F}_p) \xrightarrow{\Psi_*} \pi_{2n-1}(\hat{\mathbb{H}}(S^1; J); \mathbb{F}_p)$$
$$\xrightarrow{\partial_*} \pi_{2n-3}(J_{hS^1}; \mathbb{F}_p) \longrightarrow 0.$$

From proposition 4.3 we get

Theorem. *The mod p homotopy groups of the S^1-Tate spectrum for J are*

$$\pi_n(\hat{\mathbb{H}}(S^1; J); \mathbb{F}_p) = \begin{cases} \mathbb{F}_p\langle x_j\rangle & , n = 2j(p-1) \\ \mathbb{F}_p\langle y_j\rangle \times (\prod_{k=0}^{\infty} \mathbb{F}_p / \bigoplus_{k=0}^{\infty} \mathbb{F}_p) & , n = 2j(p-1)+1 \\ \prod_{k=0}^{\infty} \mathbb{F}_p / \bigoplus_{k=0}^{\infty} \mathbb{F}_p & , n \text{ odd, } n \not\equiv 1 \ (2p-2) \\ 0 & , \text{else.} \end{cases}$$

Here x_j and y_j represents ${v_1}^j$ and $t^{-1}a{v_1}^{j-1}$ in $\hat{E}^\infty(S^1; J/p)$ respectively.

If we compare this result with lemma 4.2 we see that the spectral sequence misses the term $\prod \mathbb{F}_p / \bigoplus \mathbb{F}_p$ in the odd degree homotopy groups. Thus in contrast to the convergent integral spectral sequence, cf. 3.3 the mod p spectral sequence diverges very badly. We note that this is in accord with proposition 3.3. Indeed the $\mathcal{O}$-topology on $\pi_{2r-1}(J^{hS^1}; \mathbb{F}_p)$ is given by the descending sequence of submodules

$$\prod_{k=0}^{\infty} \mathbb{F}_p\langle \bar{z}_r(k)\rangle \supset \prod_{k=1}^{\infty} \mathbb{F}_p\langle \bar{z}_r(k)\rangle \supset \prod_{k=2}^{\infty} \mathbb{F}_p\langle \bar{z}_r(k)\rangle \supset \cdots$$

so the sum $\bigoplus_{k=0}^{\infty} \mathbb{F}_p\langle \bar{z}_r(k)\rangle$ is a dense subset.

4.5. The rest of this paragraph is devoted to the integral spectral sequence $\hat{E}^*(S^1; J)$ and to the proof of proposition 2.4.

Lemma. *For every $s \neq 0$ there are infinitely many non-zero differentials of the form*

$$\hat{d}^r\colon \hat{E}^r_{2s,0}(S^1; J) \to \hat{E}^r_{2s-r,r-1}(S^1; J)$$

and these are the only non-zero differentials in $\hat{E}^*(S^1; J)$. Furthermore the only elements of even total degree in $\hat{E}^\infty$ are $\hat{E}^\infty_{0,0} = \mathbb{Z}_p$.

Proof. The spectral sequence has E^2-term

$$\hat{E}^2(S^1;J) = \mathbb{Z}[t,t^{-1}] \otimes \pi_*(J)$$

and $\pi_*(J)$ is listed in 1.1. We notice that the elements in even total degrees are concentrated on the s-axis, so the differentials must either end or start here. Since all elements away from the lines $t = 0, -1$ are p-torsion the only differentials which may end on the s-axis are

$$\hat{d}^2 \colon \hat{E}^2_{2s,-1}(S^1;J) \to \hat{E}^2_{2s-2,0}(S^1;J).$$

The map $J \to \Sigma K$ allows us to compare the Tate spectral sequence for J with that for ΣK. Since there are no differentials in $\hat{E}^*(S^1;\Sigma K)$ as all elements have odd total degree we may conclude that $\hat{d}^2 = 0$ in $\hat{E}^*(S^1;J)$. Thus all non-zero differentials in $\hat{E}^*(S^1;J)$ originate on the s-axis. As a consequence

$$\Psi^r \colon E^r_{2s,0}(J^{hS^1}) \to \hat{E}^r_{2s,0}(S^1;J)$$

is an isomorphism for all $r \geq 2$. Now $\pi_{2s}(J^{hS^1}) = 0$ when $s \neq 0$ and since $E^*(J^{hS^1})$ converges strongly to $\pi_*(J^{hS^1})$ it follows that $\hat{E}^\infty_{2s,0}(S^1;J) = 0$ when $s < 0$. Since the Tate spectral sequence is multiplicative we have in particular

$$0 = d^*(1) = d^*(t^{2s}t^{-2s}) = d^*t^{2s} \cdot t^{-2s} + t^{2s} \cdot d^*t^{-2s}$$

and may conclude that $\hat{E}^\infty_{2s,0}(S^1;J) = \hat{E}^\infty_{-2s,0}(S^1;J) = 0$ when $s \neq 0$. This can only happen if infinitely many differentials leave $\hat{E}^2_{2s,0}(S^1;J)$ because all elements in the upper half plane are p-torsion. □

Proposition 2.4 is an immediate consequence of the following corollary.

Corollary. *The only non-zero even degree integral homotopy group of the S^1-Tate spectrum for J is $\pi_0\hat{\mathbb{H}}(S^1;J) = \mathbb{Z}_p$.* □

5. The S^1-Tate spectrum for J

5.1. In this paragraph we solve the extension problems associated with the calculation of $\pi_*\hat{\mathbb{H}}(S^1;J)$, e.g. 2.4. We use some structure theorems from the theory of infinite abelian groups and begin by presenting these; a standard reference is [5]. *Throughout a group will mean an abelian group.*

The sum of all divisible subgroups of a group A is again divisible and therefore maximal. Since divisible groups are injective A is the direct sum of the maximal divisible subgroup and its quotient called the reduced group. The p-completion of A is defined by

$$A^\wedge_p = \varprojlim_n A/p^nA$$

and the canonical map $c \colon A \to A^\wedge_p$ is called the completion map. We say that A is p-complete if c is an isomorphism.

Theorem. [5; p.168] *A reduced p-complete group is the p-completion of a direct sum of cyclic $\mathbb{Z}_p$-modules. The cardinal numbers of the sets of components $\mathbb{Z}_p$ resp. $\mathbb{Z}/p^n$ are invariants of the group.* $\square$

Examples. 1. $A = \prod_{k=0}^{\infty} \mathbb{Z}_p$. Since A is torsion free we have $A = (\bigoplus_{i\in I} \mathbb{Z}_p)^{\wedge}_p$, so have only left to determine $\mathrm{card}(I)$. To this end we notice that

$$A/pA = (\bigoplus_{i\in I} \mathbb{Z}_p)^{\wedge}_p / p(\bigoplus_{i\in I} \mathbb{Z}_p)^{\wedge}_p = (\bigoplus_{i\in I} \mathbb{F}_p)^{\wedge}_p = \bigoplus_{i\in I} \mathbb{F}_p.$$

But A/pA is also a countable product of $\mathbb{F}_p$'s so $\mathrm{card}(I) = 2^{\aleph_0}$.

2. $A = (\bigoplus_{k=0}^{\infty} \mathbb{Z}_p)^{\wedge}_p$. One readily shows that the image of A under the inclusion

$$A = (\bigoplus_{k=0}^{\infty} \mathbb{Z}_p)^{\wedge}_p \subset (\prod_{k=0}^{\infty} \mathbb{Z}_p)^{\wedge}_p = \prod_{k=0}^{\infty} \mathbb{Z}_p$$

is the group $\mathcal{N}(\mathbb{Z}_p)$ of null sequences in $\mathbb{Z}_p$ with respect to the p-adic valuation, i.e. sequences (x_k) such that $v_p(x_k) \to \infty$ with k.

The next theorem gives the structure of divisible groups. Basic examples are the rational group $\mathbb{Q}$ and the group $\mathbb{Q}/\mathbb{Z}_{(p)}$ of all p^nth roots of unity, $n > 0$.

Theorem. [5; p.104] *A divisible group is a direct sum of copies of $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}_{(p)}$ where p runs through all primes. The cardinal numbers of the sets of components $\mathbb{Q}$ resp. $\mathbb{Q}/\mathbb{Z}_{(p)}$ are invariants of the group.* $\square$

Example. Let $\mathbb{Q}_p$ denote the field of p-adic numbers and consider the group $\mathcal{N}(\mathbb{Q}_p)$ of null sequences in $\mathbb{Q}_p$ with respect to the p-adic valuation. It is a divisible torsion free group so a rational vector space. The dimension is $2^{2^{\aleph_0}}$.

5.2. The basic extension problem we need to solve is the following

$$0 \to \prod_{k=0}^{\infty} \mathbb{Z}_p \to E \to \bigoplus_{k=0}^{\infty} \mathbb{Q}/\mathbb{Z}_{(p)} \to 0.$$

As an obvious candidate we have the group of those sequences (x_k) in $\mathbb{Q}_p$ where all but a finite number of elements belong to $\mathbb{Z}_p$. We write $\mathcal{S}(\mathbb{Q}_p)$ for this group,

$$\mathcal{S}(\mathbb{Q}_p) = \varinjlim_{|S|<\infty} (\prod_{s\in S} \mathbb{Q}_p \times \prod_{s\notin S} \mathbb{Z}_p)$$

with the colimit taken over finite subsets of $\mathbb{N}$.

Proposition. *If the underlying group E is torsion free it is isomorphic to $\mathcal{S}(\mathbb{Q}_p)$.*

Proof. We let (E,p) denote the limit system $E \stackrel{p}{\leftarrow} E \stackrel{p}{\leftarrow} E \stackrel{p}{\leftarrow} \cdots$. Since E is torsion free we have exact sequences

$$0 \to E \xrightarrow{p^n} E \to E/p^nE \to 0$$

As n varies these give rise to an exact sequence of limit systems and this in turn supplies an exact sequence

$$0 \to \varprojlim(E,p) \to E \xrightarrow{c} E_p^\wedge \to \varprojlim{}^{(1)}(E,p) \to 0.$$

Now the definition of E as an extension gives us a six term exact sequence

$$\begin{aligned} 0 &\to \varprojlim(\prod_{k=0}^{\infty} \mathbb{Z}_p, p) \to \varprojlim(E,p) \to \varprojlim(\bigoplus_{k=0}^{\infty} \mathbb{Q}/\mathbb{Z}_{(p)}, p) \\ &\to \varprojlim{}^{(1)}(\prod_{k=0}^{\infty} \mathbb{Z}_p, p) \to \varprojlim{}^{(1)}(E,p) \to \varprojlim{}^{(1)}(\bigoplus_{k=0}^{\infty} \mathbb{Q}/\mathbb{Z}_{(p)}, p) \to 0. \end{aligned}$$

The first, fourth and sixth term vanish. The first because no elements in $\prod \mathbb{Z}_p$ may be divided by p infinitely often, the fourth term because $\prod \mathbb{Z}_p$ is compact and the sixth term because $(\bigoplus \mathbb{Q}/\mathbb{Z}_{(p)}, p)$ is Mittag-Leffler, cf. [3; p.256]. Thus c is epi. To evaluate the kernel we use

$$0 \to \mathcal{N}(\mathbb{Z}_p) \to \mathcal{N}(\mathbb{Q}_p) \to \bigoplus \mathbb{Q}/\mathbb{Z}_{(p)} \to 0$$

which gives an exact sequence

$$0 \to \mathcal{N}(\mathbb{Q}_p) \to \varprojlim(\bigoplus_{k=0}^{\infty} \mathbb{Q}/\mathbb{Z}_{(p)}, p) \to \varprojlim{}^{(1)}(\mathcal{N}(\mathbb{Z}_p), p).$$

But $\varprojlim{}^{(1)}(\mathcal{N}(\mathbb{Z}_p), p) = 0$ as $\mathcal{N}(\mathbb{Z}_p)$ is p-complete, so $\ker(c) = \mathcal{N}(\mathbb{Q}_p)$.

We claim that $\mathcal{N}(\mathbb{Q}_p)$ is the maximal divisible subgroup of E. Indeed any divisible subgroup of E is contained in $\ker(c) = \mathcal{N}(\mathbb{Q}_p)$ which is itself divible, hence the claim. The reduced group $E_p^\wedge$ is torsion free and p-complete, so by 5.1 we have only left to determine a cardinal number. We have

$$E_p^\wedge \cong E/\mathcal{N}(\mathbb{Q}_p) \cong (\prod_{k=0}^{\infty} \mathbb{Z}_p)/\mathcal{N}(\mathbb{Z}_p) = (\bigoplus_{2^{\aleph_0}} \mathbb{Z}_p)_p^\wedge / (\bigoplus_{\aleph_0} \mathbb{Z}_p)_p^\wedge = (\bigoplus_{2^{\aleph_0}} \mathbb{Z}_p / \bigoplus_{\aleph_0} \mathbb{Z}_p)_p^\wedge.$$

so the cardinal number we were looking for is $2^{\aleph_0}$. We have shown that if E is torsion free it is the direct sum of $\mathcal{N}(\mathbb{Q}_p)$ and $\prod_{k=0}^{\infty} \mathbb{Z}_p$. Now $\mathcal{S}(\mathbb{Q}_p)$ is such a group. $\square$

5.3. We can now state and prove our main theorem.

Theorem. *Abstractly the integral homotopy groups of the S^1-Tate for J are*

$$\pi_n\hat{\mathbb{H}}(S^1;J)=\begin{cases}\mathbb{Q}/\mathbb{Z}_{(p)}\times\mathcal{S}(\mathbb{Q}_p) & \textit{if } n\equiv -1 \textit{ (mod } 2p-2\textit{) and } n\neq -1\\ \mathcal{S}(\mathbb{Q}_p) & \textit{otherwise when } n \textit{ is odd}\\ \mathbb{Z}_p & \textit{if } n=0\\ 0 & \textit{if } n\neq 0 \textit{ is even.}\end{cases}$$

Proof. We have already shown that the groups in even degrees are zero, cf. lemma 4.5. If we compare this with theorem 4.4 we also obtain that $\pi_n\hat{\mathbb{H}}(S^1;J)$ is torsion free when n is odd and not congruent to -1 (mod $2p-2$) and when $n=-1$. For all these groups the result stated above follows from 5.2 or a slight modification thereof. As an example suppose $n\leq -1$ and $n\not\equiv \pm 1$ (mod $2p-2$). Then $\pi_n\hat{\mathbb{H}}(S^1;J)$ is an extension

$$0\to\prod_{k=0}^{\infty}\mathbb{Z}_p\to\pi_n\hat{\mathbb{H}}(S^1;J)\to\bigoplus_{k=0}^{\infty}\mathbb{Q}/\mathbb{Z}_{(p)}\to 0$$

and we may apply 5.2.

When $n\equiv -1$ (mod $2p-2$) we must work a little harder. If $n\leq -2p+1$ the homotopy group is an extension

$$0\to\mathbb{Z}/p^{v_p(n+1)+1}\times\prod_{k=0}^{\infty}\mathbb{Z}_p\to\pi_n\hat{\mathbb{H}}(S^1;J)\to\bigoplus_{k=0}^{\infty}\mathbb{Q}/\mathbb{Z}_{(p)}\to 0$$

but this time not torsion free. In fact the torsion product $\operatorname{Tor}(\pi_n\hat{\mathbb{H}}(S^1;J),\mathbb{F}_p)=\mathbb{F}_p$ since $\pi_{n+1}(\hat{\mathbb{H}}(S^1;J);\mathbb{F}_p)=\mathbb{F}_p$ by 4.4 and since $\pi_{n+1}\hat{\mathbb{H}}(S^1;J)=0$. However if we can show that the torsion subgroup in $\pi_n\hat{\mathbb{H}}(S^1;J)$ is divisible it follows by 5.1 that it is isomorphic to $\mathbb{Q}/\mathbb{Z}_{(p)}$. We may instead consider the p-completed Tate spectrum $\hat{\mathbb{H}}(S^1;J)^{\wedge}_p$ and ask whether $\pi_n(\hat{\mathbb{H}}(S^1;J)^{\wedge}_p)$ is torsion free. This group fits into an exact sequence (by applying π_* to the completed norm cofibration)

$$0\to\pi_{n+1}(\hat{\mathbb{H}}(S^1;J)^{\wedge}_p)\xrightarrow{\partial}(\bigoplus_{k=0}^{\infty}\mathbb{Z}_p)^{\wedge}_p\xrightarrow{N_*}\mathbb{Z}/p^{\alpha}\times\prod_{k=0}^{\infty}\mathbb{Z}_p\xrightarrow{t_*}\pi_n(\hat{\mathbb{H}}(S^1;J)^{\wedge}_p)\to 0$$

where $\alpha=v_p(n+1)+1$, cf. 2.2 and 2.4. The component $\mathbb{Z}/p^{\alpha}$ in $\pi_n((J^{hS^1})^{\wedge}_p)$ is from the first factor of the decomposition $J^{hS^1}\simeq J\times F(\mathbb{C}\mathrm{P}^{\infty},J)$. In the mod p spectral sequence such elements are located on the t-axis (s=0), and therefore some generator in $\mathbb{Z}/p^{\alpha}$ represents the class $z_r(0)$ with $2r-1=n$, cf. 4.5. Since $z_r(0)$ is in the image of N_* (mod p) by 4.3, $\mathbb{Z}/p^{\alpha}\subset\operatorname{im}N_*$ and consequently $p^{\alpha}\mathbb{Z}_p$ is a quotient of $\pi_{n+1}(\hat{\mathbb{H}}(S^1;J)^\wedge_p)$. We conclude that $\mathbb{F}_p$ is a quotient of $\pi_{n+1}(\hat{\mathbb{H}}(S^1;J)^\wedge_p)\otimes\mathbb{F}_p$. Now recall from 4.4 that $\pi_{n+1}(\hat{\mathbb{H}}(S^1;J);\mathbb{F}_p)=\mathbb{F}_p$ and from [2] the formula

$$\pi_{n+1}(\hat{\mathbb{H}}(S^1;J);\mathbb{F}_p)\cong\pi_{n+1}(\hat{\mathbb{H}}(S^1;J)^\wedge_p)\otimes\mathbb{F}_p\oplus\operatorname{Tor}(\pi_n(\hat{\mathbb{H}}(S^1;J)^\wedge_p),\mathbb{F}_p).$$

It follows that the second summand must be zero and hence $\pi_n(\hat{\mathbb{H}}(S^1;J)^\wedge_p)$ is a torsion free group. For the p-completed spectra the case $n\geq 2p-3$ is the same. □

Corollary. *The homotopy groups of the p-completed Tate spectrum are*

$$\pi_n\hat{\mathbb{H}}(S^1;J)^\wedge_p=\begin{cases}\mathbb{Z}_p\oplus\prod\mathbb{Z}_p/\mathcal{N}(\mathbb{Z}_p) & \text{if } n\equiv 1 \pmod{2p-2}\\ \prod\mathbb{Z}_p/\mathcal{N}(\mathbb{Z}_p) & \text{otherwise when } n \text{ is odd}\\ \mathbb{Z}_p & \text{if } n=0 \pmod{2p-2}\\ 0 & \text{otherwise if } n \text{ is even.}\end{cases}$$

Proof. This follows immediately from the formula on page 3 when we notice that

$$\begin{aligned}\operatorname{Hom}(\mathbb{Q}/\mathbb{Z}_{(p)},\bigoplus_{k=0}^{\infty}\mathbb{Q}/\mathbb{Z}_{(p)})&=\operatorname{Hom}(\varinjlim_n\mathbb{Z}/p^n,\bigoplus_{k=0}^{\infty}\mathbb{Q}/\mathbb{Z}_{(p)})\\ &=\varprojlim_n\operatorname{Hom}(\mathbb{Z}/p^n,\bigoplus_{k=0}^{\infty}\mathbb{Q}/\mathbb{Z}_{(p)})=\varprojlim_n\bigoplus_{k=0}^{\infty}\mathbb{Z}/p^n=\mathcal{N}(\mathbb{Z}_p).\end{aligned}$$ □

We return to the conjecture we stated in 0.2,

$$\hat{\mathbb{H}}(S^1;J)^\wedge_p\simeq K^{[1]}\vee\Sigma K^{[1]}\vee(\prod_{r=0}^{\infty}\Sigma K)/(\bigvee_{r=0}^{\infty}\Sigma K)^\wedge_p.$$

The two sides have isomorphic homotopy groups. We note that there does exist a cofibration

$$\Sigma J\vee(\bigvee_{s=0}^{\infty}\Sigma K)^\wedge_p\to J\vee\prod_{s=0}^{\infty}\Sigma K\to K^{[1]}\vee\Sigma K^{[1]}\vee(\prod_{r=0}^{\infty}\Sigma K)/(\bigvee_{r=0}^{\infty}\Sigma K)^\wedge_p.$$

which could be a candidate for the norm cofibration 0.2. The Adams splitting of p-complete K-theory is a decomposition $K\simeq K^{[1]}\vee K^{[2]}\vee\ldots\vee K^{[p-1]}$, and ψ^g-1 induces a homotopy equivalence on all components save $K^{[1]}$. Thus J may equally well be defined by the cofibration

$$J\longrightarrow K^{[1]}\xrightarrow{\psi^g-1}K^{[1]}.$$

There are equivalences $K^{[i]}=\Sigma^{2i-2}K^{[1]}$ and $\pi_*K^{[1]}=\mathbb{Z}_p[v_1,{v_1}^{-1}]$, so by an infinite exchange of Adams components we get equivalences

$$(\bigvee_{s=0}^{\infty}\Sigma K)^\wedge_p\simeq\Sigma^{-1}K^{[1]}\vee(\bigvee_{r=0}^{\infty}\Sigma K)^\wedge_p,\qquad\prod_{s=0}^{\infty}\Sigma K\simeq\Sigma K^{[[H1]}\vee\prod_{r=0}^{\infty}\Sigma K.$$

Now it is obvious how to construct the proposed cofibration.

References

1. J.M.Boardman, *Conditionally convergent spectral sequence*, preprint, The John Hopkins University (1981).
2. A.K.Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), 257-281.
3. A.K.Bousfield and D.M.Kan, *Homotopy Limits, Completions and Localizations*, LNM **304**.
4. M.Bökstedt and I.Madsen, *Topological Cyclic Homology of $\hat{\mathbb{Z}}_p$*, preprint, Aarhus University (1993).
5. L.Fuchs, *Infinite Abelian Groups I*, Academic Press (1970).
6. J.P.C.Greenlees an J.P.May, *Generalized Tate, Borel and CoBorel Cohomology*, preprint, University of Chicago (1992).
7. K.Hansen, *The K-localization of some classifying spaces*, preprint series, Aarhus University **36** (1991).
8. D.C.Ravenel, *Localization with respect to certain periodic homology theories*, Amer.J.Math. **106** (1984), 351-414.
9. L.C.Washington, *Introduction to Cyclotomic Fields*, Springer, GTM **83**.