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Modular Invariance in Mathematics and Physics

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In this talk I want to discuss some recently discovered beautiful connections of representation theory of infinite-dimensional Lie algebras with the theory of modular functions, and related progress in theoretical physics.

1. Modular functions. Consider a finite-dimensional space of complex analytic functions on the upper half-plane, having at worst a pole at $i\infty$, and suppose that this space is invariant under transformations

$$(1)_w \qquad f(\tau) \mapsto f(\tau+1) \quad \text{and} \quad f(\tau) \mapsto \tau^{-w} f(-1/\tau).$$

These functions are then called *modular functions* of *weight* w $(\in \frac{1}{2}\mathbb{Z})$.

To illustrate the idea of modular invariance, consider the classical partition function $p(n)$, the number of partitions of n into a sum of positive integers. Its generating series $1 + \sum_{n=1}^{\infty} p(n)x^n$ is equal to $1/\phi(x)$, where $\phi(x) = \prod_{n\geq 1}(1-x^n)$. The key observation is that the closely related *Dedekind η-function* $\eta(\tau) = q^{1/24}\phi(q)$, where $q = e^{2\pi i\tau}$, is a modular function of weight $1/2$ since it has the following modular invariance property:

$$(2) \qquad \eta(\tau) = (-i\tau)^{-1/2}\eta(-1/\tau).$$

Let $\tau = i\beta$ (β is the standard notation of the inverse of the temperature in statistical mechanics). Then, by (2), we have asymptotically as $\beta \downarrow 0$:

$$(3) \qquad \eta(i\beta)^{-1} \sim \beta^{1/2} e^{\pi/12\beta}.$$

Applying the standard Tauberian theorem which relates the asymptotics of a series $\sum_n a_n x^n$ as $x \to 1$ to the asymptotics of a_n as $n \to \infty$, we obtain the following classical result:

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2n/3}} \quad \text{as } n \to \infty.$$

Similarly, given an integral lattice Λ of rank l in the euclidean space

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$\mathbb{R}^l$, we can count the number of vectors of any given length by studying the associated *theta series*:

$$\theta_\Lambda(\tau) = \sum_{\gamma\in\Lambda} q^{|\gamma|^2/2}.$$

This is a modular function of weight $l/2$. Indeed, the linear span of "generalized" theta series $\{\theta_{\Lambda+a}(\tau)|a \in \Lambda^* \bmod \Lambda\}$ is invariant with respect to transformations $(1)_{l/2}$ (this is obvious for the first transformation, and follows from the Poisson summation formula for the second). For example, taking $\Lambda = \mathbb{Z}^l$, one can study the number of ways n can be represented as a sum of l squares.

Note that by Euler's identity

$$\phi(q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{(3n^2+n)/2}, \tag{4}$$

it follows that

$$\eta(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{3(n+1/6)^2/2}$$

is a difference of two (generalized) theta series, hence is a modular function (of weight $1/2$).

Another famous example is the *modular invariant* $j = \theta^3_{E_8}/\eta^{24}$ (which parametrizes elliptic curves), where E_8 is the only even unimodular lattice in eight dimensions. The function $j(\tau)$ is invariant under transformations $(1)_0$, and, moreover, generates the ring of all such functions.

On recent developments in number theory based on modular invariance (including Ribet's proof of Fermat's last theorem modulo the Taniyama-Weil conjecture), see the address of B. Gross.

Note that a modular function f of weight $-w$ has asymptotics similar to (3), namely, for some numbers d and g one has:

$$f(i\beta) \sim d\beta^{w/2} e^{\pi g/12\beta} \quad \text{as } \beta \downarrow 0. \tag{5}$$

2. Infinite-dimensional Lie algebras. The most important infinite-dimensional Lie algebras $\mathfrak{g}$ possess a derivation (Hamiltonian) H satisfying the following three properties:

(i) $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$, where $\mathfrak{g}_j = \{x \in \mathfrak{g}|[H, x] = jx\}$ and $\mathfrak{g}_1 \neq 0$;
(ii) $\mathfrak{g}$ has only trivial graded (with respect to (i)) ideals;
(iii) $\dim \mathfrak{g}_j < \text{const}$.

According to the Kac-Mathieu classification theorem, there are only two possibilities for such algebras:

(a) loop algebras (the Lie algebra of regular maps from $\mathbb{C}^\times$ to complex simple finite-dimensional Lie algebras) and their "twisted" analogs;
(b) the Lie algebra of regular vector fields on $\mathbb{C}^\times$.

In our experience, one loses most (if not all) interesting representations, unless one considers central extensions of the Lie algebras in question. We explain below the construction of (â) the *affine Kac-Moody algebras* $\hat{\mathfrak{g}}$, which are universal central extensions of (nontwisted) loop algebras, and of ($\hat{\mathrm{b}}$) the *Virasoro algebra* Vir, the universal central extension of (b).

EXAMPLE (â). $\hat{\mathfrak{g}} = (\bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{(n)}) \oplus \mathbb{C}k$, where, for all n, we have $\mathfrak{g}_{(n)} = \mathfrak{g}$, a simple Lie subalgebra of $\mathrm{gl}_N(\mathbb{C})$, with the following commutation relations:

$$(6) \qquad [x_{(m)}, y_{(n)}] = [x, y]_{(m+n)} + m\delta_{m,-n}(x, y)k, \qquad [k, \hat{\mathfrak{g}}] = 0;$$

here $x, y \in \mathfrak{g}$ and $(x, y) = \mathrm{const}\,\mathrm{tr}\, xy$ (const $= 1$ for $\mathfrak{g} = \mathrm{sl}_N(\mathbb{C})$). The Hamiltonian $H = H_0 + \mathrm{ad}\, z_{(0)}$, where $[x_{(m)}, H_0] = mx_{(m)}$ and $z \in \mathfrak{g}$ is a real diagonal matrix. Geometric interpretation:

$$\hat{\mathfrak{g}}/\mathbb{C}k = \bigoplus \mathfrak{g}_{(n)} = \mathrm{Map}(\mathbb{C}^\times, \mathfrak{g}) = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}, \qquad x_{(n)} = t^n x.$$

EXAMPLE ($\hat{\mathrm{b}}$). $\mathrm{Vir} = (\bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n) \oplus \mathbb{C}c$, with commutation relations

$$(7) \qquad [L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n}c, \qquad [c, \mathrm{Vir}] = 0.$$

The Hamiltonian $H = L_0$. Geometric interpretation:

$$\mathrm{Vir}/\mathbb{C}c = \mathrm{Vect}\,\mathbb{C}^\times, \qquad L_n = -t^{n+1}\frac{d}{dt}.$$

3. Positive energy representations. It has been clear for some time now, both to mathematicians and to physicists, that the most interesting representations of an infinite-dimensional Lie algebra are the *positive energy representations.* These are the representations in a vector space V for which the Hamiltonian H is diagonalizable and Spec H is a set of real numbers bounded below. The function $\mathrm{tr}_V q^H$ is called the *character* of this representation. An important and still somewhat mysterious fact is that in all known examples, the characters of irreducible positive energy representations have asymptotics of the form (5) (i.e., behave as modular functions at the high temperature limit), and that the numbers d, w, and g, called respectively the *asymptotic dimension*, the *weight*, and the *growth* of V, have a group theoretical interpretation. For example, the asymptotic dimension d has all the properties of usual dimension, although it is an irrational number in general. Actually, for this reason, d is a much more powerful invariant than the usual dimension.

In all known examples of irreducible positive energy representations of infinite-dimensional Lie algebras, the number w is nonnegative (and the numbers d and g are positive), in sharp contrast with the finite-dimensional case, when $\mathrm{tr}_V q^H \sim e\beta^{-d}$ as $\beta \downarrow 0$ and d is a nonnegative integer (called the Bernstein-Gelfand-Kirillov dimension).

Sometimes a character becomes a modular function when multiplied by $q^{-a/24}$, where a is some number (the *modular anomaly)*; the result is then called the *modified character.* The corresponding representation is said to be *modular invariant*. (Note that modular invariance implies energy positivity.) It is the modular invariant representations that have played a fundamental role in the recent development of representation theory of infinite-dimensional Lie algebras and groups on the one hand, and of quantum field theory on the other.

4. A toy example. An affine algebra may be viewed as a "nonabelian" generalization of the *oscillator algebra* (= Heisenberg algebra) $\hat{\mathfrak{a}}$, the affine algebra associated to the 1-dimensional Lie algebra $\mathfrak{a} = \mathbb{C}s$. We have:

$$[s_{(m)}, s_{(n)}] = m\delta_{m,-n}k, \qquad [s_{(m)}, k] = 0.$$

It is very easy to describe all irreducible positive energy representations V of $\hat{\mathfrak{a}}$: Either $k = 0$, then V is the trivial 1-dimensional representation, or $k \neq 0$, then $V = \mathbb{C}[x_1, x_2, \dots]$, and $s_{(m)} = \partial/\partial x_m$, $s_{(-m)} = kmx_m$ for $m > 0$; $s_0 = \mu$; $H_0 = \sum_{j\geq 1} jx_j\, \partial/\partial x_j$. These are the canonical commutation relations representations. All of them are modular invariant with modified character $\eta(\tau)^{-1}$, so that $d = w = g = a = 1$. This example, which is very important in quantum field theory, is, from a purely mathematical point of view, not just a toy model. It also serves as a basis for rather nontrivial constructions, such as the vertex operator construction and its variations. Along these lines one constructs the modular invariant representation of the famous Monster group, whose modified character is $j(\tau)$.

5. Modular invariant representations of the affine algebra $\hat{\mathfrak{g}}$. There are some general experimental facts (most of them conjectures) about an irreducible positive energy representation V of $\hat{\mathfrak{g}}$:

(a) $\operatorname{rank}\mathfrak{g} \leq g(V) \leq \dim\mathfrak{g}$;
(b) $0 \leq w(V) \leq \dim\mathfrak{g}$;
(c) $w(V) = 0 \Rightarrow V$ is modular invariant;
(d) $g(V) < \dim\mathfrak{g} \Leftrightarrow V$ is modular invariant with $w(V) < \dim\mathfrak{g}$.

In our experience, the "smaller" a representation is, the more interesting it becomes. Since the growth is a reasonable measure of the "size" of a representation, we may conclude that the most interesting are the irreducible modular invariant representations of zero weight. An amazing fact is that even in the simplest case $\mathfrak{g} = \mathrm{sl}_2(\mathbb{C})$, the theory of such representations is remarkably rich (in sharp contrast with the "abelian case" when no such representation exists).

Positive energy representations V of $\hat{\mathfrak{g}}$ are parametrized by the central charge k (= the eigenvalue of the operator k) and by the (finite-dimensional irreducible) representation λ, on the minimal energy subspace V_h of the Hamiltonian $H_0 + z$, of the algebra of "internal symmetries" $\{x \in \mathfrak{g}_{(0)} | [z, x] = 0\}$. (In unitary theories one always takes $z = 0$, otherwise one needs to take $z \neq 0$ to avoid divergences.)

The first problem I am going to address is to determine for which pairs (k, λ) the corresponding representations $V(k, \lambda)$ is modular invariant of zero weight. An easy necessary condition is

(8) $k + h'$ is a nonnegative rational number, where h' is the dual Coxeter number of $\mathfrak{g}$ ($h' = N$ for $\mathfrak{g} = \mathrm{sl}_N(\mathbb{C})$).

Consider also the following condition:

(9)
$$\langle \hat{\lambda} + \hat{\rho}, \alpha \rangle \in \mathbb{Q} \backslash \{0, -1, -2, \dots\} \quad \text{for all } \alpha \in R,$$
where R is the set of "positive real coroots" of $\hat{\mathfrak{g}}$.

Provided that (8) and (9) hold, one has an explicit character formula (due to Kac and Wakimoto) of the following form:

(10)
$$\mathrm{tr}_{V(k,\lambda)} q^H = \sum_{y \in W^{k,\lambda}} \pm \Theta_{y,k,\lambda} / \prod_\alpha,$$

where $\Theta_{y,k,\lambda}$ are certain theta series and $\prod_\alpha$ is certain infinite product over all "positive roots" of $\hat{\mathfrak{g}}$. Here $\hat{\lambda} + \hat{\rho}$ is the "shifted highest weight" of $V(k, \lambda)$ (which is simply expressed in terms of k and λ) and $W^{k,\lambda}$ is the associated subgroup of the "Weyl group" W of $\mathfrak{g}$.

But if λ is the trivial 1-dimensional representation of $\hat{\mathfrak{g}}$ and $k = 0$, then $\dim V(k, \lambda) = 1$, hence $\mathrm{tr}_V q^H = 1$ and we obtain from (10) an identity of the form:

(11)
$$\prod_\alpha = \sum_{y \in W} \pm \Theta_y.$$

These are the well-known Macdonald identities.

It follows from (10) and (11) that, provided that (8) and (9) hold, the representation $V(k, \lambda)$ is modular invariant; moreover, the following condition guarantees that $w(V(k, \lambda)) = 0$:

(12) the $\mathbb{Q}$-span of the set $\{\alpha \in R | \langle \hat{\lambda} + \hat{\rho}, \alpha \rangle \in \mathbb{Z}\}$ contains R.

The main *conjecture* asserts that conditions (8), (9), and (12) give us the complete list of all modular invariant representations of $\hat{\mathfrak{g}}$ of weight 0 with $k \neq -h'$. This conjecture has been checked for $\widehat{\mathrm{sl}}_2$, Vir, and in some other cases.

6. The example of $\widehat{\mathrm{sl}}_2$. To avoid technicalities, I have given only a flavor of the main results for general affine algebras. Now I will state these results explicitly in the simplest case of the affine algebra $\widehat{\mathrm{sl}}_2$.

The complete list of zero weight modular invariant irreducible representations $V(k, \lambda)$ is as follows: the central charge k is a rational number u'/u, in lowest terms, such that

(13)
$$k + 2 \geq 2/u;$$

all possible λ (= the maximal eigenvalue of the element $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ on the subspace of minimal energy) for this k are

(14)
$$\lambda = n - s(k+2),$$

where $n, s \in \mathbb{Z}$, $0 \le n \le u(k+2)-2$, $0 \le s \le u-1$.

The character formula (10) (taking into account (11)) for such a representation is:

$$\text{(15)} \qquad \operatorname{tr}_{V(k,\lambda)} q^{H_0 + \frac{1}{2} z \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) - \left(\frac{1}{24} a - \frac{1}{4} z^2\right)} = \frac{\left(\Theta_{b_+, b}(\tau, \frac{z}{u}) - \Theta_{b_-, b}(\tau, \frac{z}{u})\right)}{\left(\Theta_{1,2}(\tau, z) - \Theta_{-1,2}(\tau, z)\right)}.$$

Here $\Theta_{n,m}(\tau, z) = \theta_{(n/m+z)/2+\mathbb{Z}}(m\tau)$ is a Jacobi-Riemann theta function, and

$$b = u^2(k+2), \quad b_\pm = u(\pm(n+1) - s(k+2)), \quad a = 3 - 6b_+^2/b.$$

One also has:

$$d(V(k,\lambda)) = \sqrt{\frac{2}{b}} \sin \frac{\pi(n+1)}{u(k+2)} \sin \frac{\pi(s+z)}{u} / \sin \pi z;$$
$$g(V(k,\lambda)) = 3 - 6/b.$$

Given k, the space spanned by all modified characters of modular invariant representations is invariant with respect to modular transformations $(1)_0$. Explicitly, denoting the left-hand side of (15) by χ_λ, one has

$$\text{(16)} \qquad \begin{aligned} \chi_\lambda\left(-\frac{1}{\tau}, -\tau z\right) &= \left(\exp \frac{i\pi k z^2 \tau}{2}\right) \left(\frac{2}{b}\right)^{1/2} \\ &\quad \times \sum_{\lambda'} \frac{e^{-i\pi b_+ b'_- / b} - e^{-i\pi b_+ b'_+ / b}}{2i} \chi_{\lambda'}(\tau, z), \end{aligned}$$

where λ' runs over the list (14).

Results similar to these hold for all affine algebras (see Kac-Peterson and Kac-Wakimoto).

All experimental facts (a)–(d) stated at the beginning of this section hold for $\widehat{\mathrm{sl}}_2$. In particular, the following statements are equivalent for an irreducible positive energy representation V of $\widehat{\mathrm{sl}}_2$:

(i) $g(V) < 3$;
(ii) $w(V) = 0$;
(iii) V is modular invariant of weight 0.

The Macdonald identity (11) is, in this case, the celebrated Jacobi triple product identity:

$$\prod_{n=1}^{\infty} (1 - u^n v^n)(1 - u^n v^{n-1})(1 - u^{n-1} v^n) = \sum_{n \in \mathbb{Z}} (-1)^n u^{n(n+1)/2} v^{n(n-1)/2}.$$

Letting $u = q$, $v = q^2$ we deduce Euler's identity (4), and letting $u = v = -q$, we obtain a product expansion $\theta_{\mathbb{Z}}(\tau) = \eta^2((\tau+1)/2)/\eta(\tau+1)$.

7. Some applications. The most popular representation is the modular invariant representation of minimal possible growth $l = \operatorname{rank} \mathfrak{g}$. This is the *basic representation* $V = V(1, \lambda = \text{trivial})$ (and the ones obtained from V by a simple twist) of the affine algebra $\hat{\mathfrak{g}}$, where $\mathfrak{g}$ is one of the simple classical Lie algebras $\mathrm{sl}_{l+1}(\mathbb{C})$, $\mathrm{so}_{2l}(\mathbb{C})$, or the exceptional ones E_6, E_7, E_8. The modified character of this representation is given by an especially simple formula (Kac):

$$\mathrm{tr}_V q^{H_0+z-(l/24-(z,z)/2)} = \frac{\theta_{Q+z}(\tau)}{\eta(\tau)^l}, \tag{17}$$

where Q is the root lattice of $\mathfrak{g}$.

Note that since H_0 commutes with $\mathfrak{g} \subset \hat{\mathfrak{g}}$, the H_0-eigenspace decomposition $V = \bigoplus_{j\in\mathbb{Z}_+} V_j$ is invariant with respect to $\mathfrak{g}$. Thus we obtain a series of finite-dimensional representations of $\mathfrak{g}$ ($\dim V_0 = 1$, $\dim V_1 = \dim \mathfrak{g}, \dots$) such that the generating series $\sum_{j\in\mathbb{Z}_+}(\dim V_j)q^{j-l/24}$ is equal to θ_Q/η^l. In particular, for E_8 this series is equal to $j(\tau)^{1/3}$. Moreover, replacing $(\dim V_j)$ by $\mathrm{tr}_{V_j} g$, where g is a finite order element of the Lie group G with Lie algebra $\mathfrak{g}$, we obtain a modular function of weight 0.

In a remarkable parallel development, McKay, Thompson, Conway, and Norton observed that the sporadic finite simple Monster group F_1 has a series of finite-dimensional representations V_0, $V_1 = 0$, V_2, $V_3, \dots,$ such that the generating series $\sum_{j\in\mathbb{Z}_+}(\dim V_j)q^{j-1}$ equals $j(\tau) - 744$ and that, moreover, the series $\sum_{j\in\mathbb{Z}_+}(\mathrm{tr}_{V_j} g)q^{j-1}$ is a modular function of weight 0 and "genus" 0 for all $g \in F_1$; almost all modular functions of "genus" 0 occur in this way. The latter phenomenon is yet to be explained.

The basic representation admits a large variety of explicit constructions. One starts with a regular loop $s(t) \in \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ $(= \hat{\mathfrak{g}}/\mathbb{C}k)$ (i.e., for any $t_0 \in \mathbb{C}^\times$, the centralizer of $s(t_0)$ in $\mathfrak{g}$ consists of commuting diagonalizable elements); its centralizer $\bar{\mathfrak{s}}$ is commutative and its preimage $\mathfrak{s}$ in $\hat{\mathfrak{g}}$ is a Heisenberg algebra. Let $\overline{S}$ denote the centralizer of $\bar{\mathfrak{s}}$ in $G(\mathbb{C}[t, t^{-1}])$. An important property of the basic representation (proved by Kac and Peterson) is its irreducibility with respect to the pair $(\mathfrak{s}, \overline{S})$. (For example, taking $s(t)$ to be a regular constant diagonal matrix, we obtain the homogeneous Heisenberg subalgebra and the irreducibility theorem follows from (17).) This allows one to identify V with the space of the oscillator representation. The oscillator representation extends to the basic representation of $\hat{\mathfrak{g}}$ by making use of the vertex operators of string theory. The vertex operators are characterized by the property of being eigenvectors with respect to $\mathfrak{s}$ and S, and are, up to a simple factor, of the form

$$\left(\exp \sum \lambda_i x_i\right)\left(\exp \sum \mu_i \frac{\partial}{\partial x_i}\right).$$

The vertex operator construction attached to the homogeneous Heisenberg subalgebra (the Frenkel-Kac-Segal construction) in the case $\mathfrak{g} = E_8$ is an important part of the "heterotic" string model used in compactification from 26 to 10 dimensions by Gross-Harvey-Martinec-Rohm. A twist of this construction applied to Griess's algebra produces the series of representations of the Monster group mentioned above (Frenkel-Lepowsky-Meurman and Borcherds).

Another beautiful application of the basic representation, to the soliton theory, was discovered by Sato, Date, Jimbo, Kashiwara, and Miwa. An analysis of their work shows that their approach is based on the following simple observation: Let G be a group acting on a vector space V, let $v_0 \in V$, and let Ω be an operator on $V \otimes V$ commuting with the (diagonal) action of G and such that $v_0 \otimes v_0$ is its eigenvector with eigenvalue λ. Then an element f of the orbit $G \cdot v_0$ satisfies the equation

$$\Omega(f \otimes f) = \lambda f \otimes f. \tag{18}$$

For example, we can take Ω to be the "Casimir operator" and v_0 to be the vacuum vector.

Applying this construction to $G = \mathrm{SL}_2(\mathbb{C}[t, t^{-1}])$ and V its basic (projective) representation, we obtain two quite different systems of partial differential equations depending on the construction of V. If the construction is based on the homogeneous Heisenberg subalgebra of $\widehat{\mathrm{sl}}_2$, then equation (18) turns into the NLS hierarchy, the simplest equations being (after a change of functions) the coupled nonlinear Schrödinger equations on functions $g(t, x)$ and $g^*(t, x)$:

$$g_t = g_{xx} - 2g^2 g^*, \qquad g_t^* = -g_{xx}^* + 2g g^{*2}.$$

If the construction is based on the other Heisenberg subalgebra, associated to $s(t) = \left(\begin{smallmatrix} 0 & 1 \\ t & 0 \end{smallmatrix}\right)$ (the principal construction), the equation (18) turns into the KdV hierarchy, the simplest equation being (after a change of functions) the celebrated KdV equation:

$$u_t = \tfrac{3}{2} u u_x + \tfrac{1}{4} u_{xxx}.$$

Moreover, the N-soliton solutions of these equations can be obtained by making use of vertex operators. For example, if $X(z)$ is the vertex operator of the principal construction

$$X(z) = \left(\exp 2 \sum_{\substack{j \geq 1 \\ j \text{ odd}}} z^j x_j\right) \left(\exp -2 \sum_{\substack{j \geq 1 \\ j \text{ odd}}} \frac{z^{-j}}{j} \frac{\partial}{\partial x_j}\right)^{\bullet},$$

then N-soliton solutions of the KdV equation are given by the following formula (in which $x = x_1$, $t = x_3$):

$$u(t, x) = 2(\log((1 + a_N X(z_N)) \cdots (1 + a_1 X(z_1)) \cdot 1))_{xx}.$$

These solutions describe the interaction of N solitary waves.

This theory has an important connection to the theory of theta-functions. Namely, equation (18) in the case of the basic representation of the group

GL_∞ is nothing else but the KP hierarchy, which characterizes theta functions of algebraic curves among all theta functions (Arbarello-De Concini, Mulase, Shiota).

Apart from the basic representation, the only positive energy irreducible representations of growth $< 2l$ have central charge k equal to 2. This case plays an important role in various supersymmetric theories.

The modular invariance constraint is an important ingredient of the representation theory itself. The basic observation here is that given an affine algebra $\hat{\mathfrak{g}}$ and its affine subalgebra $\hat{\mathfrak{g}}_1$ the branching coefficients of an irreducible unitary (and hence modular invariant) positive energy representation of $\hat{\mathfrak{g}}$ restricted to $\hat{\mathfrak{g}}_1$ are coefficients of modular functions, called *branching functions*, with well studied transformation properties. In the simplest example $\mathfrak{g} = sl_2(\mathbb{C})$ and $\mathfrak{g}_1 = \mathbb{C}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, these modular functions turn out to be the classical indefinite Hecke modular forms divided by η^2 (Kac-Peterson). This has been exploited by Date, Jimbo, Miwa, and their collaborators to calculate explicitly the local height probabilities in solvable lattice models. It should be pointed out, however, that even in this simplest example, the branching functions of a nonunitary modular invariant representation are not always modular functions (Lu), and their transformation properties are unknown.

The $SL_2(\mathbb{Z})$-invariance of the space of modified characters of modular invariant representations with given central charge k has important applications to 2-dimensional quantum field theories, since it allows one to compute explicitly their partition functions (Gepner, Capelli-Itzykson-Zuber, ...).

In general, a conformally invariant 2-dimensional quantum field theory (CFT) produces a finite set of modular functions $f_0, \dots, f_N$ whose $\mathbb{C}$-span is invariant with respect to transformations $(1)_0$ and such that

$$f_i(\tau) = \sum_{n \in \mathbb{Z}_+} a_n^{(i)} q^{n - r_i}, \quad \text{where } a_n^{(i)} \in \mathbb{Z}_+, \ r_i \in \mathbb{Q}, \text{ and } r_0 > r_1 \geq r_2 \geq \dots .$$

These data contain the most important information of this field theory, such as conformal anomaly, conformal dimensions, the partition function (Cardy), the fusion rules (Verlinde), etc. These properties of $f_0, \dots, f_N$ alone turn out to be very restrictive, and, for small N, allow an effective classification algorithm (Mathur-Mukhi-Sen). Given $\hat{\mathfrak{g}} \supset \hat{\mathfrak{g}}_1$ and $k \in \mathbb{Z}_+$, taking branching functions of all unitary modular invariant representations of $\hat{\mathfrak{g}}$ with central charge k with respect to $\hat{\mathfrak{g}}_1$ we get a finite set of functions satisfying the above properties (Kac-Peterson), obtaining thus a large variety of CFT.

8. Modular invariant representations of Vir. Positive energy irreducible representations of Vir are parametrized by two numbers, the *conformal anomaly* c (= the eigenvalue of c) and the minimal energy h of the Hamiltonian L_0. A complete list of pairs (c, h) such that the corresponding representation $V_{c,h}$ is modular invariant of weight 0 is as follows:

$$c = 1 - \frac{6(m-n)^2}{mn}, \qquad h = \frac{(mr-ns)^2 - (m-n)^2}{4mn}, \tag{19}$$

where m, n, r, s are positive integers such that $(m, n) = 1$, $r < n$, $s < m$, $sn \leq rm$.

The character formula for arbitrary $V_{c,h}$ was given by Feigin and Fuchs. For representations from the list (19) this formula turns into

$$\operatorname{tr}_{V_{c,h}} q^{L_0 - c/24} = (\theta_{(mr-ns)/2mn+\mathbb{Z}}(mn\tau) - \theta_{(mr+ns)/2mn+\mathbb{Z}}(mn\tau))/\eta(\tau). \tag{20}$$

For these representations we have

$$g(V_{c,h}) = 1 - \frac{6}{mn}. \tag{21}$$

The following properties of a representation $V = V_{c,h}$ are equivalent:

(i) $g(V) < 1$;
(ii) $w(V) = 0$;
(iii) V is modular invariant.

According to Belavin, Polyakov, and Zamolodchikov each c from the list (19) corresponds to a solvable 2-dimensional statistical model, the h's corresponding to the critical exponents of this model.

Note that $g(V) \leq 1/2$ in two cases only: $c = 1/2$, which corresponds to the Ising model and $c = -22/5$, which corresponds to the Lee-Yang edge singularity model (Cardy). The Ising model is the first member of the sequence of unitary statistical models (Friedan-Qiu-Shenker) and the Lee-Yang model is the first member of a very interesting sequence of nonunitary models with

$$c = 1 - \frac{3(m-2)}{m} \quad (m = 5, 7, 9, \ldots),$$
$$h = h_s = -\frac{(m-s-1)(s-1)}{2m} \qquad \left(s = 1, 2, \ldots, \frac{m-1}{2}\right).$$

In this case the character admits a simple product decomposition:

$$\operatorname{tr}_{V_{c,h}} q^{L_0 - h_s} = \prod_{\substack{n \geq 1 \\ n \not\equiv 0, \pm s \bmod m}} (1 - q^j)^{-1}. \tag{22}$$

Note that this product, in the case $m = 5$, is precisely the product part of the celebrated Rogers-Ramanujan identities (and in the cases $m > 5$ that of their Gordon generalizations). A natural conjecture is that these identities provide bases of representation spaces; in particular, in the case $m = 5$, vectors $\cdots L_{-j_3} L_{-j_2} L_{-j_1} |h_s\rangle$ with $j_1 + j_2 + \cdots = n$, $j_2 \geq j_1 + 2$, $j_3 \geq j_2 + 2, \ldots$ and $j_1 \geq 2$ (resp. ≥ 1) when $s = 0$ (resp. $= 1$), should form a basis of the subspace of energy $h_s + n$. In the same spirit, every modular invariant representation of Vir should produce a Rogers-Ramanujan type identity.

Further development of the Belavin-Polyakov-Zamolodchikov approach to the 2-dimensional conformal field theory gave the following simple construction of the partition function $Z(\tau)$ on a torus (Cardy). Fix the conformal anomaly c, and let $\chi_{(h)}(\tau) = \operatorname{tr}_{V_{c,h}} q^{L_0 - c/24}$ be the modified character. Then $Z(\tau)$ must be a (real analytic) function of the form

$$Z(\tau) = \sum_{h,h'} a_{h,h'} \chi_{(h)}(\tau) \overline{\chi_{(h')}(\tau)}, \tag{23}$$

where $a_{h,h'}$ are nonnegative integers, $a_{0,0} = 1$, and $Z(\tau)$ is modular invariant (i.e., invariant under transformations $(1)_0$).

If (c, h) is from the list (19), then all modified characters (with given c) form a basis of a vector space invariant under transformations $(1)_0$ and the corresponding matrices are unitary in this basis. It follows that $Z(\tau) = \sum_h |\chi_{(h)}(\tau)|^2$ is a partition function. A complete classification of partition functions (23) with $c < 1$ (and hence from the list (19)) is obtained in two steps. First, the problem is reduced to a similar problem for $\widehat{\mathrm{sl}}_2$ and unitary characters (Gepner). Fix a positive integer k, and for an integer λ from (14), i.e., $0 \leq \lambda \leq k$, let

$$\chi_\lambda(\tau) = \operatorname{tr}_{V(k,\lambda)} q^{H_0 - a/24}.$$

Then we have the following special case of (16) (Kac-Peterson):

$$\chi_\lambda\left(-\frac{1}{\tau}\right) = \left(\frac{2}{k+2}\right)^{1/2} \sum_{\lambda'=0}^{k} \left(\sin \pi \frac{(\lambda+1)(\lambda'+1)}{k+2}\right) \chi_{\lambda'}(\tau). \tag{24}$$

The problem of determining all partition functions of the form (23) with $\chi_{(h)}$ replaced by χ_λ was solved by Cappelli-Itzykson-Zuber. A remarkable fact (which is yet to be explained) is that these partition functions are labeled by Dynkin diagrams, so that the constants $a_{\lambda,\lambda'}$ in (23) are expressed in terms of "exponents" associated to these diagrams.

The transformation law (24) has reappeared recently in the beautiful work of Witten on Jones polynomials (see addresses of Witten and Jones).

9. The Ising model. This is a very instructive example of an explicit construction of all modular invariant irreducible representations of the Virasoro algebra with conformal anomaly $c = 1/2$.

Fix $\delta = 0$ (the "Ramond sector") or $\delta = 1/2$ (the "Neveu-Schwarz sector"), and let U_δ be the algebra over $\mathbb{C}$ on anticommuting indeterminates ξ_j $(j \in \delta + \mathbb{Z}_+)$. Define the following operators on U_δ:

$$\psi_0 = 2^{-1/2}\left(\xi_0 + \frac{\partial}{\partial \xi_0}\right); \qquad \psi_n = \frac{\partial}{\partial \xi_n}, \quad \psi_{-n} = \xi_n \quad \text{for } n > 0;$$

$$L_0 = \frac{1-2\delta}{16} + \frac{1}{2} \sum_{j \in \delta + \mathbb{Z}_+} j \psi_{-j} \psi_j, \qquad L_k = \frac{1}{2} \sum_{j \in \delta + \mathbb{Z}} j \psi_{-j} \psi_{j+k} \quad \text{for } k \neq 0.$$

Using that $[\psi_m, L_k] = (m + \frac{1}{2}k)\psi_{m+k}$, we find

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3 - m}{24}.$$

Thus, we have constructed a representation of Vir on U_δ with conformal anomaly $c = 1/2$. This representation is not irreducible: the subspaces U_δ^+ and U_δ^- of elements of even and odd degree respectively are invariant. One can show that

$$(25) \qquad U_{1/2}^+ = V_{1/2,0}, \quad U_{1/2}^- = V_{1/2,1/2}, \quad U_0^+ = U_0^- = V_{1/2,1/16},$$

obtaining thereby all modular invariant irreducible representations of Vir with $c = 1/2$. In fact, we see from (25) that

$$(26) \ \chi_{(0)} \pm \chi_{(1/2)} = q^{-1/48} \prod_{n \in \mathbb{Z}_+} (1 \pm q^{n+1/2}), \qquad \chi_{(1/16)} = q^{1/24} \prod_{n \in \mathbb{Z}_+} (1 + q^{n+1}).$$

This can be rewritten in terms of the η-function:

$$(27) \qquad \begin{gathered} \chi_{(0)} + \chi_{(1/2)} = \frac{\eta(\tau)^2}{\eta(\frac{1}{2}\tau)\eta(2\tau)}, \qquad \chi_{(0)} - \chi_{(1/2)} = \frac{\eta(\frac{1}{2}\tau)}{\eta(\tau)}, \\ \chi_{(1/16)} = \frac{\eta(2\tau)}{\eta(\tau)}. \end{gathered}$$

It is clear from (26) and (27) that the 3-dimensional subspace spanned by $\chi_{(0)}$, $\chi_{(1/2)}$, and $\chi_{(1/16)}$ is invariant with respect to the modular transformations $(1)_0$.

10. Modular invariance versus unitarity. Let ω be a conjugate linear anti-automorphism of $\hat{\mathfrak{g}}$ (resp. Vir) defined by $\omega(x_{(n)}) = {}^t\overline{x}_{(-n)}$, $\omega(k) = k$ (resp. $\omega(L_n) = L_{-n}$, $\omega(c) = c$). A representation of $\hat{\mathfrak{g}}$ or Vir in a vector space V is called unitary if V carries a positive definite Hermitian form for which the operators g and $\omega(g)$ are adjoint ($g \in \hat{\mathfrak{g}}$ or Vir). (For example, taking in the Ising model all monomials in ξ_i for an orthonormal basis, we see that the representations of Vir in U_δ are unitary).

Unitary irreducible positive energy representations of an affine algebra $\hat{\mathfrak{g}}$ are all modular invariant (with weight 0). We have seen that modular invariance is given by certain rationality conditions ((8) and (9)). Unitarity turns out to be given by certain integrality conditions. For example, the modular invariant representations $V(k, \lambda)$ of $\hat{\mathrm{sl}}_2$, given by (13) and (14), are unitary if and only if k and λ are nonnegative integers and $\lambda \leq k$. Representation $V_{c,h}$ of Vir for $c \geq 1$ is always unitary (Kac), and, for $c < 1$ is unitary if and only if (c, h) is one of the pairs from the list (19) such that $|m-n| = 1$ (Friedan-Qiu-Shenker, Goddard-Kent-Olive, Kac-Wakimoto).

The unitary representations of the affine algebra $\hat{\mathfrak{g}}$ always give rise to a representation of the corresponding group $\widehat{G}$ (= a central extension of $G(\mathbb{C}[t, t^{-1}])$ by $\mathbb{C}^\times$). This is not true for nonunitary modular invariant representations. The representation theory of affine algebras is thus much richer than that of the corresponding groups.

The unitary positive energy representations have been much studied in the past decade both by mathematicians and physicists. One may expect that the more universal class of modular invariant representations will keep both groups busy for years to come. I have discussed here only the genus one case and have not even touched the superalgebra case!

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