1. Prime spectrum.
(a) Given a commutative ring \( A \), the **prime spectrum** of \( A \) is given by
\[
\text{Spec}(A) = \{ \text{prime ideals } p \subseteq A \}.
\]
The **Zariski topology** on \( \text{Spec}(A) \) is given by defining the closed subsets to be precisely those the form \( V_S = \{ p \in \text{Spec}(A) : S \subseteq p \} \) for some subset \( S \subseteq A \).

(i) Verify the axioms of topology for this definition.
(ii) A point \( p \in \text{Spec}(A) \) is a **closed point** if \( \{ p \} \) is closed under the Zariski topology on \( \text{Spec}(A) \). Show that the closed points of \( \text{Spec}(A) \) are precisely the maximal ideals of \( A \).

(b) Let \( \phi : A \to B \) be a homomorphism of commutative rings.

(i) Prove that if \( p \subseteq B \) is a prime ideal then \( \phi^{-1}(p) \) is a prime ideal of \( A \). This gives rise to a map \( \phi^\sharp : \text{Spec}(B) \to \text{Spec}(A) \) given by \( \phi^\sharp(p) = \phi^{-1}p \).
(ii) Show that the map \( \phi^\sharp \) defined above is continuous with respect to the Zariski topology on \( \text{Spec}(A) \) and \( \text{Spec}(B) \). (Hint: Given a subset \( S \subseteq A \), how does \( V_S \subseteq \text{Spec} A \) relate to \( V_{\phi(S)} \subseteq \text{Spec} B \)?)
(iii) Given an example of a ring homomorphism \( \varphi : A \to B \) such that there is a maximal ideal \( m \in \text{Spec}(B) \) whose image \( \varphi^\sharp(m) \in \text{Spec}(A) \) is not a maximal ideal.

At first approximation, we may define an **affine scheme** to be a pair of the form \( (\text{Spec}(A), A) \) where \( A \) is a commutative ring and \( \text{Spec}(A) \) is the prime spectrum of \( A \) with the Zariski topology. \( (A \) is the “coordinate ring” of \( \text{Spec}(A) \)). A **morphism** between affine schemes
\[
(\varphi^\sharp, \varphi) : (\text{Spec}(B), B) \to (\text{Spec}(A), A)
\]
is a pair consisting of ring homomorphism \( \varphi : A \to B \) and induced continuous map \( \varphi^\sharp : \text{Spec}(B) \to \text{Spec}(A) \).

(c) Show that \( \text{Spec} \mathbb{Z} \) has dimension 1 as a topological space (one might say that \( \text{Spec} \mathbb{Z} \) is an “arithmetic curve”). Describe all morphisms of affine schemes
\[
(\text{Spec} \mathbb{Z}, \mathbb{Z}) \to (\text{Spec} A, A)
\]
where \( A = \mathbb{Z}[x, y]/(x^2 + y^2 - 1) \).

2. Tangent space.
Let \( k = \overline{k} \) be an algebraically closed field, and let \( V \subseteq \mathbb{A}^n \) be an affine variety over \( k \). Fix a homomorphism \( \varphi_0 : k[V] \to k \) of \( k \)-algebras.
(a) The homomorphism \( \varphi_0 \) corresponds uniquely to some \( P = (a_1, \ldots, a_n) \in V(k) \). Describe the coordinates \( a_i \) of \( P \) in terms of \( \varphi_0 \) and \( k[V] = k[x_1, \ldots, x_n]/I(V) \).

(b) Consider the non-reduced \( k \)-algebra \( k[\epsilon]/(\epsilon^2) \), and let \( \pi : k[\epsilon]/(\epsilon^2) \to k \) denote the homomorphism sending \( \epsilon \) to 0. Note that \( k[\epsilon]/(\epsilon^2) \) is a \( k \)-algebra homomorphism such that the composition

\[
\pi \circ \varphi : k[V] \to k[\epsilon]/(\epsilon^2) \to k
\]

is equal to the homomorphism \( \varphi_0 \) fixed above. If \( m_P = \ker(\varphi_0) \) is the maximal ideal of \( k[V] \) corresponding to \( P \), show that \( m_P^2 \subseteq \ker \varphi \). Deduce that \( \varphi \) induces a \( k \)-linear map

\[
m_P/m_P^2 \to k\epsilon \subseteq k[\epsilon]/(\epsilon^2)
\]

and hence corresponds to a vector in the tangent space \((m_P/m_P^2)^\vee \) of \( V \) at \( P \).

(c) Conversely, if \( T \in (m_P/m_P^2)^\vee \) is a tangent vector, show that

\[
f \mapsto (f(P), T(d_P f)\epsilon) \in k \oplus k\epsilon
\]

defines a \( k \)-algebra homomorphism \( k[V] \to k[\epsilon]/(\epsilon^2) \), and moreover its composition with \( \pi \) recovers \( \varphi_0 \).\(^1\)

3. Projective varieties. Let \( k \) be a field, and \( \bar{k} \) its algebraic closure.

(a) Prove or disprove:

(i) If \( I \leq k[X_0, \ldots, X_n] \) is a homogeneous ideal, so is its radical \( \sqrt{I} \).

(ii) If \( V \subseteq \mathbb{P}^n \) is a projective variety over \( k \) with the 0th affine patch \( V \cap U_0 \), then the projective closure \( (V \cap U_0) \) in \( \mathbb{P}^n \) is \( V \).

(b) Recall that a projective variety \( V \subseteq \mathbb{P}^n \) is smooth if each of its affine patches is smooth (viewed as an affine variety over \( \bar{k} \)). Using the Jacobian criterion for affine varieties, prove the following version of the Jacobian criterion for projective hypersurfaces:

If \( F \in k[X_0, \ldots, X_n] \) is a homogeneous polynomial of degree \( d \geq 1 \), then \( V_F \) is smooth if and only if

\[
\left( \frac{\partial F}{\partial X_0}, \ldots, \frac{\partial F}{\partial X_n} \right)
\]

is nowhere vanishing on \( V_F(\bar{k}) \). (Hint: use Euler’s lemma, which states that

\[
\sum_i X_i \frac{\partial F}{\partial X_i} = d \cdot F.
\]

\(^1\)Combining this with Question 1, we can deduce that the set of morphisms of affine schemes Spec \( k[\epsilon]/(\epsilon^2) \to \text{Spec } k[V] \) arising from \( k \)-algebra homomorphisms is in bijection with the set of tangent vectors to \( V \) at its \( k \)-points. Geometrically, we may view Spec \( k[\epsilon]/(\epsilon^2) \) as a point endowed with an “infinitesimal arrow.”