Please report any typos/mistakes in the problems to the instructor for bonus marks!

1. **Prime spectrum.**
   (a) Given a commutative ring $A$, the **prime spectrum** of $A$ is given by
   \[ \text{Spec}(A) = \{ \text{prime ideals } p \subseteq A \} \].
   
   The **Zariski topology** on $\text{Spec}(A)$ is given by defining the closed subsets to be precisely those the form $V_S = \{ p \in \text{Spec}(A) : S \subseteq p \}$ for some subset $S \subseteq A$.
   
   (i) Verify the axioms of topology for this definition.
   
   (ii) A point $p \in \text{Spec}(A)$ is a **closed point** if $\{ p \}$ is closed under the Zariski topology on $\text{Spec}(A)$. Show that the closed points of $\text{Spec}(A)$ are precisely the maximal ideals of $A$.

   (b) Let $\phi : A \to B$ be a homomorphism of commutative rings.
      
      (i) Prove that if $p \subseteq B$ is a prime ideal then $\phi^{-1}(p)$ is a prime ideal of $A$. This gives rise to a map $\phi^\sharp : \text{Spec}(B) \to \text{Spec}(A)$ given by $\phi^\sharp(p) = \phi^{-1}(p)$.
      
      (ii) Show that the map $\phi^\sharp$ defined above is continuous with respect to the Zariski topology on $\text{Spec}(A)$ and $\text{Spec}(B)$. (Hint: Given a subset $S \subseteq A$, how does $V_S \subseteq \text{Spec} A$ relate to $V_{\phi(S)} \subseteq \text{Spec} B$?)
      
      (iii) Given an example of a ring homomorphism $\phi : A \to B$ such that there is a maximal ideal $m \in \text{Spec}(B)$ whose image $\phi^\sharp(m) \in \text{Spec}(A)$ is not a maximal ideal.

   At first approximation, we may define an **affine scheme** to be a pair of the form $(\text{Spec}(A), A)$ where $A$ is a commutative ring and $\text{Spec}(A)$ is the prime spectrum of $A$ with the Zariski topology. ($A$ is the “coordinate ring” of $\text{Spec}(A)$). A **morphism** between affine schemes
   \[ (\phi^\sharp, \phi) : (\text{Spec}(B), B) \to (\text{Spec}(A), A) \]
   is a pair consisting of ring homomorphism $\phi : A \to B$ and induced continuous map $\phi^\sharp : \text{Spec}(B) \to \text{Spec}(A)$.

   (c) Show that $\text{Spec} \mathbb{Z}$ has dimension 1 as a topological space (one might say that $\text{Spec} \mathbb{Z}$ is an “arithmetic curve”). Describe all morphisms of affine schemes
   \[ (\text{Spec} \mathbb{Z}, \mathbb{Z}) \to (\text{Spec} A, A) \]
   where $A = \mathbb{Z}[x, y]/(x^2 + y^2 - 1)$.

2. **Tangent space.**
   Let $k = \bar{k}$ be an algebraically closed field, and let $V \subseteq \mathbb{A}^n$ be an affine variety over $k$. Fix a homomorphism $\phi_0 : k[V] \to k$ of $k$-algebras.
(a) The homomorphism $\varphi_0$ corresponds uniquely to some $P = (a_1, \ldots, a_n) \in V(k)$. Describe the coordinates $a_i$ of $P$ in terms of $\varphi_0$ and $k[V] = k[x_1, \ldots, x_n]/I(V)$.

(b) Consider the non-reduced $k$-algebra $k[\epsilon]/(\epsilon^2)$, and let $\pi : k[\epsilon]/(\epsilon^2) \to k$ denote the homomorphism sending $\epsilon$ to 0. Note that $k[\epsilon]/(\epsilon^2) = k \oplus k\epsilon$ as a $k$-vector space. Suppose now that $\varphi : k[V] \to k[\epsilon]/(\epsilon^2)$ is a $k$-algebra homomorphism such that the composition $\pi \circ \varphi : k[V] \to k[\epsilon]/(\epsilon^2) \to k$ is equal to the homomorphism $\varphi_0$ fixed above. If $m_P = \ker(\varphi_0)$ is the maximal ideal of $k[V]$ corresponding to $P$, show that $m_P^2 \subseteq \ker \varphi$. Deduce that $\varphi$ induces a $k$-linear map $m_P/m_P^2 \to k\epsilon \subseteq k[\epsilon]/(\epsilon^2)$ and hence corresponds to a vector in the tangent space $(m_P/m_P^2)^*$ of $V$ at $P$.

(c) Conversely, if $T \in (m_P/m_P^2)^*$ is a tangent vector, show that $f \mapsto (f(P), T(d_P f)\epsilon) \in k \oplus k\epsilon$ defines a $k$-algebra homomorphism $k[V] \to k[\epsilon]/(\epsilon^2)$, and moreover its composition with $\pi$ recovers $\varphi_0$.

3. Projective varieties. Let $k$ be a field, and $\bar{k}$ its algebraic closure.

(a) Prove or disprove:

(i) If $I \leq k[X_0, \ldots, X_n]$ is a homogeneous ideal, so is its radical $\sqrt{I}$.

(ii) If $V \subseteq \mathbb{P}^n$ is a projective variety over $k$ with the 0th affine patch $V \cap U_0$, then the projective closure $(V \cap U_0)$ in $\mathbb{P}^n$ is $V$.

(b) Recall that a projective variety $V \subseteq \mathbb{P}^n$ is smooth if each of its affine patches is smooth (viewed as an affine variety over $\bar{k}$). Using the Jacobian criterion for affine varieties, prove the following version of the Jacobian criterion for projective hypersurfaces:

If $F \in k[X_0, \ldots, X_n]$ is a homogeneous polynomial of degree $d \geq 1$, then $V_F$ is smooth if and only if

$$\left( \frac{\partial F}{\partial X_0}, \ldots, \frac{\partial F}{\partial X_n} \right)$$

is not identically zero on $V_F(\bar{k})$. (Hint: use Euler’s lemma, which states that

$$\sum_i X_i \frac{\partial F}{\partial X_i} = d \cdot F.$$"

Combining this with Question 1, we can deduce that the set of morphisms of affine schemes $\text{Spec} k[\epsilon]/(\epsilon^2) \to \text{Spec} k[V]$ arising from $k$-algebra homomorphisms is in bijection with the set of tangent vectors to $V$ at its $k$-points. Geometrically, we may view $\text{Spec} k[\epsilon]/(\epsilon^2)$ as a point endowed with an “infinitesimal arrow.”