1 Syllabus

Spin geometry

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Positive mass theorem

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References


[PT] Parker, T., & Taubes, C. H. (1982). On Witten’s proof of the positive energy theorem. *Communications in Mathematical Physics, 84*(2), 223-238.

Summary of topics covered

**Clifford algebras (Dec 23).** Definition of Clifford algebras, universal property, isomorphism with exterior algebra.

**Pin and Spin groups (Jan 3).** Definition of adjoint representation, definition of spin group, spin group as double cover of special orthogonal group, example: Spin(3) \(\cong\) SU(2) and explicit computation of the map SU(2) → SO(3).

**The algebras \(C\ell_n\), \(C\ell_{r,s}\), and \(C\ell_n\) (Jan 4).** Definition of volume element, the complexification isomorphism \(C\ell_{r,s} \otimes \mathbb{R} \cong C\ell_{r+s}\).

**Classification of Clifford algebras (Jan 6).** Classification of real and complex Clifford algebras, examples in low dimensions: \(C\ell_3 \cong \mathbb{H} \oplus \mathbb{H}\) and \(C\ell_{3,1} \cong \mathbb{H}(2)\), \(C\ell_{2n} \cong \mathbb{C}(2^n)\).

**Representations, part 1 (Jan 8).** Basic representation theory definitions, real and complex spin representation.

**Representations, part 2 (Jan 11).** Existence of the canonical inner product on Clifford modules.

**Lie algebra structures (Jan 13).** The isomorphism \(\text{spin}(n) \cong \mathfrak{so}(n)\).

**Clifford and spin bundles (Jan 15).** Construction of the spin bundle by lifting SO\(n\)-cocycles.
2 Exercises

Exercises labeled with a ⋆ are used in the proof of the positive mass theorem.

Algebraic aspects

1 (Clifford vs. exterior algebra). Let $V$ be a vector space over the field $K = \mathbb{R}$ or $K = \mathbb{C}$, and let $q$ be a quadratic form on $V$. Show that if $e_1, \ldots, e_n$ is a $q$-orthogonal basis of $V$, then the following map is an isomorphism of vector spaces

$$\text{Cl}(V, q) \rightarrow \Lambda^\bullet V, \quad e_{j_1} \cdots e_{j_p} \mapsto e_{j_1} \wedge \cdots \wedge e_{j_p}.$$ 

2 (Connected components of SO). Show that for all $n \geq 1$, SO($n$) is connected, and that SO($n-1, 1$) has exactly two connected components, where SO($r, s$) is the Lie group

$$\text{SO}(r, s) = \{ \lambda \in \text{GL}(\mathbb{R}^n) \mid \lambda^* q = q, \det(\lambda) = 1 \},$$

and $q$ is the quadratic form on $\mathbb{R}^n$ given by

$$q(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2.$$ 

Hint: Use the Cartan-Dieudonné theorem to find paths connecting elements in SO($r, s$) to either plus or minus the identity.

3 (⋆ SU(2) is double cover of SO(3)). Prove that there exists a homomorphism $\xi : \text{SU}(2) \rightarrow \text{SO}(3)$ which is surjective and has kernel $\{1, -1\} \subset \text{SU}(2)$.

Hint: Show first that the Lie algebra $\mathfrak{su}(2)$ of $\text{SU}(2)$ is isomorphic to the 3-dimensional real vector space of traceless, skew-hermitian $2 \times 2$ complex matrices, which has a basis given by

$$(2.1) \quad \sigma_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$ 

Then show that for all $U \in \text{SU}(2)$, the adjoint action $\text{Ad}_U : \mathfrak{su}(2) \rightarrow \mathfrak{su}(2)$ by $U$, defined for $X = X_i \sigma_i$, by

$$(2.2) \quad \text{Ad}_U(X) = UXU^{-1},$$

is an element of SO(3), is a homomorphism $\text{SU}(2) \rightarrow \text{SO}(3)$, is surjective, and has kernel $\{1, -1\}$.

4 (Real vs. complex Clifford algebras). Let $q$ and $q^\mathbb{C}$ be the non-degenerate quadratic forms on $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively, defined by

$$q(x) = \sum_{i=1}^r x_i^2 - \sum_{i=r+1}^s x_i^2, \quad q^\mathbb{C}(z) = \sum_{i=1}^n z_i^2.$$
Consider the Clifford algebras
\[ \mathbb{C}\ell_{r,s} = \mathbb{C}\ell(\mathbb{R}^n, q), \quad \mathbb{C}\ell_n = \mathbb{C}\ell(\mathbb{C}^n, q^\mathbb{C}). \]

Show that there exists an isomorphism
\[ \mathbb{C}\ell_{r,s} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\ell_n. \]

In particular,
\[ \mathbb{C}\ell_n \cong \mathbb{C}\ell_{n,0} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\ell_{n-1,1} \otimes_{\mathbb{R}} \mathbb{C} \cong \ldots \cong \mathbb{C}\ell_{0,n} \otimes_{\mathbb{R}} \mathbb{C}. \]

Hint: Use the universal property of Clifford algebras.

5 (Canonical representation of $\mathbb{C}\ell_n$). Prove that the complex Clifford algebra $\mathbb{C}\ell_{2n}$ is isomorphic to the matrix algebra $\mathbb{C}(2^n)$, and that $\mathbb{C}\ell_{2n+1}$ is isomorphic to $\mathbb{C}(2^n) \oplus \mathbb{C}(2^n)$.

6 (Canonical representation of $\mathbb{C}\ell_3$ and $\mathbb{C}\ell_{3,1}$). Prove that $\mathbb{C}\ell_3$ is isomorphic to $\mathbb{H} \oplus \mathbb{H}$ and that $\mathbb{C}\ell_{3,1}$ is isomorphic to the matrix algebra $\mathbb{H}(2)$.

7 (★ Exceptional isomorphisms). Prove that there exists an isomorphism $\text{Spin}(3) \cong \text{SU}(2)$.

8 (★ Exceptional isomorphisms). Prove that there exists a diffeomorphism $\text{SU}(2) \to S^3$, where $S^3 \subset \mathbb{R}^4$ is the set of unit vectors.

9 (★ Exceptional isomorphisms). Prove that there exists an isomorphism $\text{Spin}(3, 1) \cong \text{SL}(2, \mathbb{C})$.

10 (Complex volume element). Let $e_1, \ldots, e_n$ be a positively oriented orthonormal basis of $\mathbb{R}^n$ (with respect to the standard inner product) and let
\[ \omega^\mathbb{C} := i^{n+1} e_1 \cdots e_n \in \mathbb{C}\ell_n \]
be the complex volume element, where "·" denotes the product in the Clifford algebra $\mathbb{C}\ell_n$. Show that
\[ (\omega^\mathbb{C})^2 = 1, \]
and, for all $x \in \mathbb{R}^n$, there holds
\[ x \cdot \omega^\mathbb{C} = (-1)^{n-1} \omega^\mathbb{C} \cdot x. \]

Hint: Use the Clifford algebra relation $e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}1$.

11 (★ 3D Spin representation). Prove that the spin representation of $\text{Spin}(3)$ is the standard representation of $\text{SU}(2)$ on $\mathbb{C}^2$. 

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12 (Spinorial inner product). For $n$ even, let $\rho : \mathbb{C}\ell_n \to \text{End}(\Sigma_n)$ be the unique irreducible representation for the complex Clifford algebra, and for $n$ odd, let $\rho_{\pm} : \mathbb{C}\ell_n \to \text{End}(\Sigma_n)$ be the two inequivalent irreducible representations, where $\Sigma_n$ is a complex vector space with

\begin{equation}
\dim_{\mathbb{C}} \Sigma_n = 2^{\lfloor \frac{n}{2} \rfloor}.
\end{equation}

Construct a Hermitian inner product $\langle \cdot , \cdot \rangle$ on $\Sigma_n$ with respect to which Clifford multiplication is orthogonal, i.e. such that for all $x \in \mathbb{R}^n$ and all $\varphi, \psi \in \Sigma_n$, there holds

\begin{equation}
\langle \rho(x)\varphi, \rho(x)\psi \rangle = \|x\|^2\langle \varphi, \psi \rangle.
\end{equation}

Show that this inner product is unique, up to scaling by a constant factor.

*Hint:* See Proposition I.5.16 of [LM] or Proposition 1.35 of [B] for help.

13 (★ 3D Clifford multiplication). Prove that the Clifford multiplication map $c : \mathbb{R}^3 \to \text{End}(\mathbb{C}^2)$ is given by

$$c(x, y, z) = \begin{pmatrix} ix & iy + z \\ iy - z & -ix \end{pmatrix} \in \mathfrak{su}(2).$$

14 (★ Lie algebra representation of Spin($n$)). Prove formula (A.1) of Lee-Parker [LP]; that is, prove that the Lie algebra representation $\text{spin}(n) \to \text{End}(V)$ can be written in terms of Clifford multiplication as follows:

$$A \mapsto -\frac{1}{4}A_{ij}c(e^i)c(e^j) = -\frac{1}{8}A_{ij}[c(e^i), c(e^j)],$$

where $\{e^i\}$ is the standard basis of $\mathbb{R}^n$.

*Hint:* For help, see Proposition 6.2 of Lawson-Michelsohn [LM] or Theorem 1.25 of Bourguignon et al. [B].

**Geometric aspects**

Let $(M, g)$ be a Riemannian spin $n$-manifold and let $\{e_i\}$ be a local orthonormal frame of $TM$ around $p \in M$, with dual coframe $\{e^i\}$.

15 (★ Spin connection in local coordinates). Prove that the spin connection $\nabla$ can locally be written as

$$\nabla_i \psi = \partial_i \psi + \frac{1}{4} \sum_{j,k=1}^n \Gamma^k_{ij} c(e_j)c(e_k)\psi,$$

where $\Gamma^k_{ij} := g(\nabla_i e_j, e_k)$ are the Christoffel symbols and $c$ denotes Clifford multiplication.

*Hint:* See Theorem 2.7 of Bourguignon et al. [B] for help.
16 (★ Spin curvature in terms of Riemannian curvature). Prove that if
\[ \mathcal{R}_{X,Y} := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \]
is the curvature of the spin connection, then for any spinor \( \psi \),
\[ \mathcal{R}_{X,Y} \psi = \frac{1}{4} \sum_{i,j=1}^{n} g(R_{X,Y}e_i, e_j)c(e_i)c(e_j)\psi, \]
where \( R \) is the curvature of the Levi-Civita connection on \( TM \).

*Hint*: Use Exercise 14. See Theorem 2.7 of Bourguignon et al. [B] for help.

17 (★ Lichnerowicz’ vanishing theorem). Prove that if \( M \) is closed (i.e. compact with empty boundary) and has positive scalar curvature, then the Dirac operator of \((M, g)\) has trivial kernel.

*Hint*: Use the Schrödinger-Lichnerowicz formula for the Dirac operator.