18.100Q Recitation - Continuity

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Let X and Y be metric spaces.

Definition 1 (Continuity; limit points)**.** A function $f : X \to Y$ is *continuous on* X if for all $x \in X$ and all $\epsilon > 0$, there exists $\delta = \delta(x, \epsilon) > 0$ such that if $x' \in X$ satisfies $d_X(x, x') < \delta$, then $d_Y(f(x), f(x')) < \epsilon$.

Definition 2 (Continuity; open sets)**.** A function $f : X \to Y$ is *continuous on* X if for all open subsets $U \subset Y$, the preimage $f^{-1}(U)$ is open in X.

Definition 3 (Uniform continuity)**.** A function $f : X \to Y$ is *uniformly continuous on* X if for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $x, x' \in X$ have $d_X(x, x') < \delta$, then $d_Y(f(x), f(x')) < \epsilon$.

Exercises

1. (Continuous functions on compact sets.) Let X and Y be metric spaces with X compact and let $f : X \to Y$ be a continuous function.

   (a) Prove that $f(X) \subset Y$ is compact. (*Hint:* Use the definition of continuity in terms of open sets.)

   (b) Let $Y = \mathbf{R}$. Show that f attains a maximum and a minimum on X, i.e. there exists points $x_*, x^* \in X$ such that

   $$f(x_*) = \inf_{x \in X} f(x), \qquad f(x^*) = \sup_{x \in X} f(x). \tag{1}$$

   (*Hint:* Use the Heine-Borel theorem.)

   (c) Give an example of a continuous function $f : (-1, 1) \to \mathbf{R}$ which is bounded, but which does not attain a maximum on $(-1, 1)$.

2. (Continuity vs. uniform continuity.) Give an example of a continuous function which is *not* uniformly continuous.

3. (Equivalent definition of uniform continuity.) Let X and Y be metric spaces. Prove that a function $f : X \to Y$ is uniformly continuous if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $A \subset X$ has $\text{diam}(A) < \delta$, then $\text{diam} f(A) < \epsilon$.

4. (Fixed point theorem.) Let $f : [a, b] \to [a, b]$ be a continuous function. Prove that there exists $x_0 \in [a, b]$ such that $f(x_0) = x_0$. (*Hint:* Use the intermediate value theorem.)