18.100Q Recitation - Proof techniques

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Instructions

Please write up proofs of *five* of the following exercises. Do *at least three* of the last five exercises (concerning supremum and infimum). Write proofs as if you were writing them for a paper. Explain the result in a level of detail befitting a cutting-edge result. Unlike an actual paper, you will not need to introduce the result in any way, or write any kind of history of the problem, or any follow-up/future work section, etc.

Readings

For background on set theory and logic, see for example Appendices A and B of Arthur Mattuck's textbook *Introduction to Analysis*. For background on mathematical proof techniques, see Appendices A and B of Mattuck's book, as well as the notes on proof writing in the Exercise 1 page on Canvas.

Exercises

1. Let n be a composite (non-prime) integer greater than 1. Show that n has a factor x such that $1 < x \leq \sqrt{n}$.

2. Discuss the flaw in the argument below.

   Here is a proof that all cows are the same colour. We will use induction on the size of a herd of cows. Consider any herd of just 1 cow. Clearly, every cow in this herd has the same colour. Now suppose we have shown that any herd of n cows is monochromatic. Consider a herd of n + 1 cows. Take one of the cows out of the herd and put it out to pasture, leaving a herd of n cows; by the inductive hypothesis, all n have the same colour (for the sake of argument, let's suppose this colour is purple). Now take one of these purple cows out, and bring back the cow from the pasture; the herd again has n cows, and so all n must be purple (again by the inductive hypothesis). This means that the originally removed cow must be purple, and so all n + 1 cows are purple. This shows that any group of n + 1 cows is monochromatic, and concludes the proof.

3. Show that if X is an infinite subset of $\mathbf{N}$, then there is a bijection from X to $\mathbf{N}$.

4. (Cantor's diagonal argument.) Show that there exists no surjection $\mathbf{N} \to \mathbf{R}$ and therefore that there exist distinct infinite cardinalities.

5. This problem shows that the sets $2^{\mathbf{N}}$ and $\mathbf{R}$ are in bijection.

   (a) Show that for any nonempty set A, there exists no surjection from A to the power set 2^A, i.e. the cardinality of the power set of A is strictly greater than the cardinality of A.

   (b) Argue that the relation of sets $A \sim B$ if there is a bijection $A \to B$ is an equivalence relation among sets (i.e. it is reflexive, symmetric, and transitive).

   (c) Argue there is a bijection $2^{\mathbf{N}} \to [0,1]$. *Hint:* Let f be the map given in recitation, defined by

   $$\begin{aligned} f : 2^{\mathbf{N}} &\to [0,1] \\ (\sigma_i)_{i=1}^{\infty} &\mapsto \sum_{i=1}^{\infty} \sigma_i \frac{1}{2^i} \end{aligned}$$

   where $\sigma_i \in \{0,1\}$.

   Given f, one can define the set $S = \left\{\sigma \in 2^{\mathbf{N}} \mid \text{s.t. there exists } \sigma' \neq \sigma \text{ with } f(\sigma) = f(\sigma')\right\}$. For instance, $011111\ldots$ and 10000 are in this set because $f(01111\ldots) = f(10000\ldots) = 1/2$. But, $10101010\ldots$ is not a member of S. Describe the set S, and its image, $f(S)$.

   Note that f is not a bijection along S. Construct a bijection $g : S \to f(S)$, and define

   $$\begin{aligned} \hat{f} : 2^{\mathbf{N}} &\to [0,1] \\ \sigma &\mapsto \begin{cases} f(\sigma) & \sigma \notin S \\ g(\sigma) & \sigma \in S \end{cases} \end{aligned}$$

   and deduce that $\hat{f}$ is a bijection.

   (d) Argue that there is a bijection $[0,1] \to (0,1)$.

   (e) Find a bijection $(0,1) \to \mathbf{R}$.

   (f) Conclude that there exists a bijection $2^N \to \mathbf{R}$.

6. The below result is a useful source of counterexamples.

   (a) Prove

   $$\lim_{x \to \infty} p(x) e^{-x} \;=\; 0$$

   for any polynomial $p(x)$.

   (b) Show that the piecewise function

   $$f(x) \;=\; \begin{cases} 0 & x \leq 0 \\ e^{-1/x} & x > 0 \end{cases}$$

   is infinitely differentiable at 0, with all derivatives at the origin being 0. You may assume it is smooth on $\{x > 0\}$.

(c) Deduce that $f(x)$ is infinitely differentiable everywhere, but is not equal to its power series in any ball of positive radius centered at 0.

7. Show that an ordered set satisfies the least upper bound property if and only if it satisfies the greatest lower bound property.

8. Let $A \subset \mathbf{R}$ and let $f, g : A \to \mathbf{R}$ be bounded functions. Show that

   (a) $\sup (f+g)(A) \leq \sup f(A) + \sup g(A)$,

   (b) $\inf (f+g)(A) \geq \inf f(A) + \inf g(A)$,

   (c) $\sup -f(A) = -\inf f(A)$,

   (d) $\sup (f-g)(A) \leq \sup f(A) - \inf g(A)$.

9. Let $A \subset \mathbf{R}$ be a bounded set.

   (a) Let $\alpha = \sup A$. Show that for any $\epsilon > 0$, there exists $x \in A$ such that $x > \alpha - \epsilon$.

   (b) Let $\beta = \inf A$. Show that for any $\epsilon > 0$, there exists $x \in A$ such that $x < \beta + \epsilon$.

   (c) Show that there exist sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ in A such that $x_i \to \alpha$ and $y_i \to \beta$ as $i \to \infty$.

10. Show that any bounded and monotone sequence $(x_i)_{i \in \mathbf{N}} \subset \mathbf{R}$ is convergent.

11. (Nested interval theorem.) For each $n \in \mathbf{N}$, let $I_n = [a_n, b_n]$ be nonempty and such that $I_{n+1} \subset I_n$. Then $\bigcap_{n \in \mathbf{N}} I_n \neq \emptyset$.