18.100Q RECITATION - ORDER OF QUANTIFIERS

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1. Part A

Instructions: Spend no more than 25 minutes on this section. Unless there are other instructions, do the following for each item:

For the Definitions:

- Write out the definition in formal logical notation.
- Write out the negation of the definition in formal logical notation.
- Give an example and a non-example for the definition.
- Follow any further instructions on a problem.

For the Theorems: (these will be labeled "Theorem")

- Write out the Theorem in formal logical notation.
- Show how the Theorem fails to hold if you relax one or more of the assumptions.
- Write the contrapositive of the Theorem (the contrapositive of $A \Rightarrow B$ is $\neg B \Rightarrow \neg A$.)

Below, unless otherwise specified, X and Y are metric spaces.

(1) The map $f : A \to B$ is *surjective*
(2) (Theorem) The real number line is the unique ordered field which contains the rationals and satisfies the least upper bound property.
(3) For $A \subset \mathbb{R}$, a is the *least upper bound* of A.
(4) The set $A \subset X$ is *open*.
(5) The *closure* of $A \subset X$ is the smallest closed set containing A.
(6) (Theorem) Closed subsets of compact sets are compact.
(7) The point $a \in A \subset X$ is a *limit point* of A.
(8) A subset $A \subset X$ is *connected*.
(9) The sequence $(x_n)_{n\in\mathbf{N}} \subset X$ has *limit* x.
(10) The sequence $(x_n)_{n\in\mathbf{N}} \subset X$ is *bounded*.
(11) The sequence $(x_n)_{n\in\mathbf{N}} \subset X$ has a *convergent subsequence*.
(12) The sequence $(x_n)_{n\in\mathbf{N}} \subset X$ is *Cauchy*.
(13) The sequence $(x_n)_{n\in\mathbf{N}} \subset \mathbb{R}$ has *upper limit* s^* (or $\limsup_n x_n = s^*$, if you prefer).
(14) The point x is an *accumulation point* of $A \subset X$.
(15) The space X is *complete*.
(16) (Theorem) The Monotone Convergence Theorem.
(17) The function $f : \mathbb{R} \to \mathbb{R}$ is *monotone increasing*.
(18) (Theorem) The set of discontinuities of a monotone function $f : (a, b) \to \mathbb{R}$ is at most countable.
(19) The function $f : X \to Y$ is *continuous*.
(20) The function $f : X \to Y$ is *uniformly continuous*.
(21) (Theorem) If X is compact and $f : X \to Y$ is continuous, then $f(X) \subset Y$ is compact.
(22) (Theorem) If $f : (a, b) \to \mathbb{R}$ is uniformly continuous, then f is bounded.
(23) The real number $f'(x)$ is the *derivative* of $f : (a, b) \to \mathbb{R}$ at $x \in (a, b)$.

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(24) (Theorem) If $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq |x - y|^\alpha$ for all $x, y \in \mathbb{R}$, where $\alpha > 1$ is a constant, then f is constant.

2. Part B

Instructions: Spend no more than 15 minutes on the following section.

A *sequence of real numbers* is a function $a : \mathbf{N} \to \mathbf{R}$; by convention, the value $a(n)$ is denoted a_n and the sequence itself is often denoted $(a_n)_{n\in\mathbf{N}}$.

Following is a list of six formal statements, (a)-(f), about a sequence of real numbers $(a_n)_{n\in\mathbf{N}}$, along with six example sequences (1)-(6). For each of (a) through (f), determine which if any of the sequences (1) through (6) satisfy the given property. Then describe in informal (English) language what each property means.

Properties of a sequence a_n

(a) $\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N\ |a_n| < \epsilon$

(b) $\forall \epsilon > 0\ \forall N \in \mathbb{N}\ \forall n \geq N\ |a_n| < \epsilon$

(c) $\exists N \in \mathbb{N}\ \forall n \geq N\ \forall \epsilon > 0\ |a_n| < \epsilon$

(d) $\forall N \in \mathbb{N}\ \exists n \geq N\ \forall \epsilon > 0\ |a_n| < \epsilon$

(e) $\forall N \in \mathbb{N}\ \exists n \geq N\ \exists \epsilon > 0\ |a_n| < \epsilon$

(f) $\exists \epsilon > 0\ \forall N \in \mathbb{N}\ \exists n \geq N\ |a_n| < \epsilon$

Sequences a_n

(1) $a_n = 0$ if n is prime, $a_n = n$ otherwise.

(2) $a_n = 0$

(3) $a_n = 1/(n+1)$

(4) $a_0 = a_1 = a_2 = a_3 = 3$, $a_n = 0$ for $n \geq 4$

(5) $a_n = \frac{\sqrt{2}}{2} + \sin(\pi n/100)$

(6) $a_n = 10^{10^n}$