18.100Q RECITATION - EXAM 2 REVIEW

JULIUS BALDAUF

1. Exam remarks

1.1. **Exam content.** The purpose of the second midterm is to test your familiarity with some material we have learned since the first midterm. Thus, the exam will focus on *Chapters 4 and 5 of Rudin's textbook*. However, since this is a math class, the material builds on itself, so you still need to know the concepts from the first part of the course. For example, you cannot know that the image of a compact set under a continuous function is compact without knowing what a compact set is. Likewise, you cannot prove that something is countable without knowing what cardinality is. So, while I omit the previous material from the study guide, you should still review those concepts, especially in their relation to the more recent concepts we have learned.

1.2. **On definitions.** You have to know the definitions in order to do well on a pure math exam. If you know the definitions but nothing else, you have a shot at solving every problem. If you don't know the definitions, you won't even be able to start. Trying to memorize all the definitions will be quite difficult. Instead, do lots of practice problems. Create examples which satisfy parts of the definition but not other parts. Working with the definitions will give you a feel for them, and make it easier to remember them. If you don't know how to solve a problem on an exam, state relevant definitions. You may be awarded partial credit.

2. Continuity

2.1. **Definitions/Theorems you should know.**

(1) The $\epsilon - \delta$ definition of limit for a function, including for left/right limit.
(2) The $\epsilon - \delta$ definition of continuity.
(3) A function is continuous at an accumulation point x_0 of the domain, iff $\lim_{x \to x_0} f(x) = f(x_0)$.
(4) A function is continuous iff the pre-image of open sets are open. Same with closed sets.
(5) The image of a compact set under a continuous map is compact.
(6) The intermediate value theorem.
(7) The definition of uniform continuity
(8) A continuous function with compact domain is uniformly continuous.
(9) A continuous function from a compact set to the real numbers attains its inf and sup.
(10) Classification of discontinuities
(11) Monotonic functions from $\mathbb{R}$ to $\mathbb{R}$ can only have jump discontinuities and there are at most countably many discontinuities.

Date: November 5, 2021.

2.2. **Things you should be able to do.**

1. Check that a function given to you explicitly (e.g. $f(x) = x^2$) is continuous.
2. Prove that compositions, sums, products etc of continuous functions are continuous.
3. Prove topological properties using continuity.
4. Use continuity to prove that a limit exists.
5. Give an example of a continuous function which isn't uniformly continuous. Give an example of a continuous function which doesn't attain a max or a min. Etc.
6. Find extrema using continuity and compactness (e.g. find a closest point in a set or a point with smallest x-coordinate etc).
7. Use the intermediate value theorem to solve an equation.
8. Use definition to prove a function is uniformly continuous.
9. Determine the type of discontinuities.

2.3. **Practice Problems.**

1. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous at $x = x_0$.
   (a) Show that if $f(x_0) > g(x_0)$, then there is a neighborhood of x_0 such that $f > g$ in that neighborhood.
   (b) If there exists a neighborhood $N_r(x_0)$ such that $f(x) > g(x)$ for each $x \in N_r(x_0) \setminus \{x_0\}$, show that $f(x_0) \geq g(x_0)$.
2. Suppose that for each $\varepsilon > 0$, the function f is continuous in $(a+\varepsilon, b-\varepsilon)$. Is it true that f is continuous in (a, b)? Prove your claim.
3. Show $\sin(x)$ is a uniformly continuous function on $(-\infty, \infty)$. (*Hint:* you might want to use the Mean Value Theorem in "Derivatives".)
4. Let f, g be uniformly continuous. Prove that $f \circ g$ is uniformly continuous. What can you say about $f \circ g$ if f is uniformly continuous but g is merely continuous? What if f is continuous and g is uniformly continuous?
5. Prove that the set of discontinuities of a monotone function $f : (a, b) \to \mathbb{R}$ is at most countable.
6. Suppose f is a monotone increasing function on $[a, b]$ and $f([a, b]) = [f(a), f(b)]$. Is f necessarily continuous? Prove your claim.
7. Let $f : X \to Y$ be a continuous function between metric spaces.
   (a) Prove that if X is compact, then $f(X) \subset Y$ is compact.
   (b) Give an example demonstrating that $f(X)$ may fail to be compact if X is not compact.
8. A real continuous function defined on a compact metric space attains both a maximum and a minimum.
9. If $f : (a, b) \to \mathbb{R}$ is uniformly continuous, then f is bounded.

3. Derivatives

3.1. **Definitions/Theorems you should know.**

1. What does it mean to be differentiable at a point.
2. Chain rule, product rule, quotient rule, etc.
3. The mean value theorem
4. L'Hospital's rule (also spelled as L'Hôpital, which justifies the pronunciation)
5. Higher order derivatives.

(6) Taylor's theorem

3.2. Things you should be able to do.

(1) Compute the derivative of a function given to you explicitly.
(2) Compute the derivative of a function using chain rule or product rule etc
(3) Give an example of a function which is continuous but not differentiable.
(4) Give an example of a function which is differentiable but the derivative is not continuous.
(5) Apply the mean value theorem to prove that a derivative has a certain value.
(6) Use the mean value theorem to estimate the differences of a function at two points.
(7) Use the derivative of a function to say something about its limit at ∞.
(8) Apply L'Hospital's rule.
(9) Compute the higher order derivatives of a function given to you explicitly.

3.3. Practice Problems.

(1) If $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq |x - y|^\alpha$ for all $x, y \in \mathbb{R}$, where $\alpha > 1$ is a constant, then f is constant.
(2) (Inverse function theorem.) Let $f : (a, b) \to \mathbb{R}$ be differentiable with $f'(x) > 0$ on (a, b).
   (a) Prove that f is strictly increasing in (a, b).
   (b) Let g be the inverse of f (which exists by the monotonicity of f). Prove that g is differentiable and that, for all $x \in (a, b)$, there holds
$$g'(f(x)) = \frac{1}{f'(x)}.$$
(3) Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to \infty$. Put $g(x) = f(x + 1) - f(x)$. Prove that $g(x) \to 0$ as $x \to \infty$.
(4) Let $f : (a, \infty) \to \mathbb{R}$ be a twice-differentiable and let M_0, M_1, M_2 be the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively, on (a, ∞). Assuming M_0 and M_2 are both finite, prove that
$$M_1^2 \leq 4M_0M_2.$$
(5) Give an example of a function $f : [a, b] \to \mathbb{R}$ which is differentiable, but whose derivative is unbounded. (*Hint:* The derivative of such a function is necessarily discontinuous, so that the function is *not* continuously differentiable. You may consider the function $x^2 \sin(x^{-2})$ on $[-1, 1]$.)
(6) Let $f : [a, b] \to \mathbb{R}$ be continuously differentiable on $[a, b]$.
   (a) Show that f is Lipschitz continuous, i.e. that there exists a constant $L \geq 0$ such that
$$|f(x) - f(y)| \leq L|x - y|$$
   holds on $[a, b]$, and state the optimal Lipschitz constant.[1] (*Hint:* Use the mean value theorem.)
   (b) Give an example of a function which is Lipschitz continuous on $[-1, 1]$, but not differentiable on $[-1, 1]$.
(7) Suppose $f : [a, b] \to \mathbb{R}$ is differentiable, $f(a) = 0$, and there is a real number $A \geq 0$ such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (*Hint:* Prove that for any $x_0 \in [a, b]$, there holds
$$|f(x)| \leq A|x_0 - a| \max_{x \in [a, x_0]} |f(x)|.$$

[1] The constant $L \geq 0$ is the *optimal* Lipschitz constant of f if $L \leq L'$ holds for any other Lipschitz constant L' of f.

Hence $f = 0$ on $[a, x_0]$ if $A|x_0 - a| < 1$. Proceed.)