18.100Q RECITATION - EXAM 3 REVIEW: INTEGRATION AND UNIFORM CONTINUITY

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1. Exam remarks

1.1. **Exam content.** The purpose of the second midterm is to test your familiarity with some material we have learned since the first and second midterms. Thus, the exam will focus on *Chapters 6 and 7 of Rudin's textbook*. However, since this is a math class, the material builds on itself, so you still need to know the concepts from the previous parts of the course. So, while I omit the previous material from the study guide, you should still review those concepts, especially in their relation to the more recent concepts we have learned.

1.2. **On definitions.** You have to know the definitions in order to do well on a pure math exam. If you know the definitions but nothing else, you have a shot at solving every problem. If you don't know the definitions, you won't even be able to start. Trying to memorize all the definitions will be quite difficult. Instead, do lots of practice problems. Create examples which satisfy parts of the definition but not other parts. Working with the definitions will give you a feel for them, and make it easier to remember them. If you don't know how to solve a problem on an exam, state relevant definitions. You may be awarded partial credit.

2. Integration

2.1. **Definitions/Theorems you should know.**

(1) What is a partition
(2) What is an upper/lower sum
(3) What are the upper/lower integrals? What does it mean for a function to be integrable?
(4) Characterizations of integrability with sequences of partitions
(5) Linearity of the integral
(6) Triangle inequality for the integral
(7) Change of variables formula for integrals
(8) Integration by parts
(9) Fundamental Theorem of Calculus
(10) Properties of $F(x) = \int_a^x f(t)dt$ (e.g. continuity, when does the derivative exist? What is it equal to?)

2.2. **Things you should be able to do.**

(1) Work with the partition defintion to prove that an integral exists

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(2) Work with the partition definition to prove that an integral has certain properties (i.e. is large or small, or grows a certain way)
(3) Prove that a given function is not integrable
(4) Give an example of a function which is integrable but not continuous
(5) Show that a continuous function is integrable.
(6) Show that a bounded function with finitely many discontinuities is integrable.
(7) Evaluate integrals using the change of variables formula
(8) Evaluate integrals using the Fundamental Theorem of Calculus and/or integration by parts
(9) Prove properties of the function $F(x) = \int_a^x f(t)dt$. Understand when it is differentiable.

2.3. **Practice Problems.**

(1) Suppose $x_0 \in [a, b]$, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f\,dx = 0$.
(2) Are the following functions in $\mathcal{R}$? If they are, argue what the integral should be.
   (a) The function f, defined on $[0, \infty)$ by
   $$f(x) = \begin{cases} 1 & x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$
   (b) The function g, defined on $[0, 1]$ by
   $$g(x) = \begin{cases} 1 & x = 1/2^k \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$
   (c) The function h defined on $[0, 1]$ by
   $$h(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$
(3) Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f\,dx = 0$. Prove that $f = 0$ on $[a, b]$.
(4) Suppose f is a bounded *real* function on $[a, b]$, and $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$? Does the answer change if we assume $f^3 \in \mathcal{R}$? What about f^4?.
(5) Suppose $f \in \mathcal{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define
   $$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx \tag{2.1}$$
   if this limit exists and is finite. In that case, we say the integral on the left *converges*. If the integral also converges after f has been replaced by $|f|$, it is said to *converge absolutely.*
   (a) In the setup above, give an example of an f, defined on $[a, \infty)$ such that $f \in \mathcal{R}$ on $[a, b]$ for every $b > a$, but such that the integral in (2.1) does not converge.
   (b) Assume that $f(x) \geq 0$ and that f decreases monotonically. Prove that
   $$\int_1^\infty f(x)dx$$
   converges if and only if
   $$\sum_{n=1}^\infty f(n)$$
   converges. This is the "integral test" for convergence of series.

3. Uniform convergence

3.1. Definitions/Theorems you should know.

(1) Pointwise convergence
(2) Uniform convergence
(3) Equicontinuity
(4) The uniform limit of continuous functions is continuous.
(5) Uniform limits and integrals commute.
(6) How uniform limits interact with differentiability
(7) Uniform limits and limits of sequences

3.2. Things you should be able to do.

(1) Verify a convergence is uniform.
(2) Construct a sequence of functions which converge pointwise but not uniformly and for which you cannot interchange limits (e.g. $f_n(x_n) \not\to f(x)$ even though $f_n \to f$ pointwise and $x_n \to x$). Similarly with interchanging the integral and the limit, the derivative and the limt, etc.
(3) Verify equicontinuity.

3.3. Practice Problems.

(1) Let $f_n : X \to \mathbb{R}$ be a sequence of bounded functions on a set X converging uniformly to a function $f : X \to \mathbb{R}$. Prove that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded.
(2) Construct sequences of functions $\{f_n\}$ and $\{g_n\}$ which converge uniformly on some set X, but for which the sequence $\{f_n g_n\}$ does not converge uniformly on X.
(3) Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a metric space X.
   (a) Prove that
$$\lim_{n\to\infty} f_n(x_n) = f(x)$$
   for every sequence of points $\{x_n\} \subset X$ such that $x_n \to x$.
   (b) Is the converse of (a) true?
(4) Let $f_n : \mathbb{Q} \to \mathbb{R}$ be continuous functions which converge uniformly to f, which is continuous on $\mathbb{R}$. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $(q_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$ be a sequence such that $q_n \to \alpha$. Show that $\lim_{n\to\infty} f_n(q_n)$ exists.
(5) Let $\alpha \in (0, 1)$. Recall that a function $f : [a, b] \to \mathbb{R}$ is α-Hölder continuous if there exists $M \geq 0$ such that
$$|f(x) - f(y)| \leq M|x - y|^\alpha, \tag{3.1}$$
for all $x, y \in [a, b]$.

   Let f_n be a sequence of α-Hölder continuous functions which converge uniformly to f on $[a, b]$. Is f necessarily α-Hölder continuous? If so, provide a proof; if not, provide a counter-example.
(6) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, 2π-periodic function. For each $n \in \mathbb{N}$, define $f_n(x) = f(x + \frac{1}{n})$ for $x \in \mathbb{R}$.
   (a) Show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges *uniformly* to f on $\mathbb{R}$.
(7) Let f_n be a decreasing sequence of positive continuous functions on $[0, 1]$; that is, $f_n(x) \geq f_{n+1}(x)$ for all $x \in [0, 1]$ and $f_n \in C([0, 1])$.

(a) Prove that f_n converges pointwise to some function f on $[0, 1]$.
(b) Assume further that f is continuous. Prove that $\int_0^1 f_n \, dx \to \int_0^1 f \, dx$.
(c) Does $f_n \to f$ uniformly on $[0, 1]$?