MATH 18.177 EXERCISES

Problem 1 Recall that for a finite, connected graph $G = (V \cup \partial V, E)$ with $\partial V \neq \emptyset$, the discrete Gaussian free field (DGFF) h on G is the random function $h \colon V \cup \partial V \to \mathbf{R}$ with density

$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)$$

with respect to Lebesgue measure on $\mathbf{R}^{|V|}$ and constrained to satisfy $h|_{\partial V} \equiv 0$. The sum is over all pairs $\{x, y\} \in E$ and $\mathcal{Z}$ is a normalizing constant so that this defines a probability measure.

Suppose that $D \subseteq \mathbf{C}$ is bounded. Recall also that for $s \in \mathbf{R}$ we defined

$$(-\Delta)^s f = \sum_n (-\lambda_n)^s \alpha_n \phi_n \quad \text{for} \quad f = \sum_n \alpha_n \phi_n$$

where (λ_n) are the negative eigenvalues of Δ and (ϕ_n) are the corresponding eigenvectors, normalized to be an orthonormal basis of $L^2(D)$. We then set

$$(f, g)_{H^s} = ((-\Delta)^{s/2} f, (-\Delta)^{s/2} g)_{L^2}$$

and let $H^s(D)$ be the closure of $C_0^\infty(D)$ under $(\cdot, \cdot)_{H^s}$.

For each n, let h_n be a DGFF on the subgraph $G_n = (V_n \cup \partial V_n, E_n)$ of $\frac{1}{n}\mathbf{Z}^2$ which consists of those vertices in $\frac{1}{n}\mathbf{Z}^2$ which are contained in $[0,1]^2$. The boundary vertices are those which are contained in $\partial [0,1]^2$. For $f \in C_0^\infty([0,1]^2)$, let

$$\xi_n(f) = \frac{1}{\sqrt{2\pi}} \sum_{b \in E_n} \nabla f(b) \nabla h_n(b)$$

where $\nabla g(b) = g(y) - g(x)$ for $b = (x, y) \in E_n$.

Show that ξ_n converges to the continuum GFF on $[0,1]^2$ (with zero boundary conditions) using the following steps.

(1) Explain why, for $f \in C_0^\infty(D)$, we have that $\xi_n(f)$ is a Gaussian random variable with mean 0 and variance $(2\pi)^{-1} \sum_{b \in E_n} (\nabla f(b))^2$

(2) Explain why

$$\frac{1}{2\pi} \sum_{b \in E_n} (\nabla f(b))^2 \to \|f\|_\nabla^2 \equiv \frac{1}{2\pi} \iint_{[0,1]^2} (\nabla f(x))^2 dx \quad \text{as} \quad n \to \infty.$$

(3) Explain why $\psi_j = (-\lambda_j)^{-2} \phi_j$ is an orthonormal basis of $H^4([0,2]^2)$.

(4) Assume that there exists a constant $a_0 > 0$ such that $\sum_{b \in E_n} (\nabla \phi_j(b))^2 \leq a_0 j^2$ for all $n, j \in \mathbf{N}$. Show that there exists $a > 0$ such that

$$\sum_j \mathbf{P}[\xi_n(\psi_j) \geq j^{-1/2-a}] < \infty$$

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(Recall the Weyl asymptotics: $-\lambda_j/j \to c_0$ as $j \to \infty$ for a constant $c_0 \in (0,\infty)$ depending only on D. You may use without proof the fact that if $Z \sim N(0,1)$ then

$$\mathbf{P}[Z \geq \lambda] \sim \sqrt{\frac{2}{\pi}}\lambda^{-1}e^{-\lambda^2/2} \quad \text{as} \quad \lambda \to \infty$$

where $f(\lambda) \sim g(\lambda)$ if and only if $f(\lambda)/g(\lambda) \to 1$ as $\lambda \to \infty$).

(5) Using the Borel-Cantelli lemma, show that there almost surely exists $C < \infty$ (random)

$$|\xi_n(\psi_j)| \leq \frac{C}{j^{1/2+a}} \quad \text{for all} \quad j$$

Explain why the law of C can be made to be tight in n.

(6) Suppose that $f \in H^4([0,1]^2)$. Using the previous part, explain why there almost surely exists $C < \infty$ (random but whose law is tight with n) such that

$$|\xi_n(f)| \leq C\|f\|_{H^4([0,1]^2)}$$

Use this to conclude that $\|\xi_n\|_{H^{-4}([0,1]^2)} \leq C$.

(7) Explain why all of the parts together imply the result.

(8) Is it possible to prove convergence in $H^{-s}([0,1]^2)$ for some $s < 4$ (i.e., is $s = 4$ the best possible)?

Problem 2 Using the notation of the previous problem, show that for each $\delta > 0$ we have that

$$\mathbf{P}\left[\max_{x \in V_n} h(x) \geq \left(\sqrt{\frac{2}{\pi}} + \delta\right)\log n\right] \to 0 \quad \text{as} \quad n \to \infty$$

using the following steps. (Letting $G_n(x,y)$ be given by the expected number of steps that a simple random walk starting from x spends at y before exiting V_n, recall from class that $\mathrm{cov}(h(x),h(y)) = d_y^{-1}G_n(x,y)$ where d_y is the degree at y.)

(1) Explain why $G_n(x,y) = \sum_{t=0}^{\infty} p_{V_n}^t(x,y)$ where $p_n^t(x,y)$ is the transition kernel for simple random walk on V_n stopped upon exiting V_n. That is, $p_n^t(x,y)$ is the probability that a simple random walk started from x hits y at time t before exiting V_n.

(2) Let p^t be the transition kernel for simple random walk on $\mathbf{Z}^2$. That is, $p^t(x,y)$ is the probability that a simple random walk starting from x is at y at time t. (This differs from p_n^t because p_n^t is for the walk stopped on exiting V_n). The local central limit theorem[1] implies that there exists a constant $c_0 > 0$ such that

$$|p^n(x,y) + p^{n+1}(x,y) - 2\overline{p}^n(x,y)| \leq \frac{c_0}{n^2}$$

where

$$\overline{p}^n(x,y) = \frac{1}{\pi n}e^{-\|x-y\|^2/n}.$$

Using the local central limit theorem and the Markov property for simple random walk, explain why there exists a constant $c_1 > 0$ such that

$$p_{V_n}^{\alpha n^2}(x,x) \leq e^{-c_1\alpha} \quad \text{for all} \quad n \in \mathbf{N} \quad \text{and} \quad \alpha > 0.$$

[1] For a reference, see [1, Theorem 2.1.3]; the actual local central limit theorem is significantly stronger than the result stated here.

(3) Use the previous part and the local central limit theorem to show that

$$G(x,x) \le \frac{2}{\pi}(1+o(1))\log n \quad \text{for all} \quad x \in V_n \quad \text{and} \quad n \in \mathbf{N}$$

where the $o(1)$ term tends to 0 as $n \to \infty$.

(4) Combine the previous steps with a union bound to complete the proof. You may use without proof the Gaussian tail bound from the previous problem.

Problem 3 Suppose that f is a distribution on D with $\Delta f = 0$ in the distributional sense. That is, if $g \in C_0^\infty(D)$, then $(f, \Delta g) = 0$. Show that $f \in C^\infty(D)$ and $\Delta f = 0$ using the following steps.

(1) Let ϕ be a radially symmetric C_0^∞ bump function supported in $\mathbf{D}$. In other words, $\phi(x) \ge 0$ for all x, $\phi(x)$ depends only on $|x|$, $\phi(x) = 0$ for $|x| \ge 1$, and $\int \phi = 1$. For each $\epsilon > 0$, let

$$f_\epsilon(x) = \epsilon^{-2} \int f(y)\phi\left(\frac{x-y}{\epsilon}\right) dy.$$

Explain why f_ϵ is C^∞ in $D_\epsilon = \{z \in D : \text{dist}(z, \partial D) \ge \epsilon\}$.

(2) Fix $\delta > 0$ and let $x \in D_\delta$. Explain why $f_\epsilon(x)$ does not depend on the value of ϵ for $\epsilon \in (0, \delta)$. Hint: compute the derivative with respect to ϵ, recall the form of Δ when expressed in polar coordinates, and consider the radially symmetric function $\psi(r) = \int r\phi(r)dr$.

(3) Conclude that if $g \in C_0^\infty(D)$, then the value of (f_ϵ, g) does not depend on ϵ for sufficiently small values of ϵ.

(4) Explain why the previous parts imply that $f \in C^\infty(D)$ and $\Delta f = 0$ (in the usual sense).

Problem 4 Suppose that $D \subseteq \mathbf{C}$ is a domain, $U \subseteq D$, and fix $f \in H(D)$.

(1) Explain why there exists a unique minimizer $h \in H(D)$ to the variational problem

$$\inf\{\|g\|_\nabla^2 : g \in H(D),\ f|_{D \setminus U} \equiv g|_{D \setminus U}\}.$$

(Hint: think of this variational problem in terms of orthogonal projection.)

(2) Show that h is harmonic in U by first showing that $(h, \Delta g) = 0$ for $g \in C_0^\infty(U)$ and then invoking the previous problem.

Problem 5 Suppose that $D \subseteq \mathbf{C}$ is a simply connected domain and fix $z \in D$. Recall that the **conformal radius** $C(z; D)$ of z is given by $|\phi'(0)|$ where ϕ is any conformal transformation from $\mathbf{D}$ to D with $\phi(0) = z$. Let $G_z(y)$ be the function which is harmonic D and equal to $y \mapsto -\log|z-y|$ on ∂D. Show that $G_z(z) = -\log C(z; D)$ using the following steps.

(1) Explain why, if $f\colon D \to \mathbf{C}$ is a conformal map with $f(z) \neq 0$ for all $z \in D$ then $\log|f(z)|$ is harmonic in D.

(2) Using the previous part, explain why the map $\psi\colon \mathbf{D} \to \mathbf{R}$ given by

$$w \mapsto \begin{cases} \log\left|\frac{\phi(w)-\phi(0)}{w}\right| & \text{for} \quad w \neq 0 \\ \log|\phi'(0)| & \text{for} \quad w = 0 \end{cases}$$

is harmonic in $\mathbf{D}$.

(3) Using the previous part, explain why

$$\log|\phi'(0)| = \frac{1}{2\pi}\int_{\partial \mathbf{D}} \log|\phi(w) - \phi(0)| dw$$

where dw denotes Lebesgue measure on $\partial \mathbf{D}$.

(4) Explain why $G_z(y)$ (as defined above) is equal to $-\psi(\phi^{-1}(y))$ and use this to explain why $G_z(z) = -\log C(z; D)$. (Hint: use that if u is harmonic and φ is a conformal map then $u \circ \varphi$ is harmonic.)

Problem 6 Recall that if $K \subseteq \mathbf{H}$ is a compact hull (i.e., $\mathbf{H} \setminus K$ is simply connected and K is bounded), and $g_K \colon \mathbf{H} \setminus K \to \mathbf{H}$ is the unique conformal map with $|g_K(z) - z| \to 0$ as $z \to \infty$, then $\text{hcap}(K)$ is given by the coefficient of z^{-1} in the Laurent expansion

$$g_K(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

at ∞. That is, $\text{hcap}(K) = \lim_{z\to\infty} z(g_K(z) - z)$. Show using the following steps that

$$\text{hcap}(K) = \lim_{y\to\infty} y\mathbf{P}^{iy}[\text{Im}(B_\tau)]$$

where B is a standard Brownian motion in $\mathbf{H}$, $\mathbf{P}^{iy}$ denotes the law under which $B_0 = y$, and τ is the first time that B exits $\mathbf{H} \setminus K$.

(1) Let B_t be a standard Brownian motion. Explain why $g_K(B_t) - B_t$ is a martingale.
(2) Explain why $g_K(z) - z = \mathbf{E}^z[g_K(B_\tau) - B_\tau]$ where τ is as above.
(3) Finish the proof by making the particular choice $z = iy$ and use the representation $\text{hcap}(A) = \lim_{z\to\infty} z(g_K(z) - z)$.

Problem 7 Recall that a **Bessel process** of dimension δ is given by the solution to the SDE:

$$dX_t = \frac{\delta - 1}{2} \cdot \frac{1}{X_t} dt + dB_t, \quad X_0 > 0$$

where B is a standard Brownian motion, at least up until the first time that $X_t \leq 0$. Recall that $M_t = X_t^{2-\delta}$ is a local martingale.

(1) For each a, let $\tau_a = \inf\{t \geq 0 : X_t = a\}$. For $a < X_0 < b$, compute $\mathbf{P}[\tau_a < \tau_b]$ using that M is a local martingale.
(2) Assume that $\delta < 2$. For $b > 1$, explain how one can condition on the event that $\tau_b < \tau_0$ by using M.
(3) Using the previous part and the Girsanov theorem, describe the law of $X|_{[0,\tau_b]}$ conditioned on $\tau_b < \tau_0$.
(4) Explain why, informally, the statement "A standard Brownian motion conditioned to be positive is a 3-dimensional Bessel process" is true.

Problem 8 Suppose that (g_t) is the chordal Loewner flow associated with an SLE_κ process with driving function $W_t = \sqrt{\kappa} B_t$ and that $\rho \in \mathbf{R}$.

(1) Fix $z \in \mathbf{H}$, let $z_t = x_t + iy_t = g_t(z)$. Using Ito's formula, show that $M_t = |g_t'(z)|^{(8-2\kappa+\rho)\rho} y_t^{\rho^2/(8\kappa)} |W_t - z_t|^{\rho/\kappa}$ is a local martingale. (Hint: let

$$Z_t = \frac{(8 - 2\kappa + \rho)\rho}{8\kappa} \log g_t'(z) + \frac{\rho^2}{8\kappa} \log y_t + \frac{\rho}{\kappa} \log(W_t - z_t).$$

Then compute dZ_t, take its real part, exponentiate, and apply Ito's formula.)[2]

[2]Your answer should match the formula for M_t given in the proof [2, Theorem 6].

(2) Using the previous part and the Girsanov theorem, show that the law of $W|_{[0,t]}$ weighted by M_t is given by

$$dW_s = \mathrm{Re}\left(\frac{\rho}{W_s - z_s}\right) ds + \sqrt{\kappa} d\widetilde{B}_s$$

where $\widetilde{B}|_{[0,t]}$ is a standard Brownian motion under the law $\widetilde{\mathbf{P}}$ weighted by M_t. That is,

$$\frac{d\widetilde{\mathbf{P}}}{d\mathbf{P}} = \frac{M_t}{M_0}.$$

References

[1] Gregory F. Lawler and Vlada Limic. *Random walk: a modern introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.
[2] Oded Schramm and David B. Wilson. SLE coordinate changes. *New York J. Math.*, 11:659–669 (electronic), 2005.