# 18.03 PDE.3: The Wave Equation

**1.** $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$
**2.** Normal Modes.
**3.** Wave fronts and wave speed (d'Alembert solution).
**4.** Real life waves.

I showed you an elastic band which oscillated. In the literature this is usually referred to as a vibrating string. Let $u(x,t)$ represent the vertical displacement of the string. In other words, for each fixed time $t$, the graph of the string is $y = y(x) = u(x,t)$.

Consider, as in the case of the heat equation, the equilibrium position in which the elastic string is not moving. In that case, it's in a straight line. If the string is concave down (curved above the equilibrium) then the elasticity pulls the string back down in the middle. This tendency is a restoring force like that of a spring. Since force is proportional to acceleration ($F = ma$), and acceleration is $\partial^2 u/\partial t^2$, we have

$$\frac{\partial^2 u}{\partial x^2} < 0 \implies \frac{\partial^2 u}{\partial t^2} < 0.$$

Similarly for concave up configurations,

$$\frac{\partial^2 u}{\partial x^2} > 0 \implies \frac{\partial^2 u}{\partial t^2} > 0.$$

The simplest rule that realizes this effect is proportionality,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

We wrote the constant first as $c > 0$ and rewrote it later as $c^2 > 0$ when it turned out that the right units of $c$ are meters per second so that it represents a speed.

Let us take $c = 1$ for simplicity and fix the ends of the string at 0 and $\pi$. The normal modes have the form

$$u(x,t) = v(x)w(t), \quad v(0) = v(\pi) = 0.$$

Substituting into the equation, we find

$$\ddot{w}(t)v(x) = w(t)v''(x),$$

which leads (via the separation of variables method, $\ddot{w}(t)/w(t) = v''(x)/v(x) = \lambda$) to the equations

$$v''(x) = \lambda v(x), \quad v(0) = v(\pi) = 0.$$

These are the same as for the heat equation with fixed ends, and we already found a complete list of solutions (up to multiples)

$$v_n(x) = \sin(nx), \quad n = 1, 2, 3, \ldots .$$

We also have $v_n''(x) = -n^2 v_n(x)$, so that $\lambda_n = -n^2$. What is different this time is that the equation for $w_n$ is second order

$$\ddot{w}_n(t) = -n^2 w_n(t)$$

A second order equation has a two dimensional family of solutions. In this case, they are

$$w_n(t) = a\cos(nt) + b\sin(nt)$$

This highlights the main conceptual differences between the heat and wave equations. These wave equation solutions are oscillatory in $t$ not exponential. Also the extra degree of freedom means we have to specify not only the initial position $u(x,0)$, but also the initial velocity $(\partial/\partial t)u(x,0)$.

We will take the simplest initial velocity, namely, initial velocity 0 (also the most realistic choice when we pluck a string). Thus we impose the conditions

$$0 = \dot{w}_n(0) = -an\sin(0) + bn\cos 0 = bn \implies b = 0$$

and (for simplicity) $w_n(0) = a = 1$. Now $a = 1$ and $b = 0$, so that the normal modes are

$$u_n(x,t) = \cos(nt)\sin(nx)$$

The principle of superposition says that if

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \quad 0 < x < \pi$$

then

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos(nt)\sin(nx), \quad 0 < x < \pi$$

solves the wave equation with constant $c = 1$, initial condition $u(x,0) = f(x)$ and initial velocity $(\partial/\partial t)u(x,0) = 0$ and endpoint conditions $u(0,t) = u(\pi,t) = 0$, $t > 0$. (Actually, the wave equation is reversible, and these equations are satisfied for $-\infty < t < \infty$.)

Notice that there are now **two** inputs at time $t = 0$, the initial position $f(x)$ and the initial velocity which we have set equal to 0 for simplicity.[1] This is consistent with the fact that the equation is second order in the $t$ variable.

**Wave fronts.** D'Alembert figured out another formula for solutions to the one (space) dimensional wave equation. This works for initial conditions $v(x)$ is defined for all $x$, $-\infty < x < \infty$. The solution (for $c = 1$) is

$$u_1(x,t) = v(x-t)$$

We can check that this is a solution by plugging it into the equation,

$$\frac{\partial^2}{\partial t^2}u(x,t) = (-1)^2 v''(x-t) = v''(x-t) = \frac{\partial^2}{\partial x^2}u(x,t).$$

Similarly, $u_2(x,t) = v(x+t)$ is a solution.

We plot the behavior of this solution using a space-time diagram and taking the simplest initial condition, namely the step function,

$$v(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

The solution

$$u_1(x,t) = v(x-t)$$

[1] If the initial velocity is not zero, one can write a series solution involving, in addition, the other solution to the equation for $w_n(t)$, namely $\sin(nt)$

takes on only two values, 0 and 1. Therefore, we can draw a picture showing how the solution behaves by drawing the $(x, t)$ plane and dividing the plane into the region where $u_1(x, t) = 1$ versus $u_1(x, t) = 0$. This kind of space-time diagram is often used to describe the behavior of waves.[2] We have

$$u_1(x,t) = v(x - t) = 1 \iff x - t < 0 \iff t > x,$$

and

$$u_1(x,t) = v(x - t) = 0 \iff x - t > 0 \iff t < x.$$

The divider between the places where $u_1 = 1$ and $u_1 = 0$ is known as the *wave front* and it is located at the line of slope 1,

$$t = x$$

We drew this, indicating $u_1 = 1$ above the line $t = x$ and $u_1 = 0$ below.

The only feature we want to extract from this picture is that as time increases, the wave moves at a constant speed 1. An observer at $x = 10$ will see the wave front pass (the value of $u(x, t)$ switch from 0 to 1) at time $t = 10$. If we were to change the constant $c$ we would obtain a solution $v(x - ct)$ whose wave front travels at the speed $c$.

In order to understand what we are looking at in simulations and real life, we need to enforce both initial conditions, position and velocity. If

$$u(x,t) = av(x - t) + bv(x + t)$$

and $u(x, 0) = v(x)$, $(\partial/\partial t)u(x, 0) = 0$, then we have

$$u(x, 0) = av(x) + bv(x) = v(x) \implies a + b = 1;$$

$$(\partial/\partial t)u(x, 0) = -av'(x) + bv'(x) = 0 \implies -a + b = 0.$$

Hence,

$$a = b = \frac{1}{2}; \quad u(x,t) = \frac{1}{2}(v(x - t) + v(x + t))$$

is the solution of interest to us. Plotting the three regions $u = 1$, $u = 1/2$ and $u = 0$, we see the plane divided by a V shape with 1 on the left 1/2 in the middle and 0 on the right. This says that there is a wave front travelling both left and right from the source. This is like what happens with point sources in higher dimensions. In two dimensions, a pebble dropped in a quiet pond will send a disturbance (wave front) outward in all directions with equal speed, forming a circular wave front. The space-time picture of this wave front looks like an ice-cream cone. In three dimensions, a source of sound our light will send out wave in all directions. The wave front is an expanding sphere. In one dimension, the geometry is less evident. What is happening is that there are only two possible directions (left and right) instead of a whole circle or sphere of directions.

The next step is to note that it is not realistic for a string to have a jump discontinuity like the step function. But any feature of the graph will travel at the wave speed. For example, if we strech a string to a triangular shape with a kink, then the kink will travel (in both directions) at the wave speed. We looked at this in the applet, and also saw that when the kink hits the fixed ends it bounces off and returns. (The kinks always go at the speed $c$; take a look in the applet, in which you can adjust the wave speed.)

**Real life waves.** Finally, we looked at a slow motion film of an vibrating elastic band at

[2] In the general theory of relativity, certain rescaled space-time diagrams are used to keep track of light as it travels into black holes. In that case, they are called Penrose diagrams.

`http://www.acoustics.salford.ac.uk/feschools/waves/quicktime/elastic2512K_Stream.mov`

The video shows the kink(s) in the band propagating at a steady speed and bouncing off the ends. This resembles what we saw in the applet. Then we witnessed a new phenomenon: *damping*. As the system loses energy, all the modes are damped.

The damped wave equation is

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \quad b > 0$$

This introduces the factor $e^{-bt/2}$ in the solutions.

Closer examination indicates (not entirely clear without some more detailed numerical simulation!) that the main mode(s) are revealed more quickly than would be the case using a linear damped wave equation. The higher frequency modes are being damped more quickly than the lower frequency ones. A scientist or engineer might say that this system is exhibiting some *nonlinearity* in its response. The modes of a linear wave equation would all have the same damping constant $b$. This suggests that one can't explain fully this rubber band using linear differential equations alone. It satisfies some *nonlinear* differential equation that shares many features with the linear equation, including the wave speed and the normal modes.

# M.I.T. 18.03 Ordinary Differential Equations
# 18.03 Extra Notes and Exercises