18.03 LA.3: Complete Solutions, Nullspace, Space, Dimension, Basis

[1] Particular solutions

Matrix Example

Consider the matrix equation

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \end{bmatrix}$$

The complete solution to this equation is the line $x_1 + x_2 = 8$. The homogeneous solution, or the *nullspace* is the set of solutions $x_1 + x_2 = 0$. This is all of the points on the line through the origin. The homogeneous and complete solutions are picture in the figure below.

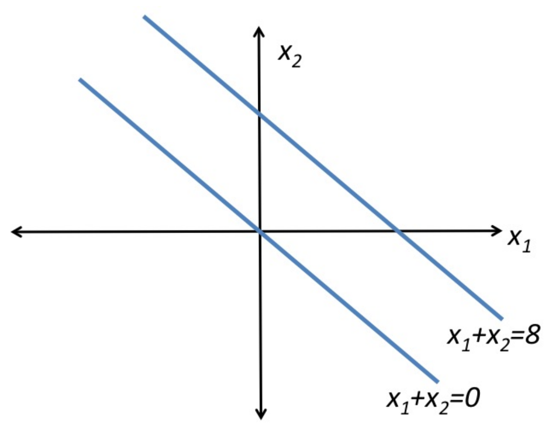

Figure 1: The homogeneous and complete solutions

To describe a complete solution it suffices to choose one particular solution, and add to it, any homogeneous solution. For our particular solution, we might choose

$$\begin{bmatrix} 8 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ 8 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

If we add any homogeneous solution to this particular solution, you move along the line $x_1 + x_2 = 8$. All this equation does is take the equation for the homogeneous line, and move the origin of that line to the particular solution!

How do solve this equation in Matlab? We type

```
x = [1 1] \ [8]
```

In general we write

```
x = A \ b
```

Differential Equations Example

Let's consider the linear differential equation with initial condition given:

$$\frac{dy}{dt} + y = 1$$
$$y(0)$$

To solve this equation, we can find one particular solution and add to it any homogeneous solution. The homogeneous solution that satisfies the initial condition is $x_h = y(0)e^{-t}$. So then a particular solution must satisfy $y_p(0) = 0$ so that $x_p(0) + x_h(0) = y(0)$, and such a particular solution is $y_p = 1 - e^{-t}$. The complete solution is then:

complete solution		**particular solution**		**homogeneous solution**
y	$=$	$1 - e^{-t}$	$+$	$y(0)e^{-t}$

However, maybe you prefer to take the steady state solution. The steady state solution is when the derivative term vanishes, $\frac{dy}{dt} = 0$. So instead we can choose the particular solution $y_p = 1$. That's an excellent solution to choose. Then in order to add to this an homogeneous solution, we add some multiple of e^{-t} so that at $t = 0$ the complete solution is equal to $y(0)$ and we find

complete solution		**particular solution**		**homogeneous solution**
y	$=$	1	$+$	$(y(0)-1)e^{-t}$
		$\uparrow$		$\uparrow$
		steady state solution		*transient solution*

The solution 1 is an important solution, because all solutions, no matter what initial condition, will approach the steady state solution $y = 1$.

There is not only 1 particular solution. There are many, but we have to choose 1 and live with it. But any particular solution will do.

[2] Complete Solutions

Matrix Example

Let's solve the system:

$$\begin{array}{lllll} x_1 & & +cx_3 & = b_1 \\ & x_2 & +dx_3 & = b_2 \end{array}$$

What is the matrix for this system of equations?

$$A = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \end{bmatrix}$$

Notice that A is already in row echelon form! But we could start with any system

$$\begin{array}{llll} x_1 & +3x_2 & +5x_3 & = b_1 \\ 4x_1 & +7x_2 & +19x_3 & = b_2 \end{array}$$

and first do a sequence of row operations to obtain a row echelon matrix. (Don't forget to do the same operations to b_1 and b_2:

$$\begin{bmatrix} 2 & 3 & 5 & \vdots & b_1 \\ 4 & 7 & 17 & \vdots & b_2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 & 5 & \vdots & b_1 \\ 0 & 1 & 9 & \vdots & b_2 - 2b_1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3/2 & 5/2 & \vdots & b_1/2 \\ 0 & 1 & 9 & \vdots & b_2 - 2b_1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -11 & \vdots & 5b_1/2 - 3b_2/2 \\ 0 & 1 & 9 & \vdots & b_2 - 2b_1 \end{bmatrix}$$

Let's find the complete solution to $Ax = b$ for the matrix

$$A = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \end{bmatrix}.$$

Geometrically, what are we talking about?

The solution to each equation is a plane, and the planes intersect in a line. That line is the complete solution. It doesn't go through 0! Only solutions to the equation $A\mathbf{x} = \mathbf{0}$ will go through 0!

So let's find 1 particular solution, and all homogeneous solutions.

Recommended particular solution: Set the free variable $\mathbf{x}_3 = 0$. Then

$$\mathbf{x}_p = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}.$$

We could let the free variable be any value, but 0 is a nice choice because with a reduced echelon matrix, it is easy to read off the solution.

So what about the homogenous, or null solution. I will write $\mathbf{x}_n$ instead of $\mathbf{x}_h$ for the null solution of a linear system, but this is the same as the homogeneous solution. So now we are solving $A\mathbf{x} = \mathbf{0}$. The only bad choice is $x_3 = 0$, since that is the zero solution, which we already know. So instead we choose $\mathbf{x}_3 = 1$. We get

$$\mathbf{x}_n = C \begin{bmatrix} -c \\ -d \\ 1 \end{bmatrix}$$

The complete solution is

$$x_{complete} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} + C \begin{bmatrix} -c \\ -d \\ 1 \end{bmatrix}.$$

This is the power of the row reduced echelon form. Once in this form, you can read everything off!

Differential Equations Example

Let's consider the differential equation $y'' + y = 1$. We can choose the steady state solution for the particular solution $y_p = 1$.

Let's focus on solving $y'' + y = 0$. What is the nullspace of this equation?

We can't say vectors here. We have to say functions. But that's OK. We can add functions and we can multiply them by constants. That's all we could do with vectors too. Linear combinations are the key.

So what are the homogeneous solutions to this equation? Give me just enough, but not too many.

One answer is $y_h = c_1 \cos(t) + c_2 \sin(t)$. Using linear algebra terminology, I would say there is a *2-dimensional* nullspace. There are two independent solutions $\cos(t)$ and $\sin(t)$, and linear combinations of these two solutions gives all solutions!

$\sin(t)$ and $\cos(t)$ are a *basis* for the nullspace.

A **basis** means each element of the basis is a solution to $A\mathbf{x} = \mathbf{0}$. Can multiply by a constant and we still get a solution. And we can add together and still get a solution. Together we get all solutions, but the $\sin(t)$ and $\cos(t)$ are different or *indepenedent* solutions.

What's another description of the nullspace?

$$C_1 e^{it} + C_2 e^{-it}$$

This description is just as good. Better in some ways (fulfills the pattern better), not as good in others (involves complex numers). The basis in this case is e^{it} and e^{-it}. They are independent solutions, but linear combinations give all null solutions.

If you wanted to mess with your TA, you could choose $y_h = Ce^{it} + D\cos(t)$. This is just as good.

We've introduced some important words. The *basis for the nullspace.* In this example, the beauty is that the nullspace will always have 2 functions in it. 2 is a very important number.

- The degree of the ODE is 2
- There are 2 constants
- 2 initial conditions are needed
- The dimension of the nullspace is 2.

[3] The nullspace

Suppose we have the equation $R\mathbf{x} = 0$. The collection of $\mathbf{x}$ that solve this equation form the *nullspace.* The nullspace always goes through the origin.

Example

Suppose we have a 5 by 5 matrix. Does it have an inverse or doesn't it? Look at the nullspace! If only solution in the nullspace is $\mathbf{0}$, then yes, it is invertible. However, if there is some nonzero solution, then the matrix is not invertible.

The other importan work we used is space.

Matrix Example

Let NR denote the nullspace of R:

$$R = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \end{bmatrix}$$

What's a basis for the nullspace? A basis could be $\begin{bmatrix} -c \\ -d \\ 1 \end{bmatrix}$. Or we could take $\begin{bmatrix} -2c \\ -2d \\ 2 \end{bmatrix}$. The dimension is $3 - 2 = 1$. So there is only one element in the basis.

Why can't we take 2 vectors in the basis?

Because they won't be independent elements!

Differential Equations Example

For example, $Ce^{it} + D\cos(t) + E\sin(t)$ does not form a basis because they are not independent! Euler's formula tells us that $e^{it} = \cos(t) + i\sin(t)$, so e^{it} *depends* on $\cos(t)$ and $\sin(t)$.

[4] **Space, Basis, Dimension** There are a lot of important words that have been introduced.

- Space
- Basis for a Space
- Dimension of a Space

We have been looking at small sized examples, but these ideas are not small, they are very central to what we are studying.

First let's consider the word space. We have two main examples. The column space and the nullspace.

A	**Column Space**	**Nullspace**
Definition	All linear cominations of the columns of A	All solutions to $A\mathbf{x} = \mathbf{0}$
50×70 matrix	Column space lives in $\mathbb{R}^{50}$	Nullspace lives in $\mathbb{R}^{70}$
$m \times n$ matrix	Column space lives in $\mathbb{R}^m$	Nullspace lives in $\mathbb{R}^n$

Definition V is a space (or *vector* space) when: if x and y are in the space, then for any constant c, cx is in the space, and $x + y$ is also in the space. That's superposition!

Let's make sense of these terms for a larger matrix that is in row echelon form.

Larger Matrix Example

$$R = \begin{bmatrix} = 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first and third columns are the *pivot* columns. The second and forth are *free* columns.

What is the column space, $C(R)$?

All linear combinations of the columns. Is $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ in the column space? No it's not. The column space is the xy-plane, all vectors $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$. The dimension is 2, and a basis for the column space can be taken to be the pivot columns.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Note, if your original matrix wasn't in rref form, you must take the original form of the pivot columns as your basis, not the row reduced form of them!

What is a basis for the nullspace, $N(R)$?

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

The reduced echelon form makes explicit the linear relations between the columns.

The relationships between the columns of A are the same as the linear relationships between the columns of any row-equivalent matrix, such as the reduced echelon form R. So a pivot indicates that this column is independent of the previous columns; and, for example, the 2 in the second column in this reduced form is a record of the fact that the second column is 2 times the first. This is why the reduced row echelon form is so useful to us. It allows us to immediately read off a basis for both the independent columns, and the nullspace.

Note that this line of thought is how you see that the reduced echelon form is well-defined, independent of the sequence of row operations used to obtain it.

M.I.T. 18.03 Ordinary Differential Equations
18.03 Extra Notes and Exercises