FOURIER SERIES

7A-1)

a) For \( \sin kt \), use the frequency \( k \),
and \((\text{frequency})(\text{period}) = 2\pi\).
\[ \frac{\pi}{3} \cdot P = 2\pi, \quad P = 6 \]

b) Period is \( \frac{2\pi}{3} \): \( \sin(kx) = \sin(kt) \)

\( \cos kt \) has period \( \frac{2\pi}{k} \). (As problem 4)
\( \cos^2 kt \) has period \( \frac{\pi}{k} \). (As in part b).
\[ \cos^2 x = \left( \frac{\cos 2x}{2} \right)^2 = \left( \cos^2 x \right)^2 = \left( \cos (3x) \right)^2 \]

7A-2)

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nt \cos \frac{t}{n} \, dt = \frac{\sin \frac{2\pi}{n}}{\pi n} \int_{-\pi}^{\pi} dt = 0 \]
\[ (a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} dt = \pi) \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin nt \sin \frac{t}{n} \, dt = \frac{\cos \frac{2\pi}{n}}{\pi n} \int_{-\pi}^{\pi} dt = \frac{(-1)^n - (-1)^{n-1}}{\pi n} \]
\[ = \frac{1}{n}, \quad n \text{ even} \]
\[ = \frac{2}{n}, \quad n \text{ odd} \]

Then: (b)
\[ a = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, dt + \frac{1}{\pi} \int_{0}^{\pi} f(t) \, dt \]
\[ = \frac{1}{\pi} \int_{0}^{\pi} f(t) \, dt + \frac{1}{\pi} \int_{0}^{\pi} f(t) \, dt \quad \text{by the first part} \]
\[ f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{nt}{n} + b_n \sin \frac{nt}{n} \right] \]

f(t) \sim \frac{\pi}{2} - \frac{4}{\pi} \left( \cos \frac{3t}{3^2} + \cos \frac{5t}{5^2} + \ldots \right)
\[ 7B-1. \]
\[ a_n = 2 \int_0^1 (1-t) \cos nt \, dt = 2t - t^2 \bigg|_0^1 = 1 \]

\[ b_n = 2 \int_0^1 (1-t) \sin nt \, dt \quad \text{Integ by parts:} \]
\[ = 2 \left[ (1-t) \frac{\sin nt}{nt} - \int_{-1}^1 \cos nt \, dt \right]_0^1 \]
\[ = 2 \left[ (1-t) \frac{\sin nt}{nt} \bigg|_0^1 + \cos nt \bigg|_0^1 \right] \]
\[ = \frac{2}{n \pi} \left[ (-1)^n - 1 \right] = \begin{cases} 0, & n \text{ even} \\ \frac{4}{n \pi^2}, & n \text{ odd} \end{cases} \]

\[ f(t) \sim \frac{1}{2} + \frac{4}{\pi^2} \left( \frac{\cos t + \cos 3t + \cos 5t + \cdots}{3} \right) \]

\[ \text{Fourier cosine series (picture below)} \]

\[ b_n = 2 \int_0^1 (1-t) \sin nt \, dt \quad \text{Integ by parts:} \]
\[ = 2 \left[ (1-t) \frac{\cos nt}{nt} - \int_{-1}^1 \sin nt \, dt \right]_0^1 \]
\[ = 2 \left[ (0 + \frac{1}{nt}) \right] \]
\[ = \frac{2}{nt} \left[ \sin nt + \sin 3nt + \sin 5nt + \cdots \right] \]

\[ \text{Fourier sine series (picture below)} \]

\[ 7B-3 \]
\[ \int_0^1 f(t) \, dt = \int_0^1 (f(u)) \, du = \int_a^b (f(u)) \, du \]
\[ f\text{ even} \quad (t + t = -u) \quad f(-u) = f(u) \]
\[ f\text{ odd} \quad -a \quad a \quad -a \quad a \quad -a \quad a \]

\[ 7B-1a \]
\[ t \sim \frac{\pi}{2} - \frac{4}{\pi} \left( \cos t + \cos 3t + \cos 5t + \cdots \right) \]

\[ x(t) = \frac{A_0}{2} + \sum A_n \cos nt \]
\[ x'' = -2n^2 A_n \cos nt \quad \text{Adding,} \]
\[ t = A_0 + \sum A_n \cos nt \]
\[ A_0 = \frac{\pi}{2}, \quad A_n = 0 \] if \( n \) even \( A_n = -\frac{4}{\pi n(n^2 - \frac{1}{4})} \] if \( n \) odd
\[ f(t) = \frac{1}{\pi} + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos \pi n t}{n^2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin \pi n t}{n} \]

This series doesn't converge (the sine terms don't add up to a finite sum). So it certainly can't converge to \( f(t) \).

**7C-1**

**Preliminary remarks**

\[ mx'' + kx = F(t) \]

The natural frequency of the spring-mass system is \( \omega_0 = \sqrt{\frac{k}{m}} \).

The typical term of the Fourier expansion of \( F(t) \) is \( \cos \pi n t \), \( \sin \pi n t \); thus we get pure resonance if and only if the Fourier series has a \( \cos \pi n t \) or \( \sin \pi n t \) term where \( n \pi = \omega \).

a) \( \omega_0 = \sqrt{5} \) for spring-mass system

\[ L = 1 \]

Fourier series is \( \sum b_n \sin \pi n t \)

\[ n \pi = \sqrt{5} \]

\( \therefore \) no resonance

b) \( \omega_0 = 2\pi \)

\[ L = 1 \]

Fourier series is \( \sum b_n \sin \pi n t \), and \( \pi n = 2\pi \) if \( n = 2 \).

Example 1, 8.4 shows that this term actually occurs in the Fourier series for \( f(t) \) just change scale. \( \therefore \) get resonance.

c) \( \omega_0 = 3 \)

Fourier series is a sine series (\( F(t) \) is odd):

\[ F(t) = \sum b_n \sin \pi n t \]

all odd \( n \) occur (see Problem 8.3/11, or ex 1, 8.1)

\( n = 3 \) occurs, \( \therefore \) we get resonance.

**7C-2**

**Fourier series for \( f(t) \)**

will be same up to factor 2 as the Fourier sine series in Example 1, 8.3

\[ F(t) = 4 \left( \sin t - \frac{1}{2} \sin 2t + \frac{1}{3} \sin 3t - \ldots \right) \]

\[ x' = \sum b_n \sin nt \]

\[ x'' = \sum -b_n n^2 \sin nt \]

Adding:

\[ F(t) = \sum b_n \left( 3 - n^2 \right) \sin nt \]

\[ b_n = (-1)^{n+1} \frac{1}{n} \left( \frac{1}{3 - n^2} \right) = (-1)^n \frac{4}{n(n^2 - 3)} \]

**7C-3a**

The natural frequency of the undamped spring is \( \omega_0 = \sqrt{18/2} = 3 \).

This frequency occurs in the Fourier series for \( F(t) \) (Exercise 8.3). Thus the \( n = 3 \) term should dominate. (This actual series is)

\[ x_{sp}(t) = \frac{4}{3} \sin (3t - 0.005) - 0.20 \sin (3t + 0.12) \]

\[ 4.14 \sin (3t - 1.5708) \]

\[ t = 0 \quad \text{amplitude} = 0.7 \quad \text{sin} (4t - 3.1416) \]

**7C-3b**

The natural frequency of the undamped spring is \( \sqrt{30/3} = \sqrt{10} \).

Expanding the force \( f(t) \) as a Fourier series, since \( L = 1 \) (half-period), \( \therefore \) odd, it will be

\[ F(t) = \sum b_n \sin \pi n t \]

It's virtually certain all terms will occur (since \( F(t) \) is so messy). (Check soln to 8.4/5 wi tech for work)

\[ b_n \sin n t \]

should be the dominant term in the series (use checks with answer given in tech of book)

**Note:** Edwards, Penney 4th edn:
8.4 (16), p. 590 has a sign error in denominators - cf. (13), which is correct.