

Section 1 SOLUTIONS

1A-1 a) $y = c_1 e^x + c_2 x e^x$
 (x-2) $y' = (c_1 + c_2) e^x + c_2 x e^x$
 $y'' = (c_1 + 2c_2) e^x + c_2 x e^x$
 Add $y'' - 2y' + y = 0$ ✓ (easily checked)

b) $y' = -\frac{(\sin x + a)}{x^2} + \frac{\cos x}{x} + \sin x$
 $\frac{y}{x} = \frac{\sin x + a}{x^2} - \frac{\cos x}{x}$
 $\therefore y' + \frac{y}{x} = \sin x$

1A-2 a) $c_1 e^{kx}$ and $c_1' e^{k'x}$ are the same only if $c_1 = c_1'$, $k = k'$

b) let $k = c_1 e^a$
 then $y = k e^x$

c) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$
 $\therefore y = c_1 + c_2(2\cos^2 x - 1) + c_3 \cos^2 x$
 $= (c_1 - c_2) + (2c_2 + c_3) \cos^2 x$
 $= k_1 + k_2 \cos^2 x$

d) $y = \ln(ax+b)(cx+d)$
 $= \ln(acx^2 + (ad+bc)x + bd)$
 $\therefore y = \ln(k_1 x^2 + k_2 x + k_3)$

1A-3 a) Separating variables gives
 $y^2 dy = \frac{dx}{\ln x}$ Integrate both sides from 2 to x:
 $\frac{y^3}{3} \Big|_2^x = \int_2^x \frac{dt}{\ln t}$ Now use $y(2) = 0$:
 $\frac{y(x)^3}{3} - \frac{0^3}{3} = \int_2^x \frac{dt}{\ln t}$
 $\therefore y = \left[3 \int_2^x \frac{dt}{\ln t} \right]^{1/3}$

b) Separate variables: $\frac{dy}{y} = \frac{e^x}{x} dx$
 Can either use same method as in (a), or else: integrate both sides, using a definite integral as the antiderivative on the right:
 $\ln y + c = \int_1^x \frac{e^t}{t} dt$ *

Evaluate c by using $y(1) = 1$. This gives
 $\ln y(1) + c = \int_1^1 \frac{e^t}{t} dt = 0$
 $\therefore c = 0$
 So $y = e^{\int_1^x \frac{e^t}{t} dt}$
 from *

1A-4 a) $\frac{y dy}{y+1} = x dx$ Integrate, noting that $\frac{y}{y+1} = 1 - \frac{1}{y+1}$

$\therefore dy - \frac{dy}{y+1} = x dx$
 $y - \ln(y+1) = c + \frac{1}{2} x^2$ Put $x=2$ to evaluate c:
 $0 - \ln(1) = c + \frac{1}{2} \cdot 2^2$ [$y(2) = 0$]
 $\therefore c = -2$

Solu: $y - \ln(y+1) = \frac{1}{2} x^2 - 2$

b) $\sec^2 u du = \sin t dt$
 $\therefore \tan u = -\cos t + c$ Put $t=0$:
 $\therefore \tan 0 = -1 + c$ $u(0) = 0$
 so $c = 1$

Solu: $u = \tan^{-1}(1 - \cos t)$

1A-9a) $\frac{dy}{y^2-2y} = -\frac{dx}{x^2}$ Integrate left side by partial fractions

$$\frac{1}{2} \frac{dy}{y-2} - \frac{1}{2} \frac{dy}{y} = -\frac{dx}{x^2}$$

$$\frac{1}{2} \ln\left(\frac{y-2}{y}\right) = C_1 + \frac{1}{x}$$

Multiply by 2, exponentiate

$$= 1 - \frac{2}{y} \rightarrow \frac{y-2}{y} = C_2 e^{2/x}$$

algebra; (replace left side by $1 - \frac{2}{y}$)

$$\therefore y = \frac{2}{1 - C_2 e^{2/x}}$$

b) $\frac{dv}{\sqrt{1-v^2}} = \frac{dx}{x}$

$$\sin^{-1} v = \ln x + C$$

$$v = \sin(\ln x + C)$$

c) $\frac{dy}{(y-1)^2} = \frac{dx}{(x+1)^2}$

$$-\frac{1}{y-1} = -\frac{1}{x+1} + C$$

Solve for y by ordinary algebra.

$$y = 1 + \frac{x+1}{1-C(x+1)}$$

d) $\frac{dx}{\sqrt{1+x}} = \frac{dt}{t^2+4}$

$$2\sqrt{1+x} = \frac{1}{2} \tan^{-1}\left(\frac{t}{2}\right) + C$$

$$\therefore x = \frac{1}{4} \left(\frac{1}{2} \tan^{-1}\left(\frac{t}{2}\right) + C \right)^2 - 1$$

These problems all take for granted that you know the standard integration formulae and methods from 18.01. Review them if you are having trouble.

You need also the laws of exponentials and logarithms.

1B-1 a) $\frac{\partial M}{\partial y} = 3x^2 = \frac{\partial N}{\partial x} \therefore$ exact. what's $f(x,y)$?

$$\frac{\partial f}{\partial x} = 3x^2 y \therefore f = x^3 y + g(y)$$

$$\frac{\partial f}{\partial y} = x^3 + g'(y) = x^3 + y^3 \therefore g = \frac{1}{4} y^4 + C$$

so that $f = x^3 y + \frac{1}{4} y^4 + C$.

Solution: $x^3 y + \frac{y^4}{4} = C_1$

b) $\frac{\partial M}{\partial y} = -2y, \frac{\partial N}{\partial x} = -2x$ not exact.

c) $\frac{\partial M}{\partial v} = e^{uv} + v e^{uv} = \frac{\partial N}{\partial u} \therefore$ exact

$$\frac{\partial f}{\partial u} = v e^{uv}, \therefore f = e^{uv} + g(v)$$

$$\frac{\partial f}{\partial v} = u e^{uv} + g'(v) = u e^{uv} \therefore g = C$$

so $f = e^{uv} + C$. Soln: $e^{uv} = C_1$

or taking \ln of both sides:

$$uv = C_2$$

d) $\frac{\partial M}{\partial y} = 2x, \frac{\partial N}{\partial x} = -2x$ not exact.

1B-2

a) Multiply by y - this gives $2xy'dx + x^2 dy = 0$

or $d(x^2 y) = 0 \therefore x^2 y = C$
so $y = C/x^2$

b) Integrating factor is $\frac{1}{y^2}$:

$$\frac{y dx - x dy}{y^2} - \frac{dy}{y} = 0$$

$$d\left(\frac{x}{y}\right) - d(\ln y) = 0$$

$$\frac{x}{y} - \ln y = C$$

Evaluate C by setting $x=1$ (so $y(1)=1$)

$$\therefore \frac{1}{1} - \ln 1 = C, \text{ so } C=1$$

$$\therefore x - y \ln y = y$$

or $x = y(\ln y + 1)$

1B-2

c) Divide by t^2 (so integrating factor is $1/t^2$)

$$\left(1 + \frac{4}{t^2}\right) dt = \frac{x dt - t dx}{t^2}$$

$$\therefore d\left(t - \frac{4}{t}\right) = d\left(\frac{-x}{t}\right)$$

$$t - \frac{4}{t} = \frac{-x}{t} + C$$

$$\therefore \boxed{x = 4 - t^2 + Ct}$$

d) $\frac{1}{u^2+v^2}$ is an integrating factor:

$$\frac{u du + v dv}{u^2+v^2} + \frac{v du - u dv}{u^2+v^2} = 0$$

$$\frac{1}{2} \ln(u^2+v^2) + \tan^{-1}\left(\frac{u}{v}\right) = C$$

when $u=0, v=1$; $\frac{1}{2} \ln 1 + \tan^{-1}(0) = C$
 $\therefore C=0$

$$\boxed{\frac{1}{2} \ln(u^2+v^2) + \tan^{-1}\left(\frac{u}{v}\right) = 0}$$

(substitute $r = \sqrt{u^2+v^2}$, $\theta = \tan^{-1}\frac{u}{v}$
to get polar coords)

equation becomes $\ln r + \theta = 0$
 $r = \boxed{e^{-\theta}}$

1B-3

a) $z = y/x \therefore y = zx, y' = z'x + z$

Substituting:

$$z'x + z = \frac{z^2 - 1}{z+4}, \therefore z'x = -\frac{(z+1)^2}{z+4}$$

Sep. variables:

$$\frac{z+4}{(z+1)^2} dz = \frac{-dx}{x} \quad \text{For ease, write } z+1 = u$$

$$\left(\frac{u+3}{u^2}\right) du = \frac{-dx}{x} \quad \text{integrate:}$$

$$\ln u - \frac{3}{u} = -\ln x + C$$

To improve this:

$$\ln u + \ln x = \frac{3}{u} + C$$

Combine \rightarrow exponentiate: $ux = ke^{3/u}$

$$\text{Finally: } u = z+1 = \frac{y}{x} + 1 = \frac{y+x}{x}$$

$$\therefore \boxed{y+x = ke^{3/(y+x)}}$$

b) let $z = \frac{w}{u}$, so $w = zu$
 $w' = z'u + z$

Substituting:

$$z'u + z = \frac{2z}{1-z^2}$$

$$\therefore z'u = \frac{z(1+z^2)}{1-z^2}, \text{ after a little algebra}$$

Separate variables:

$$\otimes \frac{1-z^2}{z(1+z^2)} dz = \frac{du}{u} \quad \text{Use partial fractions on the left; result}$$

$$\frac{1-z^2}{z(1+z^2)} = \frac{1}{z} + \frac{-2z}{z^2+1}$$

Integrating \otimes :

$$\ln z - \ln(z^2+1) = \ln u + C$$

Combine and exponentiate both sides:

$$\frac{z}{z^2+1} = ku$$

Finally, put $z = w/u$; result is

$$\boxed{\frac{w}{w^2+u^2} = k} \quad \text{as the solution (you could also solve for } u \text{ in terms of } w)$$

c) Put $z = y/x$; so $y = zx, y' = z'x + z$

$$\text{Here } \frac{dy}{dx} = \frac{y^2 + x\sqrt{x^2-y^2}}{xy} \quad \text{Substitute } y=zx$$

$$z'x + z = \frac{z^2 + \sqrt{1-z^2}}{z}$$

$$\therefore z'x = \frac{\sqrt{1-z^2}}{z} \quad \text{Separate variables}$$

$$\frac{z dz}{\sqrt{1-z^2}} = \frac{dx}{x}$$

$$-\sqrt{1-z^2} = \ln x + C$$

$$\boxed{\sqrt{1-y^2/x^2} = C_1 - \ln x}$$

This can be solved explicitly for y :
square both sides, etc...

$$\boxed{y = x\sqrt{1-(C_1 - \ln x)^2}}$$

1B-4

$$y = ux^n$$

$$\therefore y' = x^n u' + nx^{n-1} u$$

$$x^n u' + nx^{n-1} u = \frac{4 + x^{2n+1} u^2}{x^{n+2} u}$$

$$\therefore u' = \frac{4 + (1-n)x^{2n+1} u^2}{x^{2n+2} u}$$

If $n=1$, we can separate vars:

$$u du = \frac{4 dx}{x^4}$$

$$\therefore \frac{u^2}{2} = -\frac{4}{3} \cdot \frac{1}{x^3} + C$$

Since $n=1$, $u = y/x$

$$\therefore \boxed{y^2 = -\frac{8}{3x} + 2Cx^2}$$

1B-5

a) $y' + \frac{2}{x}y = 1$ when written in normal form for linear eqn.

Integ. factor: $e^{\int 2/x dx} = e^{2 \ln x} = x^2$

$$\therefore x^2 y' + 2xy = x^2$$

or $(x^2 y)' = x^2$

$$x^2 y = \frac{1}{3} x^3 + C$$

$$\boxed{y = \frac{x}{3} + \frac{C}{x^2}}$$

b) In standard form;

integ. factor is $e^{\int -\tan t dt} = e^{\ln(\cos t)} = \cos t$

$$\therefore \cos t \frac{dx}{dt} - x \sin t = t$$

or $(x \cos t)' = t$

$$x \cos t = \frac{t^2}{2} + C$$

Since $x(0) = 0$, putting $t=0$ shows $C=0$.

$$\therefore \boxed{x = \frac{t^2}{2} \sec t}$$

1B-5

c) $(x^2-1)y' + 2xy = 1$ LHS is already exact!

$$[(x^2-1)y]' = 1$$

$$(x^2-1)y = x + C$$

$$\therefore y = \frac{x+C}{x^2-1}$$

d) Writing it in standard linear form

$$\frac{dv}{dt} + \frac{3v}{t} = 1$$

Integrating factor: $e^{\int 3/t dt} = e^{3 \ln t} = t^3$

$$\therefore t^3 v' + 3t^2 v = t^3$$

$$(t^3 v)' = t^3$$

$$t^3 v = \frac{1}{4} t^4 + C$$

$$V(1) = \frac{1}{4} \Rightarrow C = 0 \quad \left(\begin{matrix} \text{put} \\ t=1 \end{matrix} \right)$$

$$\therefore \boxed{V = \frac{1}{4} t}$$

1B-6

The integrating factor for this linear equation is $e^{\int at dt} = e^{at}$

$$(x e^{at})' = e^{at} r(t)$$

$$x = e^{-at} \left[\int_0^t e^{as} r(s) ds \right] + C$$

$$x = \frac{\int_0^t e^{as} r(s) ds}{e^{at}} + \frac{C}{e^{at}}$$

To find $\lim_{t \rightarrow \infty} x(t)$, use L'Hospital's rule, (∞/∞) [note that $C/e^{at} \rightarrow 0$]
differentiating top and bottom

$$\therefore \lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} \frac{e^{at} r(t)}{a e^{at}} = \lim_{t \rightarrow \infty} \frac{r(t)}{a}$$

$$= 0 \text{ by hypothesis}$$

[where did we need the hypothesis $a > 0$?]

[We used, in connection with L'H rule, the result $\frac{d}{dt} \int_0^t e^{as} r(s) ds = e^{at} r(t)$.

This follows from the 2nd Fundamental Theorem of calculus.]

1B-7

$$\frac{dy}{dx} = \frac{y}{y^3+x} \Rightarrow \frac{dx}{dy} = \frac{y^3+x}{y}$$

$$\therefore \frac{dx}{dy} - \frac{1}{y}x = y^2$$

This is now a linear equation in x .

$$\text{Integ. factor: } e^{-\int \frac{dy}{y}} = e^{-\ln y} = y^{-1}$$

\therefore multiply by $\frac{1}{y}$:

$$\frac{1}{y} \frac{dx}{dy} - \frac{1}{y^2}x = y$$

$$\text{or } \frac{d}{dy} \left(\frac{x}{y} \right) = y$$

$$\frac{x}{y} = \frac{y^2}{2} + C$$

$$\boxed{x = \frac{y^3}{2} + Cy}$$

1B-8

The systematic procedure - it always works, though it's a bit longer in this case - since we want to substitute for y, y' , begin by expressing them in terms of u .

(Don't just differentiate $u = y^{1-n}$ as is).

$$y = u^{\frac{1}{1-n}}$$

$$y' = \frac{1}{1-n} u^{\left(\frac{1}{1-n}-1\right)} \cdot u' = \frac{1}{1-n} u^{\frac{n}{1-n}} u'$$

Substitute into the ODE:

$$\frac{1}{1-n} u^{\frac{n}{1-n}} u' + pu^{\frac{1}{1-n}} = qu^{\frac{n}{1-n}}$$

Divide through by $u^{\frac{n}{1-n}}$:

$$\boxed{\frac{1}{1-n} u' + pu = q}$$

[Note: in this particular case, it's actually easier just to fumble around, but in general, this only leads to a mess.]

$$\text{Hence: } y' + py = qy^n$$

$$\text{Divide: } \frac{y'}{y^n} + \frac{p}{y^{n-1}} = q \quad (*)$$

$$\text{Put } u = y^{1-n} = \frac{1}{y^{n-1}}$$

$$u' = (1-n) \cdot \frac{1}{y^n} \cdot y'$$

$$\therefore (*) \text{ becomes } \frac{u'}{1-n} + pu = q, \text{ as before.}]$$

1B-9

$n=2$, so $u = y^{1-2} = y^{-1}$ (by problem 1B)

Since we want to substitute for y, y' , express them in terms of u and u' :

$$y = \frac{1}{u}, \quad y' = -\frac{1}{u^2} \cdot u'$$

\therefore the ODE becomes

$$-\frac{u'}{u^2} + \frac{1}{u} = 2x \cdot \frac{1}{u^2}$$

or $\boxed{u' - u = -2x}$ in standard linear eqn form.

$$\text{Integ. factor: } e^{\int -dx} = e^{-x}$$

Eqn becomes

$$(e^{-x}u)' = -2xe^{-x} \leftarrow \text{integrate by parts}$$

$$\therefore e^{-x}u = 2xe^{-x} + 2e^{-x} + C$$

$$u = 2x + 2 + Ce^x$$

$$\therefore \boxed{y = \frac{1}{2x + 2 + Ce^x}}$$

1B-9

$y' - y$ Here $n=3$, so by prob. 1B,

$$u = y^{1-3} = y^{-2}$$

As above, calculate y, y' in terms of u and u' (not other way around)

$$y = \frac{1}{\sqrt{u}}, \quad y' = -\frac{1}{2} u^{-3/2} \cdot u'$$

Substitute into the ODE:

$$-x^2 \cdot \frac{u'}{2u^{3/2}} - \frac{1}{u^{3/2}} = \frac{x}{u^{1/2}}$$

$$\therefore \boxed{u' + \frac{2u}{x} = -\frac{2}{x^2}}$$

This is linear ODE; integ. factor is

$$e^{\int \frac{2dx}{x}} = e^{2\ln x} = x^2$$

ODE becomes

$$x^2 u' + 2xu = -2$$

$$(x^2 u)' = -2$$

$$x^2 u = -2x + C$$

$$u = \frac{C-2x}{x^2}$$

$$\boxed{y = \frac{\pm x}{\sqrt{C-2x}}}$$

1B-10

a) $y = y_1 + u$
 $y' = y_1' + u' = A + By_1 + Cy_1^2 + u'$

Substituting into the ODE:

$$A + By_1 + Cy_1^2 + u' = A + B(y_1 + u) + C(y_1 + u)^2$$

After some algebra,

$$u' = Bu + 2Cy_1 u + Cu^2$$

$$\therefore u' - (B + 2Cy_1)u = Cu^2$$

This is a Bernoulli eq'n (problem 13) with $n = 2$.

b) By inspection, $y_1 = x$ is a solution to the ODE. \therefore put $y = x + u$
 $y' = 1 + u'$;

Substitution into the ODE gives

$$1 + u' = 1 - x^2 + (x + u)^2$$

$$\therefore \boxed{u' - 2xu = u^2}$$

a Bernoulli equation with $n = 2$.

Put $w = u^{-2} = u^{-1}$

$$\therefore u = \frac{1}{w}, \quad u' = -\frac{w'}{w^2}$$

Substituting,

$$\frac{-w'}{w^2} - \frac{2x}{w} = \frac{1}{w^2}$$

or $\boxed{w' + 2xw = -1}$

Linear ODE with integrating factor

$$e^{\int 2x dx} = e^{x^2}$$

$$\therefore (e^{x^2} w)' = -e^{x^2}$$

$$e^{x^2} w = -\int e^{x^2} dx + C$$

$$\boxed{w = -e^{-x^2} \int e^{x^2} dx + Ce^{-x^2}}$$

Finally:

$$y = x + u = x + \frac{1}{w}$$

$$\therefore y = x + \frac{e^{x^2}}{C - \int e^{x^2} dx}$$

(Actually, no value for C gives the original sol'n $y = x$; we have to take " $C = \infty$ ", or simply add $y = x$ to the above family.

1B-11

a) $y' = z$
 $y'' = \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = \frac{dz}{dy} \cdot z$

Substitute into the ODE:

$$\frac{dz}{dy} \cdot z = a^2 y; \quad \text{Sep. vars:}$$

$$z dz = a^2 y dy$$

$$z^2 = a^2 y^2 + K$$

$$z = \sqrt{a^2 y^2 + K}$$

$$\therefore y' = \sqrt{a^2 y^2 + K}$$

Separate variables again:

$$\frac{dy}{\sqrt{a^2 y^2 + K/a^2}} = a dx$$

Look this integral up!

$$\cosh^{-1}\left(\frac{ay}{\sqrt{K}}\right) = ax + C$$

$$y = \frac{\sqrt{K}}{a} \cosh(ax + C)$$

$$\therefore \boxed{y = C_1 \cosh(ax + C)}$$

1B-11

166)

Let $y' = z$

$$y'' = \frac{dz}{dx} = \frac{dz}{dy} \cdot z$$

Substituting, $y \cdot \frac{dz}{dy} \cdot z = z^2$

$$\therefore \frac{dz}{z} = \frac{dy}{y} \quad \therefore \ln z = \ln y + \text{const.}$$

$$\therefore z = y' = Ky$$

Then $\frac{dy}{y} = K dx$

$$\therefore \ln y = Kx + C$$

or $y = e^{Kx+C}$ is the solution

1B-11

(c)

Let $y' = z$

$$y'' = \frac{dz}{dy} \cdot z$$

Substituting, $\frac{dz}{dy} \cdot z = z(1+3y^2)$

$$\therefore dz = (1+3y^2) dy$$

$$\therefore z = y + y^3 + C \quad \text{Using the initial conditions, } C=0$$

$$\therefore \frac{dy}{y+y^3} = dx \quad (\text{remember: } z = \frac{dy}{dx})$$

Integrating by partial fractions:

$$\frac{1}{y+y^3} = \frac{1}{y(y^2+1)} = \frac{1}{y} - \frac{y}{y^2+1}$$

$$\therefore \frac{dy}{y} - \frac{y dy}{y^2+1} = dx$$

$$\ln y - \frac{1}{2} \ln(y^2+1) = x + C$$

Exponentiating both sides,

$$\frac{y}{\sqrt{y^2+1}} = Ke^x$$

Using the initial conditions,

$$\frac{1}{\sqrt{2}} = K$$

\therefore soln: $\frac{y}{\sqrt{y^2+1}} = \frac{e^x}{\sqrt{2}}$

$\rightarrow y = \frac{e^x}{\sqrt{2-e^{2x}}}$

(can solve for y in terms of x , if desired) (by squaring both sides)

1B-12

1. Exact; also linear (divide by $\frac{dx}{dx}$)

2. Linear; (integ. factor is e^{t^2})

3. Homogeneous: put $z = y/x$, get an ODE for z where you separate variables.

4. Separate variables; also linear in q and linear in p .

5. Exact; also linear.

6. Separate variables.

7. Bernoulli equation: $n = -1$
put $u = y^{1-(-1)} = y^2 \dots$

8. Separate variables: $\frac{dv}{e^{3v}} = e^{2u} du$

9. Divide by x - this makes it homogeneous, so put $z = y/x \dots$

10. Linear equation (integ. factor is $\frac{1}{x^2}$)

11. Think of y as indep't variable,
 x as dep't variable; then equation is $\frac{dx}{dy} = x + e^y$, which is linear in x .

12. Separate variables; also a Bernoulli equation (exercise 13)

13. When written in the form $P(x,y)dx + Q(x,y)dy = 0$, it becomes exact.

14. Linear, with int. factor e^{3x}

15. Divide by x - it becomes homogeneous, so put $z = y/x$, etc.

16. Separate variables

17. Riccati equation (exercise 15a)

A particular sol'n is $y_1 = x^2$;
make the substitution $u = y - y_1$,
get Bernoulli eq'n in u ($n = 2$), etc.

18. Autonomous - x missing.

Put $y' = v$, $y'' = v \frac{dv}{dy}$; separate variables

19. homogeneous - put $z = s/t$
($\ln s - \ln t = \ln s/t$, notice)

20. Exact when written as $Pdy + Qdx = 0$

21. Bernoulli eqn with $n = 2$. (ex. 13)

22. Make change of variable

$$u = x + y$$

(so $u' = 1 + y'$)

Then you can separate variables

23. Becomes linear if you think of y as indep't variable, s as dependent variable.

24. Linear (re dep't variable + indep't variable)

25. $y_1 = -x$ is a particular sol'n.

Riccati equation (ex. 15a) -

$$\text{put } u = y - y_1, \dots$$

OR BETTER:

$$\text{write as } y' + (x+y)^2 + (x+y) + 1 = 0.$$

$$\text{and put } u = x + y$$
$$u' = 1 + y'$$

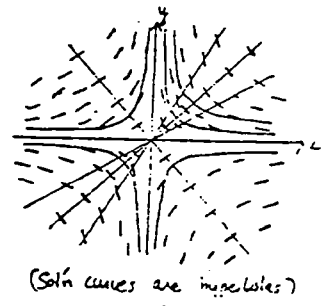
leads to separation of variables.

26. Put $y' = v$ (so $y'' = v'$)

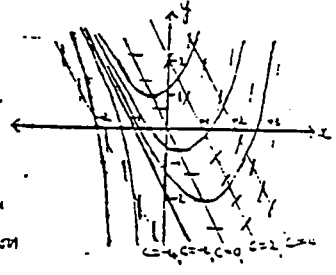
Get a first order linear eq'n in v .

1C-1

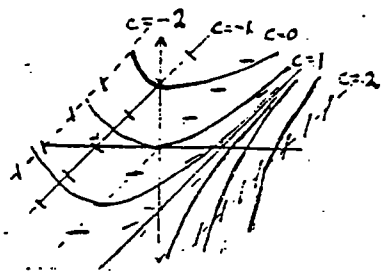
(a) Isoclines: $-\frac{y}{x} = C$
 Exact solution:
 $\frac{dy}{y} = -\frac{dx}{x}$
 $\therefore \ln y = -\ln x + K'$
 $\therefore y = \frac{K'}{x}$



(b) Isoclines:
 $2x + y = C$
 This is a solution
 if $y' = -2 = C$;
 i.e. $y + 2x + 2 = 0$ is an
 isocline which is a solution

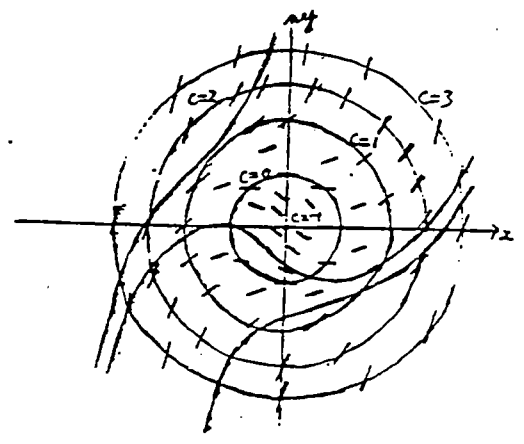


(c) Isoclines:
 $x - y = C$
 This is a solution
 if $y' = 1 = C$;
 i.e. $x - y = 1$ is
 an isocline which is
 a solution.



1C-1

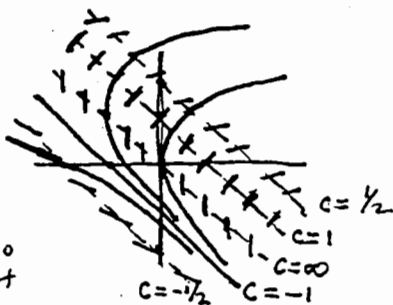
d)
 Isoclines: $x^2 + y^2 - 1 = C$
 i.e. circles centre (0,0), radius $\sqrt{1+C}$



1C-1

e) isoclines
 $x+y = \frac{1}{2}$
 or $y = -x + \frac{1}{2}$

$y = -x - 1$ is an
 integral curve, so
 other solns cannot
 cross it.

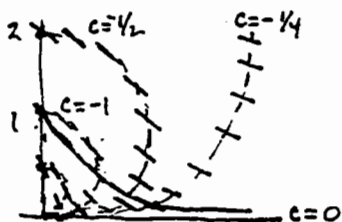


1C-2

isoclines: $x^2 + y^2 + \frac{y}{c} = 0$, or completing
 the square:

$$x^2 + (y + \frac{1}{2c})^2 = (\frac{1}{2c})^2$$

(Circles, center at $(0, -1/2c)$.)



a) decreasing, since

$$y' = -\frac{y}{x^2 + y^2} < 0$$

when $y > 0$

b) soln must have
 $y > 0$ for $x > 0$ since

it cannot cross the integral curve $y=0$.

1C-3

a) Using $\Delta y_n = h f(x_n, y_n) = h(x_n - y_n)$,

get $y_{n+1} = y_n + h(x_n - y_n)$.

Table entries:

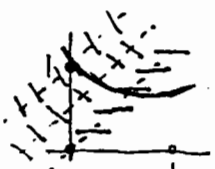
x	0	.1	.2	.3
y	1	.9	.82	.758

For example,

$$y_1 = y_0 + h(x_0 - y_0) = 1 + .1(-1) = .9$$

$$y_2 = y_1 + h(x_1 - y_1) = .9 + .1(.1 - .9) = .82$$

$$y_3 = .82 + .1(.2 - .82) = .758$$

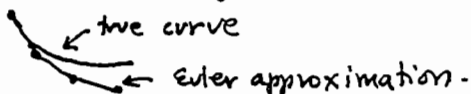


some isoclines $x-y=c$
 are drawn.

soln curve through $(0,1)$

is convex (= "concave up");

thus Euler's method gives too low a result:



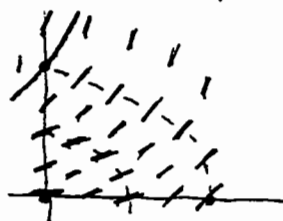
1C-4

Euler method formula:

$$y_{n+1} = y_n + h f(x_n, y_n)$$

x_n	y_n	$f(x_n, y_n)$	$h f(x_n, y_n)$
0	1	1	.1
.1	1.1	1.31	.131
.2	1.231	1.72	.172
.3	1.403		

$h = .1$
 $f(x, y) = x + y^2$



isoclines $x + y^2 = c$
 (parabolas)

Solution curve through
 $(0,1)$ is convex (concave up),

∴ Euler method gives too
 low a result (same reasoning as
 in 1C-2)

1C-3

b)

$$\Delta y_n = \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, \bar{y}_{n+1})]$$

$y_{n+1} - y_n$

For this ODE, $f(x, y) = x - y$

(\bar{y}_{n+1} is the value given by
 the next step of Euler's method).

Thus

So, $y_0 = 1$, $\bar{y}_1 = .9$ (from part a)

$$\begin{aligned} \therefore y_1 - y_0 &= \frac{.1}{2} [f(0, 1) + f(.1, .9)] \\ &= \frac{.1}{2} [-1 - .8] = -.09 \end{aligned}$$

$$\therefore y_1 = y_0 - .09$$

$$y_1 = 1 - .09 = \boxed{.91}$$

This does correct the Euler value ($\bar{y}_1 = .9$)
 in the right direction, since we predicted
 it would be too low. (.910 is actually
 the correct value of the soln to 3 places.)

1C-5

By the formula in 19a,

$$y_n = y_{n-1} + h(x_{n-1} - y_{n-1})$$

$$= (1-h)y_{n-1} + hx_{n-1}$$

But for $x_0 = 0$, we get $x_1 = h$,
 $x_2 = 2h$, and in general
 $x_{n-1} = (n-1)h$.

$$\therefore \boxed{y_n = (1-h)y_{n-1} + h^2(n-1)} \quad (**)$$

We prove by induction that the explicit formula for y_n is:

$$\textcircled{*} \quad \boxed{y_n = 2(1-h)^n - 1 + nh}$$

a) it's true if $n=0$, since
 $y_0 = 2(1-h)^0 - 1 + 0 = 1 \checkmark$

b) if true for y_n , it's true for y_{n+1} since, using $\textcircled{*}$,

$$y_{n+1} = (1-h)y_n + h^2(n+1)$$

$$= 2(1-h)^{n+1} + (1-h)(-1+nh) + h^2(n+1)$$

$$\therefore y_{n+1} = (1-h)^{n+1} - 1 + (n+1)h \checkmark$$

[Note: $\textcircled{*}$ is called a "difference equation" - there are standard ways to solve such things; here $\textcircled{*}$ is the solution].

Continuing, in our case $h = \frac{1}{n}$

$$\therefore y_n = 2\left(1 - \frac{1}{n}\right)^n - 1 + 1$$

$$= 2\left(1 - \frac{1}{n}\right)^n$$

$$\lim_{n \rightarrow \infty} y_n = 2e^{-1} \quad \left(\begin{array}{l} \text{since} \\ \lim_{k \rightarrow \infty} \left(1 + \frac{1}{k}\right)^k = e; \\ \text{put } k = -n \end{array} \right)$$

The exact sol'n to the equation is

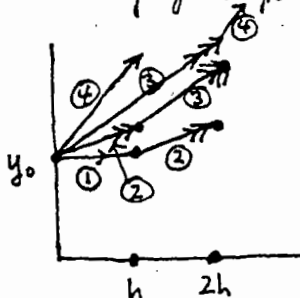
$$y = 2e^{-x} - 1 + x$$

$$\text{so } y(1) = 2e^{-1} - 1 + 1 = 2e^{-1}$$

which checks.

1C-6

It suffices to prove this is true for one step of the Runge-Kutta method and one step of Simpson's rule.



We calculate, in R-K method, the 4 slopes marked ① → ④

Then we use a weighted average of them to find $y(2h)$:

$$y_{2h} = y_0 + 2h \cdot \left(\frac{\textcircled{1} + 2 \cdot \textcircled{2} + 2 \cdot \textcircled{3} + \textcircled{4}}{6} \right)$$

Since the ODE is simply:

$$y' = f(x),$$

from the picture

$$\text{slope } \textcircled{1} = f(0)$$

$$\text{slope } \textcircled{2} = f(h)$$

$$\text{slope } \textcircled{3} = f(h)$$

$$\text{slope } \textcircled{4} = f(2h)$$

$$\therefore y_{2h} = y_0 + \frac{2h}{6} (f(0) + 4f(h) + f(2h))$$

Contrast this with the exact

$$\text{formula: } y_{2h} = y_0 + \int_0^{2h} f(x) dx$$

Evaluating the integral approximately by one step of Simpson's rule:

$$y_{2h} = y_0 + \frac{2h}{6} (f(0) + 4f(h) + f(2h)),$$

same as what Runge-Kutta gives.

1C-7

The existence and uniqueness theorem requires the equation to be written in the form

$$y' = f(x, y)$$

Doing this, we get

$$y' = -\frac{b(x)}{a(x)}y + \frac{c(x)}{a(x)}$$

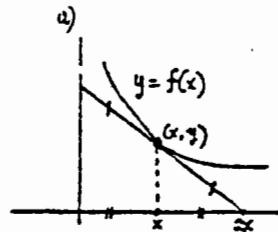
The conditions then are:

" $f(x, y)$ continuous", which will be so if $a(x), b(x), c(x)$ continuous (in an interval $[x_0-h, x_0+h]$) and $a(x) \neq 0$ in this interval.

" $f_y(x, y)$ continuous", which will be so if $\frac{b(x)}{a(x)}$ is continuous, - ~~own~~ and this is already implied by the above condition.

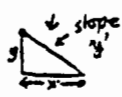
[Note that we must have $a(x) \neq 0$, a condition which is often missed.]

1D-1



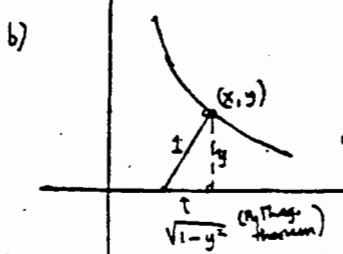
If (x, y) is a point on the curve, the geometric condition translates to:

slope of tan. line = $-\frac{y}{x}$



$\therefore y' = -\frac{y}{x}$

The solution (sep. of vars.) is $y = \frac{C}{x}$ [hyperbolas]



Since the normal is \perp to the tangent, its slope is the negative reciprocal.

$\therefore \frac{y}{1-y^2} = -\frac{1}{y'}$

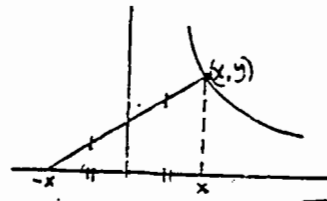
Solve by sep. of variables: $-\frac{y dy}{1-y^2} = dx$

$\therefore \sqrt{1-y^2} = x+C$ or $(x+C)^2 + y^2 = 1$ (Circles, radius 1, centre on x-axis - obvious, what?)

$y = \pm 1$ are also solutions to the problem (above assumed implicitly that $y \neq \pm 1$)

1D-1

(c)



Equating slopes of normal:

$\frac{y}{2x} = -\frac{1}{y'}$ (neg. recip. of slope of tangent)

Solve by sep. vars,

get $\frac{1}{2}y^2 + x^2 = C$ (ellipses)

(d)

The required property translates mathematically into:

$\int_a^x y(t) dt = k(y(x) - y(a))$

$k =$ constant of proportionality

Differentiate this to get an ODE for $y(x)$:

$y(x) = k y'(x)$

(by 2nd Fund. Thm of Calculus)

solution: $y = C e^{x/k}$

this is the general exponential curve.

1D-2

(a)

(i) The y -intercept of line $y = mx + c$ is $(0, c) \therefore c = 2m$

$\therefore y = mx + 2m = m(x+2)$

Differentiating $\Rightarrow y' = m$

Eliminate m : $\therefore y' = \frac{y}{x+2}$ ODE of family

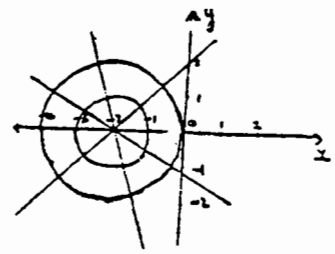
(ii) Orthogonal trajectories satisfy: $-\frac{1}{y'} = \frac{y}{x+2}$
 $\Rightarrow -\frac{dx}{dy} = \frac{y}{x+2} \Rightarrow y dy = -x dx + 2 dx$

$\therefore \frac{y^2}{2} + \frac{x^2}{2} + 2x = \text{const}$

$\therefore (x+2)^2 + y^2 = k$

\therefore Circle centre $(-2, 0)$, variable radii

(iii)



Original family: Lines thro' $(-2, 0)$
 Orthogonal trajectories: Circles centre $(-2, 0)$

1D-2

(b)

$$y = ce^x$$

$$y' = ce^x = y$$

Equation of the orthogonal family:

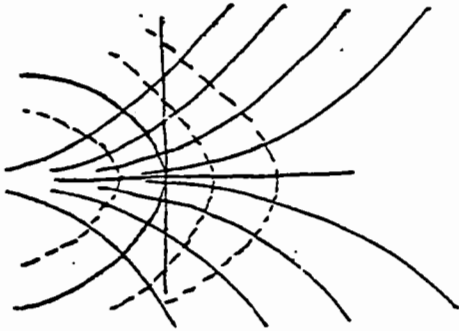
$$y' = -\frac{1}{y}$$

To find the curves, solve by separation of variables:

$$y \, dy = -dx$$

$$\frac{1}{2}y^2 = -x + C \quad \text{parabolas}$$

(all translations of one fixed parabola $\frac{1}{2}y^2 = -x$ along the x-axis)



1D-2

(c)

(i) Differentiating gives

$$2x - 2y y' = 0$$

$\therefore y' = \frac{x}{y}$ is required ODE

(ii) Orthogonal trajectories satisfy

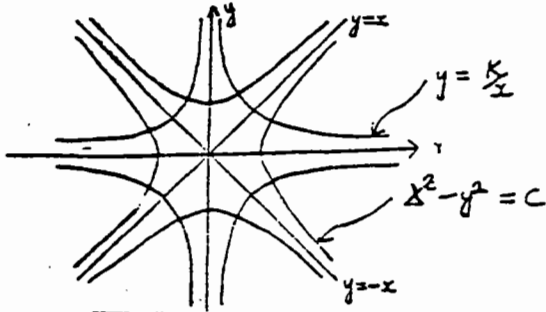
$$-\frac{1}{y'} = \frac{x}{y}$$

$$\therefore -\frac{dy}{y} = \frac{dx}{x}$$

$$\therefore -\ln y = \ln x + C_1$$

$$\therefore y = \frac{K}{x}$$

(iii)



1D-2

(d) Circles with centre on y-axis have equation $x^2 + (y-k)^2 = r^2$

Circle tangent to x-axis

$$\Rightarrow r = \pm k \therefore r^2 = k^2$$

$$\therefore x^2 + y^2 - 2yk = 0$$

$$\therefore \frac{x^2 + y^2}{2y} = k$$

Differentiate w.r.t. x:

$$\therefore \frac{2x + 2yy'}{2y} - \frac{(x^2 + y^2)y'}{2y^2} = 0$$

$$\therefore 2xy + 2y^2 y' - x^2 y' - y^2 y' = 0$$

$$\text{i.e. } y' = \frac{2xy}{x^2 - y^2}$$

(ii) Orthogonal trajectories satisfy

$$-\frac{1}{y'} = \frac{2xy}{x^2 - y^2}$$

$$\text{i.e. } y' = \frac{y^2 - x^2}{2xy} \quad \leftarrow \text{a homogeneous equation}$$

$$\text{let } y = zx \quad \therefore z = \frac{y}{x}$$

$$\text{Then } y' = xz' + z$$

$$\therefore xz' + z = \frac{z^2 x^2 - x^2}{2zx^2} = \frac{z^2 - 1}{2z}$$

$$\therefore xz' = \frac{-(z^2 + 1)}{2z} \quad \text{i.e. } \frac{2z \, dz}{z^2 + 1} = -\frac{dx}{x}$$

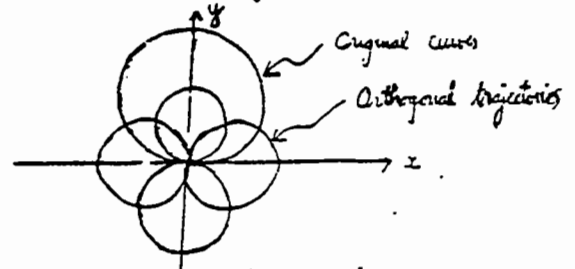
$$\therefore \ln(z^2 + 1) = -\ln x + C$$

$$\therefore z^2 + 1 = \frac{2K}{x} \quad (2K = e^C)$$

$$\therefore y^2 + x^2 = 2Kx$$

These are circles with centre on the x-axis and tangent to y-axis

(iii)



1D-3

a) $\frac{dx(t)}{dt} = \text{rate at which salt flows in} - \text{rate of salt outflow}$
 $= (\text{flow rate in}) \cdot (\text{conc. of salt in added sol'n}) - (\text{flow rate out}) \cdot (\text{conc. of salt in tank})$

$x' = kc_1 - k \cdot \frac{x}{V}$

b) $x' + ax = 0$ (since $c_1 = 0$)
 $x(0) = Vc_0$ ($a = k/v$)

Solution is, by sep. of variables
 $x = Vc_0 e^{-at}$ ($a = k/v$)

c) The general case is $\begin{cases} x' + ax = kc_1, \\ x(0) = Vc_0 \end{cases}$, which can be solved by separating variables, or as a linear equation.

Separating variables:

$\frac{dx}{dt} = kc_1 - ax$

$\frac{dx}{kc_1 - ax} = dt$

$-\frac{1}{a} \ln(kc_1 - ax) = t + A$ (const. of integration)

or $kc_1 - ax = A_1 e^{-at}$ ($A_1 = \text{arbitrary constant}$)

Using the initial condition to find A_1 :
 $kc_1 - aVc_0 = A_1$ (note that $aV = k$)

$\therefore k(c_1 - c_0) = A_1$

so soln is (note that $k/a = v$)
 $x = Vc_1 - V(c_1 - c_0)e^{-at}$

or in terms of the concentration $C(t)$:

$C = c_1 - (c_1 - c_0)e^{-at}$

As $t \rightarrow \infty$, $e^{-at} \rightarrow 0$, so $C \rightarrow c_1$

d) If $c_1 = c_0 e^{-at}$, then the ODE (IVP) becomes (as in (c))
 $\begin{cases} x' + ax = kc_0 e^{-at} \\ x(0) = Vc_0 \end{cases}$

This must be solved as a linear equation.

The integrating factor is e^{at} :
 $x'e^{at} + axe^{at} = kc_0 e^{(a-x)t}$

or $(xe^{at})' = kc_0 e^{(a-x)t}$

Integrating,
 $xe^{at} = \frac{kc_0}{a-x} e^{(a-x)t} + A$ (const. of integ.)

Using the initial condition to find A :

$Vc_0 = A + \frac{kc_0}{a-x}$

$\therefore x = \frac{kc_0}{a-x} e^{-at} + (Vc_0 - \frac{kc_0}{a-x}) e^{-at}$

Dividing by V to get concentration:

$C = \frac{ac_0}{a-x} e^{-at} + (c_0 - \frac{ac_0}{a-x}) e^{-at}$

[If $\alpha = 0$, then $c_1 = c_0$, and this agrees with part (c)]

1D-4

$\frac{dA}{dt} = -\lambda_1 A$, $\lambda_1 = \frac{\ln 2}{\text{half-life}}$

$\frac{dB}{dt} = \text{rate at which B produced by decay of A} - \text{rate at which B is lost by decay of B}$

$\therefore \frac{dB}{dt} = \lambda_1 A - \lambda_2 B$

\therefore From the first equation, $A = A_0 e^{-\lambda_1 t}$

$\therefore \frac{dB}{dt} + \lambda_2 B = \lambda_1 A_0 e^{-\lambda_1 t}$ (ODE for B(t))

Solve it as a linear equation, using $e^{\lambda_2 t}$ as integrating factor, and $B(0) = B_0$ as initial condition.

Solution is
 $B(t) = \frac{\lambda_1 A_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + (B_0 - \frac{\lambda_1 A_0}{\lambda_2 - \lambda_1}) e^{-\lambda_2 t}$

Taking $\lambda_1 = 1$, $\lambda_2 = 2$,

$B(t) = A_0 e^{-t} + (B_0 - A_0) e^{-2t}$

Differentiating to see when $B(t)$ is maximum:

$0 = B'(t) = -A_0 e^{-t} - 2(B_0 - A_0) e^{-2t}$

Solving for t :
 $\frac{A_0}{2(A_0 - B_0)} = e^{-t}$

If $A_0 > 2B_0$, then $t = -\ln(\frac{A_0}{2(A_0 - B_0)}) > 0$

If $A_0 \leq 2B_0$, no solution (the maximum is at $t=0$).

1D-5

By Newton's cooling law
 $\frac{dT}{dt} = k(T-20)$
 (k a constant of proportionality)

Solving this (by sep. of variables) - gives

$$T = \alpha e^{kt} + 20 \quad (\alpha \text{ another constant})$$

$$T(0) = 100$$

$$\therefore \alpha + 20 = 100$$

$$\therefore \alpha = 80$$

$$T(5) = \alpha e^{5k} + 20 = 80$$

$$\therefore \alpha e^{5k} = 60$$

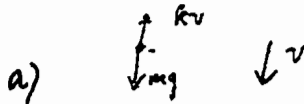
$$\therefore k = \frac{1}{5} \ln\left(\frac{60}{80}\right) = \frac{1}{5} \ln\left(\frac{3}{4}\right) < 0$$

$$\therefore T = 80 e^{-\frac{1}{5} \ln\left(\frac{3}{4}\right)t} + 20$$

When $T = 60$ we then find

$$t = \frac{5 \ln 2}{\ln\left(\frac{4}{3}\right)} \approx 12 \text{ mins.}$$

1D-6



$$\text{Downward force} = m \frac{dv}{dt} = mg - kv$$

$$\therefore \frac{dv}{dt} + \frac{k}{m} v = g$$

Solving this by separation of variables (or as a linear equation), we get

$$v = \alpha e^{-\frac{k}{m}t} + \frac{mg}{k} \quad (\alpha \text{ constant})$$

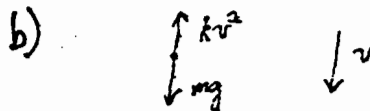
Using the initial condition

$$v(0) = 0 \quad \therefore \frac{mg}{k} + \alpha = 0$$

$$\therefore v = \frac{mg}{k} (1 - e^{-\frac{k}{m}t}) \quad \text{Soln.}$$

terminal velocity:

$$\lim_{t \rightarrow \infty} v(t) = \frac{mg}{k} \quad (\text{constant})$$



$$\text{Downward force} = m \frac{dv}{dt} = mg - kv^2$$

$$\therefore \frac{dv}{v^2 - \frac{mg}{k}} = -\frac{k}{m} dt$$

$$\text{But } \frac{1}{v^2 - \frac{mg}{k}} = \frac{1}{v^2 - a^2} = \frac{1}{2a} \left[\frac{1}{v-a} - \frac{1}{v+a} \right]$$

where $a \equiv \sqrt{\frac{mg}{k}}$

$$\therefore \frac{dv}{v-a} - \frac{dv}{v+a} = -\frac{2ak}{m} dt$$

$$\therefore \ln \left| \frac{v-a}{v+a} \right| = C - \frac{2ak}{m} t$$

$$\text{But } v(0) = 0 \quad \therefore \ln 1 = C$$

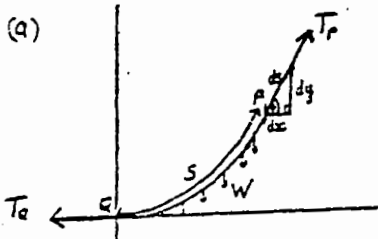
i.e., $C = 0$

$$\therefore \frac{a-v}{a+v} = e^{-\frac{2akt}{m}} \quad (\text{since L.H.S.} > 0 \text{ (at least near } t=0))$$

$$\therefore v = a \left(\frac{1 - e^{-\frac{2akt}{m}}}{1 + e^{-\frac{2akt}{m}}} \right)$$

$$\therefore \lim_{t \rightarrow \infty} v(t) = a = \sqrt{\frac{mg}{k}}$$

1D-7



$\tan \phi = \frac{dy}{dx}$

OR: the Δ s are similar:

 (Δ of forces is closed since cable is in equilibrium)
 $\therefore \frac{dx}{T_0} = \frac{dy}{W} = \frac{ds}{T_p}$
 (corresponding sides)

Balancing forces horizontally

$T_0 = T_p \cos \phi = T_p \frac{dx}{ds}$

$\therefore \frac{ds}{T_p} = \frac{dx}{T_0}$ (i)

Balancing forces vertically

$W = T_p \sin \phi = T_p \frac{dy}{ds}$

$\therefore \frac{ds}{T_p} = \frac{dy}{W}$ (ii) as required.

(b) Suppose the cable hangs under its own weight and has constant density ρ per unit length

then $W = \rho s$

Now $\frac{dx}{T_0} = \frac{dy}{W} = \frac{dy}{\rho s}$

$\therefore \frac{dy}{dx} = Ks$ (where $K = \frac{\rho}{T_0}$ is a constant)

then $\frac{d^2y}{dx^2} = K \frac{ds}{dx} = K \frac{\sqrt{(dx)^2 + (dy)^2}}{dx}$
 $= K \sqrt{1 + (y')^2}$ which gives (i)

Also, $\frac{dy}{W} = \frac{ds}{T_p}$; but $T_p = \sqrt{W^2 + T_0^2}$ (from the force triangle)

$\therefore \frac{dy}{\rho s} = \frac{ds}{\sqrt{\rho^2 s^2 + T_0^2}}$

$= \frac{dy}{ds} = \frac{s}{\sqrt{s^2 + c^2}}$ where $c = T_0/\rho$

$\therefore y = \sqrt{s^2 + c^2} + c_1$, which gives (ii)

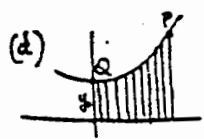
(c) Let λ be the constant weight for unit horizontal length

$\therefore W = \lambda x$

then $\frac{dy}{dx} = \frac{W}{T_0} = \frac{\lambda x}{T_0}$

$\therefore y = \frac{\lambda}{T_0} \frac{x^2}{2} + y_0$

Thus the cable takes the form of a parabola.



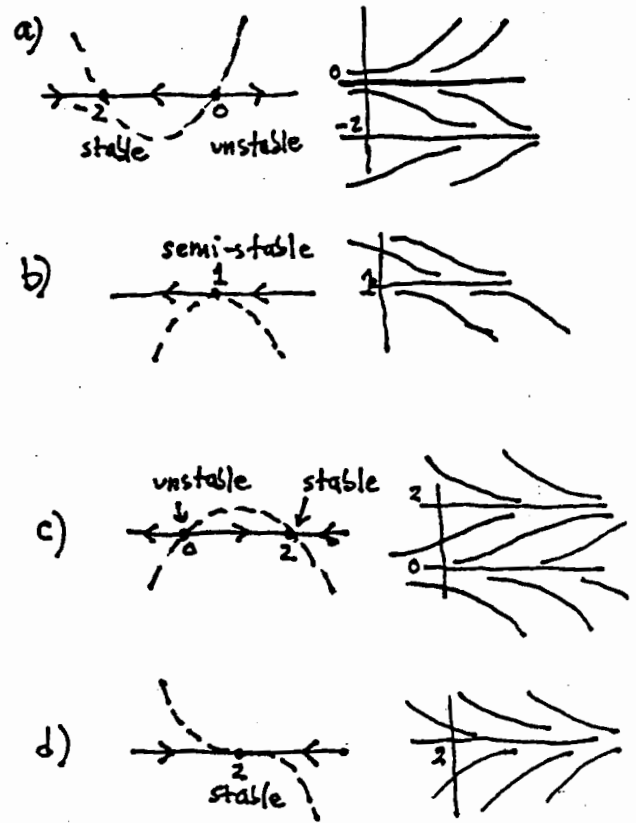
Here $W = k \cdot (\text{area under } \overline{QP})$ since rods are equally and closely spaced.

so $\frac{dy}{dx} = \frac{W}{T_0} = \frac{k}{T_0} \int_0^x y(t) dt$

$\therefore \frac{d^2y}{dx^2} = k^2 y$, by the 2nd Fund. Thm. of Calculus.
 ($k^2 = k/T_0 > 0$)

[The curve is once again of the form $y = \cosh(kx) + c_1$]

1E-1



(write: $(2-x)^3 = -(x-2)^3$)

M.I.T. 18.03 Ordinary Differential Equations
18.03 Notes and Exercises

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