V. VECTOR INTEGRAL CALCULUS

V1. Plane Vector Fields

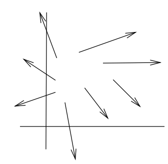
1. Vector fields in the plane; gradient fields.

We consider a function of the type

$$\mathbf{F}(x,y) \;=\; M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}\ . \tag{1}$$

where M and N are both functions of two variables. To each pair of values (x_0, y_0) for which both M and N are defined, such a function assigns a vector $\mathbf{F}(x_0, y_0)$ in the plane. $\mathbf{F}$ is therefore called a **vector function of two variables**. The set of points (x, y) for which $\mathbf{F}$ is defined is called the *domain* of $\mathbf{F}$.

To visualize the function $\mathbf{F}(x, y)$, at each point (x_0, y_0) in the domain we place the corresponding vector $\mathbf{F}(x_0, y_0)$ so that its tail is at (x_0, y_0). Thus each point of the domain is the tail end of a vector, and what we get is called a **vector field**. This vector field gives a picture of the vector function $\mathbf{F}(x, y)$.



Conversely, given a vector field in a region of the xy-plane, it determines a vector function of the type (1), by expressing each vector of the field in terms of its $\mathbf{i}$ and $\mathbf{j}$ components. Thus there is no real distinction between "vector function" and "vector field". Mindful of the applications to physics, in these notes we will mostly use "vector field". We will use the same symbol $\mathbf{F}$ to denote both the field and the function, saying "the vector field $\mathbf{F}$", rather than "the vector field corresponding to the vector function $\mathbf{F}$".

We say the vector field $\mathbf{F}$ is *continuous* in some region of the plane if both $M(x, y)$ and $N(x, y)$ are continuous functions in that region. The intuitive picture of a continuous vector field is that the vectors associated to points sufficiently near (x_0, y_0) should have direction and magnitude very close to that of $\mathbf{F}(x_0, y_0)$ — in other words, as you move around the field, the vectors should change direction and magnitude smoothly, without sudden jumps in size or direction.

In the same way, we say $\mathbf{F}$ is *differentiable* in some region if M and N are differentiable, that is, if all the partial derivatives

$$\frac{\partial M}{\partial x}\,, \qquad \frac{\partial M}{\partial y}\,, \qquad \frac{\partial N}{\partial x}\,, \qquad \frac{\partial N}{\partial y}$$

exist in the region. We say $\mathbf{F}$ is *continuously differentiable* in the region if all these partial derivatives are themselves continuous there. In general, all the commonly used vector fields are continuously differentiable, except perhaps at isolated points, or along certain curves. But as you will see, these points or curves affect the properties of the field in very important ways.

Where do vector fields arise in science and engineering?

One important way is as **gradient vector fields**. If

$$w = f(x,y) \tag{2}$$

is a differentiable function of two variables, then its *gradient*

$$\nabla w = \frac{\partial w}{\partial x}\,\mathbf{i} + \frac{\partial w}{\partial y}\,\mathbf{j} \tag{3}$$

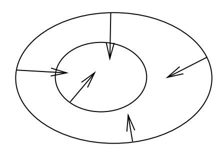
is a vector field, since both partial derivatives are functions of x and y. We recall the geometric interpretation of the gradient:

$$\begin{aligned} \text{dir } \nabla w &= \text{the direction } \mathbf{u} \text{ in which } \left.\frac{dw}{ds}\right|_{\mathbf{u}} \text{ is greatest;} \\ |\nabla w| &= \text{this greatest value of } \left.\frac{dw}{ds}\right|_{\mathbf{u}}, \end{aligned} \tag{4}$$

where $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = \nabla w \cdot \mathbf{u}$ is the directional derivative of w in the direction $\mathbf{u}$.

Another important fact about the gradient is that if one draws the contour curves of $f(x,y)$, which by definition are the curves

$$f(x,y) = c, \qquad c \text{ constant},$$

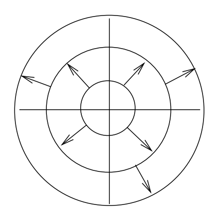


then at every point (x_0, y_0), the gradient vector ∇w at this point is perpendicular to the contour line passing through this point, i.e.,

(5) *the gradient field of f is perpendicular to the contour curves of f .*

Example 1. Let $w = \sqrt{x^2+y^2} = r$. Using the definition (3) of gradient, we find

$$\nabla w = \frac{x}{r}\,\mathbf{i} + \frac{y}{r}\,\mathbf{j} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{r}\ .$$



The domain of ∇w is the xy-plane with $(0,0)$ deleted, and it is continuously differentiable in this region. Since $|x\,i + y\,j| = r$, we see that $|\nabla w| = 1$. Thus all the vectors of the vector field ∇w are unit vectors, and they point radially outward from the origin. This makes sense by (4), since the definition of w shows that dw/ds should be greatest in the radially outward direction, and have the value 1 in that direction.

Finally, the contour curves for w are circles centered at $(0,0)$, which are perpendicular to the vectors ∇w everywhere, as (5) predicts.

2. Force and velocity fields.

Continuing our search for ways in which vector fields arise, here are two physical situations which are described mathematically by vector fields. We shall refer to them often in the sequel, using our physical intuition to suggest the sort of mathematical properties that vector fields ought to have.

Force fields.

From physics, we have the two-dimensional electrostatic force fields arising from a distribution of static (i.e., not moving) charges in the plane. At each point (x_0, y_0) of the plane, we put a vector representing the force which would act on a unit positive charge placed at that point.

In the same way, we get vector fields arising from a distribution of masses in the xy-plane, representing the gravitational force acting at each point on a unit mass. There are also the electromagnetic fields arising from moving electric charges and/or a distribution of magnets, representing the magnetic force at each point.

Any of these we shall simply refer to as a **force field**.

Example 2. Express in $\mathbf{i} - \mathbf{j}$ form the electrostatic force field $\mathbf{F}$ in the xy-plane arising from a unit positive charge placed at the origin, given that the force vector at (x, y) is directed radially away from the origin and that it has magnitude c/r^2, c constant.

Solution. Since the vector $x\,\mathbf{i} + y\,\mathbf{j}$ with tail at (x, y) is directed radially outward and has magnitude r, it has the right direction, and we need only change its magnitude to c/r^2. We do this by multiplying it by c/r^3, which gives

$$\mathbf{F} \;=\; \frac{cx}{r^3}\,\mathbf{i} + \frac{cy}{r^3}\,\mathbf{j} \;=\; c\,\frac{x\,\mathbf{i} + y\,\mathbf{j}}{(x^2+y^2)^{3/2}}\ .$$

Flow fields and velocity fields

A second way vector fields arise is as the steady-state *flow fields* and *velocity fields*.

Imagine a fluid flowing in a horizontal shallow tank of uniform depth, and assume that the flow pattern at any point is purely horizontal and not changing with time. We will call this a *two-dimensional steady-state flow* or for short, simply a *flow*. The fluid can either be compressible (like a gas), or incompressible (like water). We also allow for the possibility that at various points, fluid is beiong added to or subtracted from the flow; for instance, someone could be standing over the tank pouring in water at a certain point, or over a certain area. We also allow the density to vary from point to point, as it would for an unevenly heated gas.

With such a flow we can associate two vector fields.

There is the **velocity field** $\mathbf{v}(x, y)$ where the vector $\mathbf{v}(x, y)$ at the point (x, y) represents the velocity vector of the flow at that point — that is, its direction gives the direction of flow, and its magnitude gives the speed of the flow.

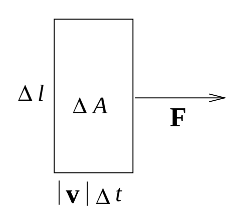
Then there is the **flow field**, defined by

$$\mathbf{F} \;=\; \delta(x, y)\,\mathbf{v}(x, y) \tag{6}$$

where $\delta(x, y)$ gives the density of the fluid at the point (x, y), in terms of mass per unit area. Assuming it is not 0 at a point (x, y), we can interpret $\mathbf{F}(x, y)$ as follows:

$$\text{dir}\,\mathbf{F} \;=\; \text{direction of fluid flow at } (x, y); \tag{7}$$

$$|\mathbf{F}| \;=\; \begin{cases} \text{rate (per unit length per second) of mass transport} \\ \text{across a line perpendicular to the flow direction at } (x, y). \end{cases}$$



Namely, we see that first by (6) and then by the picture,

$$|\mathbf{F}|\,\Delta l\,\Delta t \;=\; \delta\,|\mathbf{v}|\,\Delta t\,\Delta l \;=\; \text{mass in } \Delta A,$$

from which (7) follows by dividing by $\Delta l\,\Delta t$ and letting Δl and $\Delta t \to 0$.

If the density is a constant δ_0, as it would be for an incompressible fluid at a uniform temperature, then the flow field and velocity field are essentially the same, by (6) — the vectors of one are just a constant scalar multiple of the vectors of the other.

Example 3. Describe and interpret $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2 + y^2}$ as a flow field and a force field.

Solution. As in Example 2, the field $\mathbf{F}$ is defined everywhere except $(0,0)$ and its direction is radially outward; now, however, its magnitude is r/r^2, i.e., $|\mathbf{F}| = 1/r$.

$\mathbf{F}$ is the *flow field* for a source of magnitude 2π at the origin. To see this, look at a circle of radius a centered at the origin. At each point P on this circle, the flow is radially outward and by (7),

$$\text{mass transport rate at P} \;=\; \frac{1}{a}, \qquad \text{so that}$$

$$\text{mass transport rate across circle} \;=\; \frac{1}{a}\cdot 2\pi a \;=\; 2\pi\ .$$

This shows that in one second, 2π mass flows out through every circle centered at the origin. This is the flow field for a source of magnitude 2π at the origin — for example, one could imagine a narrow pipe placed over the tank, introducting 2π mass units per second at the point $(0,0)$.

We know that $|\mathbf{F}| = \delta\,|\mathbf{v}| = 1/r$. Two important cases are:

• if the fluid is incompressible, like water, then its density is constant, so the flow speed must decrease like $1/r$ — the flow outward gets slower the further you are from the origin;

• if it is compressible like a gas, and its flow speed stays constant, then the density must decrease like $1/r$.

We now interpret the same field as a *force field*.

Suppose we think of the z-axis in space as a long straight wire, bearing a uniform positive electrostatic charge. This gives us a vector field in space, representing the electrostatic force field.

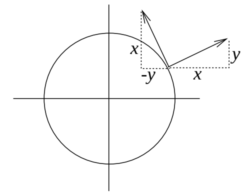
Since one part of the wire looks just like any other part, radial symmetry shows first that the vectors in the force field have 0 as their $\mathbf{k}$-component, i.e., they are pointed radially outward from the wire, and second that their magnitude depends only on their distance r from the wire. It can be shown in fact that the resulting force field is $\mathbf{F}$, up to a constant factor.

Such a field is called "two-dimensional", even though it is a vector field in space, because z and $\mathbf{k}$ don't enter into its description — once you know how it looks in the xy-plane, you know how it looks all through space.

The important thing to notice is that the magnitude of the force field in the xy-plane decreases like $1/r$, *not* like $1/r^2$, as it would if the charge were all at a point.

In the same way, the gravitational field of a uniform mass distribution along the z-axis would be $-\mathbf{F}$, up to a constant factor, and would be called a "two-dimensional gravitational field". Naturally, we don't have infinite long straight wires, but if you have a long straight wire, and stay away from its ends, or have only a short straight wire, but stay close to it, the force field will look like **F** near the wire.

Example 4. Find the velocity field of a fluid with density 1 in a shallow tank, rotating with constant angular velocity ω counterclockwise around the origin.



Solution. First we find the field direction at each point (x, y).

We know the vector $x\,\mathbf{i} + y\,\mathbf{j}$ is directed radially outward. Therefore a vector perpendicular to it in the counterclockwise direction (see picture) will be $-y\,\mathbf{i} + x\,\mathbf{j}$ (since its scalar product with $x\,\mathbf{i} + y\,\mathbf{j}$ is 0 and the signs are correct).

The preceding vector has magnitude r. If the angular velocity is ω, then the linear velocity is given by

$$|\mathbf{v}| = \omega r,$$

so to get the velocity field, we should multiply the above field by ω :

$$\mathbf{v} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j} \ .$$

Exercises: Section 4A

18.02 Notes and Exercises by A. Mattuck and Bjorn Poonen with the assistance of T.Shifrin and S. LeDuc