NON-INDEPENDENT VARIABLES

1. Introduction

Up to now, we have considered partial derivatives of n-variable functions defined on all of $\mathbb{R}^n$ or on an n-dimensional subset defined by inequalities (for example, the orthant defined by $x_1, \ldots, x_n > 0$). In such a situation, the variables are free to change independently, and in particular it makes sense to vary one variable while holding all the others constant.

But if a function is defined only on a subset defined by constraint equations, such as the unit sphere $x^2 + y^2 + z^2 = 1$, then it might be impossible to vary one variable while holding the others constant. In such a situation, extra care is needed in defining partial derivatives and working with them.

2. Constraint equations

Part of the specification of a function is its domain, the set of inputs on which it is being considered.

Example 2.1. A meteorologist studying the current world temperature is probably not interested in the temperature in deep outer space, and hence would be more likely to model temperature as a function defined on the sphere $x^2+y^2+z^2 = 1$ instead of a function defined on all of $\mathbb{R}^3$. In this case, the domain is the sphere $x^2 + y^2 + z^2 = 1$, and $x^2 + y^2 + z^2 = 1$ is called a constraint equation.

Example 2.2. The pressure P, volume V, and temperature T of a fixed amount of gas satisfy the ideal gas law $PV = nRT$, where n and R are constants (here n is the amount of gas measured in moles, and R is a universal constant). A thermodynamic quantity expressible in terms of P, V, and T (such as internal energy U or entropy S) is represented mathematically as a function whose domain is the set of points in the 3-dimensional (P, V, T)-space satisfying the constraint equation $PV = nRT$ and the constraint inequalities $P, V, T > 0$; this domain is a 2-dimensional surface inside the first orthant of $\mathbb{R}^3$. The state of the gas at a particular time is given by the numerical values of (P, V, T) then; mathematically, a state is just a point in the domain. In thermodynamics, the domain is also called the state space because it is the set of all physically possible states.

Date: March 7, 2013.

These notes were written by Bjorn Poonen, based on earlier notes by Arthur Mattuck, and incorporating suggestions by David Jerison, Haynes Miller, Gigliola Staffilani, and David Vogan.

3. Constraint equations and dimension

Although the sphere $x^2 + y^2 + z^2 = 1$ is contained in 3-dimensional space, it is really a 2-dimensional object, with surface area, not volume (the interior is not included).

Rule of thumb:

$$\boxed{\text{Each constraint equation usually reduces the dimension of the domain by 1.}}$$

Example 3.1. The space $\mathbb{R}^3$ has dimension 3, but the set of points in $\mathbb{R}^3$ satisfying the constraint equation

$$x + 2y + 3z = 5$$

is only 2-dimensional (as you know, it's a plane).

Example 3.2. Similarly, the solution set to the system

$$x^2 + y^2 + z^2 = 100$$
$$x + 2y + 3z = 5$$

with two constraint equations is only 1-dimensional. (It is the intersection of a sphere and a plane, and it turns out to be a circle.)

The rule of thumb implies that if the domain is defined by e equations in n variables, it will usually be $(n - e)$-dimensional. If $e > n$ (more equations than variables), so that $n - e$ is negative, usually this means that the constraint equations are inconsistent, so they define an *empty* domain.

Warning 3.3. It is not *always* true that each constraint equation reduces the dimension by 1. Sometimes the rule of thumb fails because of redundancy in the equations, as in the following examples.

Example 3.4. The system

$$x + 2y + 3z = 5$$
$$2x + 4y + 6z = 10$$

has two constraint equations, so one might expect them to reduce the dimension from 3 to 1, but in fact the solution set has dimension 2, because the second equation is just double the first one, so it contains no new information; it is as if there were only one constraint equation. (Geometrically, the solution is the intersection of two planes, but the two planes are the same!)

Sometimes the redundancy is a little more subtle:

Example 3.5. Usually a system of 3 linear equations in 3 variables has 0-dimensional solution set (in fact, a single point), but the system

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 3z &= 5 \\ 2x + 3y + 4z &= 6 \end{aligned}$$

has a 1-dimensional solution set since the third equation is a consequence of the first two (it is their sum) and hence contains no new information. On the other hand, the system

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 3z &= 5 \\ 2x + 3y + 4z &= 7 \end{aligned}$$

is inconsistent (no solutions), because the first two equations imply that $2x + 3y + 4z = 6$, making it impossible to satisfy all three equations simultaneously.

Remark 3.6. For a square system of linear equations, redundancy or inconsistency occurs when the coefficient matrix A satisfies $\det A = 0$. Most square matrices have *nonzero* determinant, so most square systems have a 0-dimensional solution set (in fact, we know that if $\det A \neq 0$, then the solution set to $A\mathbf{x} = \mathbf{b}$ consists of a single point, namely $A^{-1}\mathbf{b}$).

4. Constraint inequalities and dimension

What about constraint *inequalities*? These usually do not affect the dimension. For the example, the region in $\mathbb{R}^3$ defined by $x^2 + y^2 + z^2 < 9$ is still 3-dimensional (it is the interior of a ball of radius 3, and has volume).

5. Independent variables

Example 5.1. On the upper half of the circle $x^2 + y^2 = 9$ in $\mathbb{R}^2$ one can express y in terms of x, namely

$$y = \sqrt{9 - x^2}.$$

Here we think of x as an independent variable, and y depends on x. On the lower half one would use

$$y = -\sqrt{9 - x^2}$$

instead.

Example 5.2. Similarly, on the sphere $x^2 + y^2 + z^2 = 1$, one can locally express z as a function of independent variables x and y. (We said "locally" because there is not a single function that works on the whole sphere: one function works on the upper half, and a different function on the lower half.)

In general, for a D-dimensional domain defined by e constraint equations in n variables, usually $D = n - e$ and locally one can choose D variables to be the independent ones so that each of the remaining e variables can be expressed as a function of the D independent variables. In particular,

$$\boxed{\text{number of independent variables} = \text{dimension of the domain.}}$$

The precise mathematical statement along these lines is called the implicit function theorem; it is discussed in more advanced math courses.

6. Functions on a constrained domain

Consider the function

$$f(x,y,z) := x^2y^3z^5 \quad \textit{restricted to the sphere } x^2+y^2+z^2=1.$$

Since there is one equation in three variables, and $3-1=2$, the domain should be 2-dimensional, so we should be able to express f locally in terms of two independent variables.

On the part of the sphere where $z>0$, we may choose x and y as independent variables, so that $z=\sqrt{1-x^2-y^2}$ and

$$f = x^2y^3(1-x^2-y^2)^{5/2}. \tag{1}$$

The point of eliminating z is that now f is expressed in terms of variables x and y that are not related by any constraint equation.

Alternatively, on a suitable part of the sphere we could express f in terms of independent variables y and z, by substituting $x=\sqrt{1-y^2-z^2}$:

$$f = (1-y^2-z^2)y^3z^5. \tag{2}$$

Formulas (1) and (2) represent (pieces of) the *same function*, just expressed in terms of different variables.

7. Derivatives

Question 7.1. Consider the function $f(x,y) := xy$ where (x,y) is constrained to lie on the line $2x+y=7$. What is $\dfrac{df}{dx}$ at the point $(3,1)$?

(We wrote $\dfrac{df}{dx}$ instead of $\dfrac{\partial f}{\partial x}$ because there is only one independent variable: y depends on x.)

Incorrect solution: The derivative of xy with respect to x is y, whose value at $(3,1)$ is 1.

What makes this wrong? It is true that if $f(x,y) := xy$ on $\mathbb{R}^2$, then $\dfrac{\partial f}{\partial x} = y$. But the definition of partial derivative assumes that it makes sense to hold y constant while varying x, which is impossible if (x,y) is required to satisfy the constraint $2x+y=7$.

Correct solution 1 (elimination of dependent variable): There are two variables and one constraint equation, and $2 - 1 = 1$, so we should be looking to express f in terms of one independent variable. In fact, we can choose x as the independent variable, which is what we should do if we are interested in $\frac{df}{dx}$. Now use the constraint equation to eliminate the dependent variable y and express everything in terms of x:

$$\begin{aligned} y &= 7 - 2x \\ f &= x(7-2x) = 7x - 2x^2 \\ \frac{df}{dx} &= 7 - 4x, \end{aligned}$$

and the value of $\frac{df}{dx}$ at $(x, y) = (3, 1)$ is $7 - 4(3) = -5$. □

In Correct Solution 1, we were lucky that it was easy to solve for y in terms of x. In more complicated situations, this might not be possible, but one can still determine how quickly y changes as x changes, by taking the differential of the constraint equation. To see how this works, let's solve the same problem again.

Correct solution 2 (differentials): If $f = xy$ is viewed as a function on $\mathbb{R}^2$, the definition of df gives

$$df = y\,dx + x\,dy. \tag{3}$$

This expresses how f changes as x and y change.

If f is restricted to a function on the domain defined by the constraint equation, then (3) still holds, but now any change in x causes a change in y, so dx and dy are related. To find the relation, *take the differential of the constraint equation* $2x + y = 7$; this gives

$$2\,dx + dy = 0$$

so

$$dy = -2\,dx$$

(which makes sense since (x, y) is constrained to lie on the line $2x + y = 7$ of slope -2). To compute $\frac{df}{dx}$, we want to consider f as a function of the independent variable x alone, so we should express df in terms of dx alone. To eliminate the dy term, substitute $dy = -2\,dx$ into (3) to get

$$\begin{aligned} df &= y\,dx + x(-2\,dx) \\ &= (y - 2x)\,dx. \end{aligned}$$

This means that

$$\frac{df}{dx} = y - 2x.$$

At $(3,1)$, this is $1-2(3)=-5$. $\square$

8. Partial derivatives

Now let's consider what happens in a question like Question 7.1 when there is more than one independent variable.

Question 8.1. Consider the function $f(x,y,z) := x+y+x^2z$ where (x,y,z) is constrained to lie on the surface $xyz=6$. What is $\frac{\partial f}{\partial x}$ at the point $(1,2,3)$?

Answer: In the presence of the constraint equation $xyz=6$, the notation $\frac{\partial f}{\partial x}$ is meaningless, so the question does not make sense!

Here is why: Usually $\frac{\partial f}{\partial x}$ means the rate of change of f as x varies while all the other variables are held constant. But we can't hold both y and z constant while varying x, if we want the constraint equation $xyz=6$ to remain true.

Conclusions:

1. We are allowed to talk about partial derivatives of f only if f is expressed as a function of *independent* variables (independence guarantees that we can vary one variable while holding the others constant).
2. If f is initially expressed in terms of variables satisfying constraint equations, *we must choose some of the variables to be the independent ones*, and view f and all other variables as functions of the independent variables (as in Section 6), before talking about partial derivatives of f. The notation for the partial derivatives must indicate which variables are being used as the independent ones.

The notational convention is that all the independent variables are listed at the bottom of the partial derivative notation, with the variables being held constant listed as subscripts outside parentheses:

Definition 8.2. The notation

$$\left(\frac{\partial f}{\partial x}\right)_y$$

means that we are viewing f as a function of independent variables x and y, and measuring the rate of change of f as x varies while holding y constant.

Similarly, $\left(\frac{\partial f}{\partial x}\right)_z$ means that we are viewing f as a function of independent variables x and z, and measuring the rate of change of f as x varies while holding z constant.

Example 8.3 (analogous to Correct Solution 1 in Section 7)**.** In Question 8.1, we can use the constraint equation to eliminate z and express f in terms of independent variables x and y:

$$f = x + y + x^2\left(\frac{6}{xy}\right) = x + y + \frac{6x}{y}.$$

Then

$$\left(\frac{\partial f}{\partial x}\right)_y = 1 + \frac{6}{y},$$

so $\left(\frac{\partial f}{\partial x}\right)_y$ at $(1, 2, 3)$ is

$$1 + \frac{6}{2} = 4.$$

Example 8.4 (analogous to Correct Solution 1 in Section 7 again)**.** In Question 8.1, we can use the constraint equation to eliminate y and express f in terms of independent variables x and z:

$$f = x + \frac{6}{xz} + x^2 z.$$

Then

$$\left(\frac{\partial f}{\partial x}\right)_z = 1 - \frac{6}{x^2 z} + 2xz,$$

so $\left(\frac{\partial f}{\partial x}\right)_z$ at $(1, 2, 3)$ is

$$1 - \frac{6}{1^2(3)} + 2(1)(3) = 5.$$

Remark 8.5. The values of $\left(\frac{\partial f}{\partial x}\right)_y$ and $\left(\frac{\partial f}{\partial x}\right)_z$ are rates of change as one moves along two different paths in the domain: along the first path y is constant while z varies in response to x varying, but along the second path z is constant while y varies in response to x varying. So it is not surprising that the two values are different.

Remark 8.6. In Question 7.1, there was only one independent variable, namely the variable x with respect to which the derivative was being taken, so it was not necessary to specify the independent variables in the notation $\frac{df}{dx}$.

Example 8.7. Suppose that g is a function of variables s, t, u, v related by one constraint equation. Then usually g would be locally expressible as a function of three independent variables. Thus one might have partial derivatives such as $\left(\frac{\partial g}{\partial u}\right)_{t,v}$, in which g is viewed as a function of independent variables t, u, v.

9. Partial derivatives and differentials

If we cannot solve the constraint equations to eliminate the dependent variables, we can try the method of differentials.

Question 9.1. Suppose that $f(x,y,z) := x + y + x^2 z$, where x, y, z are constrained to lie on the surface $xyz = 6$. What is $\left(\frac{\partial f}{\partial x}\right)_y$ at the point $(1,2,3)$?

This is the same question as in Example 8.3, but this time we are going to answer it using differentials, in a manner similar to Correct Solution 2 in Section 7.

Solution: To compute $\left(\frac{\partial f}{\partial x}\right)_y$, in which the independent variables are x and y, we need to express df in the form

$$df = ?\,dx + ?\,dy,$$

where each ? represents a function; then $\left(\frac{\partial f}{\partial x}\right)_y$ is the first ? (and $\left(\frac{\partial f}{\partial y}\right)_x$ is the second ?).

But f is initially given as a function of dependent variables x, y, z. If $f = x + y + x^2 z$ is viewed as a function on $\mathbb{R}^3$, the definition of df gives

$$df = (1 + 2xz)\,dx + dy + x^2\,dz. \tag{4}$$

If f is restricted to a function on the domain defined by the constraint equation, then (4) still holds, but taking the differential of the constraint equation $xyz = 6$ gives a relation between dx, dy, dz:

$$yz\,dx + xz\,dy + xy\,dz = 0. \tag{5}$$

Because we want df in terms of dx and dy only, we solve (5) for dz,

$$\begin{aligned} xy\,dz &= -yz\,dx - xz\,dy \\ dz &= -\frac{z}{x}\,dx - \frac{z}{y}\,dy, \end{aligned}$$

and substitute into (4):

$$\begin{aligned} df &= (1+2xz)\,dx + dy + x^2\,dz \\ &= (1+2xz)\,dx + dy + x^2\left(-\frac{z}{x}\,dx - \frac{z}{y}\,dy\right) \\ &= (1+2xz)\,dx + dy - xz\,dx - \frac{x^2 z}{y}\,dy \\ &= (1+xz)\,dx + \left(1 - \frac{x^2 z}{y}\right)\,dy. \end{aligned}$$

This means that
$$\left(\frac{\partial f}{\partial x}\right)_y = 1 + xz,$$
and the value of $\left(\frac{\partial f}{\partial x}\right)_y$ at $(1, 2, 3)$ is $1 + 1(3) = 4$.

10. Proving rules concerning partial derivatives

There are many rules relating different partial derivatives, but they all follow from the method of differentials, so there is no need to memorize the rules. The purpose of this section is not to list rules to be memorized, but to give practice in using the method of differentials.

Problem 10.1. The cyclic rule states that if variables x, y, z are related by a constraint equation such that (as expected) any two of the variables may be taken as the independent variables, then
$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.$$
Prove this rule.

Proof. Let $f(x, y, z) = 0$ be the constraint equation, where f is a function that makes sense on $\mathbb{R}^3$. Taking the differential of the constraint equation gives
$$f_x\, dx + f_y\, dy + f_z\, dz = 0,$$
where f_x, f_y, f_z are the partial derivatives of f viewed as a function on $\mathbb{R}^3$ (or at least a 3-dimensional part of $\mathbb{R}^3$). Solving for dx gives
$$dx = -\frac{f_y}{f_x}\, dy - \frac{f_z}{f_x}\, dz, \tag{6}$$
which means that when x is viewed as a function of independent variables y, z, then
$$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{f_y}{f_x},$$
the coefficient of dy in (6). A similar argument shows that
$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{f_z}{f_y}$$
$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{f_x}{f_z},$$
and multiplying all three gives
$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = \left(-\frac{f_y}{f_x}\right)\left(-\frac{f_z}{f_y}\right)\left(-\frac{f_x}{f_z}\right) = -1. \qquad \square$$

Another example is the two-Jacobian rule. To state it, we need a definition:

Definition 10.2. If $u = u(x, y)$ and $v = v(x, y)$, then the Jacobian of (u, v) with respect to (x, y) is the function

$$\frac{\partial(u,v)}{\partial(x,y)} := \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Here u_x means $\left(\frac{\partial u}{\partial x}\right)_y$, and so on.

Two-Jacobian rule: If u, v, w, x, y are related by constraint equations such that any two of the variables may be taken as the independent variables, then

$$\left(\frac{\partial u}{\partial v}\right)_w = \frac{\partial(u,w)/\partial(x,y)}{\partial(v,w)/\partial(x,y)}.$$

(The right side could also be written out as

$$\frac{u_x w_y - w_x u_y}{v_x w_y - w_x v_y},$$

a ratio of determinants.)

How is the two-Jacobian rule used? It says that if one knows the partial derivatives of all the variables with respect to independent variables x, y, then one can calculate the partial derivatives using any other variables as the independent ones.

The two-Jacobian rule can be proved using differentials.

11. Summary

Here is a summary of some of the key points.

- The dimension of a domain in $\mathbb{R}^n$ is defined by e constraint equations (in n variables) is usually $D := n - e$.
- In that case, usually it is possible locally to choose D of the variables to be the independent variables so that the other variables becomes functions of the independent variables. Then a function f on the domain can be locally re-expressed as a function in only the independent variables.
- When discussing partial derivatives of a function defined on a domain defined by constraint equations, one must specify which variables are being used as the independent variables (and specify which independent variable we are changing while holding the other independent variables constant). For example, $\left(\frac{\partial f}{\partial s}\right)_{r,t}$ means that we are viewing f as a function of independent variables r, s, t and measuring the rate of change of f as s varies while r and t are held constant.
- There are two methods for computing a partial derivative like $\left(\frac{\partial f}{\partial s}\right)_{r,t}$:

– Method 1: Eliminate the dependent variables to express f as an explicit function of the independent variables r, s, t.
– Method 2: Start with

$$df = f_r\, dr + f_s\, ds + f_t\, dt + f_u\, du + \cdots ,$$

in which f is viewed as a function on (part of) $\mathbb{R}^n$ before taking the constraint equations into account. Use the differential of the constraint equation(s) to eliminate the differentials of the dependent variables ($du, \ldots$), so as to re-express df in terms of the differentials of the independent variables only:

$$df = ?\, dr + ?\, ds + ?\, dt.$$

Then $\left(\dfrac{\partial f}{\partial s}\right)_{r,t}$ is the function ? in front of ds.

18.02 Notes and Exercises by A. Mattuck and Bjorn Poonen with the assistance of T.Shifrin and S. LeDuc