6. Vector Integral Calculus in Space

6A. Vector Fields in Space

6A-1 Describe geometrically the following vector fields: a) $\dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\rho}$ b) $-x\,\mathbf{i} - z\,\mathbf{k}$

6A-2 Write down the vector field where each vector runs from (x, y, z) to a point half-way towards the origin.

6A-3 Write down the velocity field $\mathbf{F}$ representing a rotation about the x-axis in the direction given by the right-hand rule (thumb pointing in positive x-direction), and having constant angular velocity ω.

6A-4 Write down the most general vector field all of whose vectors are parallel to the plane $3x - 4y + z = 2$.

6B. Surface Integrals and Flux

6B-1 Without calculating, find the flux of $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ through the sphere of radius a and center at the origin. Take $\mathbf{n}$ pointing outward.

6B-2 Without calculation, find the flux of $\mathbf{k}$ through the infinite cylinder $x^2 + y^2 = 1$. (Take $\mathbf{n}$ pointing outward.)

6B-3 Without calculation, find the flux of $\mathbf{i}$ through that portion of the plane $x+y+z = 1$ lying in the first octant (take $\mathbf{n}$ pointed away from the origin).

6B-4 Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\,\mathbf{j}$, and S = the half of the sphere $x^2 + y^2 + z^2 = a^2$ for which $y \geq 0$, oriented so that $\mathbf{n}$ points away from the origin.

6B-5 Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where where $\mathbf{F} = z\,\mathbf{k}$, and S is the surface of Exercise 6B-3 above.

6B-6 Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and S is the part of the paraboloid $z = x^2 + y^2$ lying underneath the plane $z = 1$, with $\mathbf{n}$ pointing generally upwards. Explain geometrically why your answer is negative.

6B-7* Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{x^2 + y^2 + z^2}$, and S is the surface of Exercise 6B-2.

6B-8 Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\,\mathbf{j}$ and S is that portion of the cylinder $x^2 + y^2 = a^2$ between the planes $z = 0$ and $z = h$, and to the right of the xz-plane; $\mathbf{n}$ points outwards.

6B-9* Find the center of gravity of a hemispherical shell of radius a. (Assume the density is 1, and place it so its base is on the xy-plane.

6B-10* Let S be that portion of the plane $-12x + 4y + 3z = 12$ projecting vertically onto the plane region $(x - 1)^2 + y^2 \leq 4$. Evaluate

a) the area of S b) $\displaystyle\iint_S z\,dS$ c) $\displaystyle\iint_S (x^2 + y^2 + 3z)\,dS$

6B-11* Let S be that portion of the cylinder $x^2 + y^2 = a^2$ bounded below by the xy-plane and above by the cone $z = \sqrt{(x-a)^2 + y^2}$.

a) Find the area of S. Recall that $\sqrt{1-\cos\theta} = \sqrt{2}\sin(\theta/2)$. (Hint: remember that the upper limit of integration for the z-integral will be a function of θ determined by the intersection of the two surfaces.)

b) Find the moment of inertia of S about the z-axis. There should be nothing to calculate once you've done part (a).

c) Evaluate $\displaystyle\iint_S z^2\, dS$.

6B-12 Find the average height above the xy-plane of a point chosen at random on the surface of the hemisphere $x^2 + y^2 + z^2 = a^2,\ z \ge 0$.

6C. Divergence Theorem

6C-1 Calculate div **F** for each of the following fields

a) $x^2y\,\mathbf{i} + xy\,\mathbf{j} + xz\,\mathbf{k}$ b)* $3x^2yz\,\mathbf{i} + x^3z\,\mathbf{j} + x^3y\,\mathbf{k}$ c)* $\sin^3 x\,\mathbf{i} + 3y\cos^3 x\,\mathbf{j} + 2x\,\mathbf{k}$

6C-2 Calculate div **F** if $\mathbf{F} = \rho^n(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$, and tell for what value(s) of n we have div $\mathbf{F} = 0$. (Use $\rho_x = x/\rho$, etc.)

6C-3 Verify the divergence theorem when $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and S is the surface composed of the upper half of the sphere of radius a and center at the origin, together with the circular disc in the xy-plane centered at the origin and of radius a.

6C-4* Verify the divergence theorem if **F** is as in Exercise 3 and S is the surface of the unit cube having diagonally opposite vertices at (0,0,0) and (1,1,1), with three sides in the coordinate planes. (All the surface integrals are easy and do not require any formulas.)

6C-5 By using the divergence theorem, evaluate the surface integral giving the flux of $\mathbf{F} = x\,\mathbf{i} + z^2\,\mathbf{j} + y^2\,\mathbf{k}$ over the tetrahedron with vertices at the origin and the three points on the positive coordinate axes at distance 1 from the origin.

6C-6 Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$ over the closed surface S formed below by a piece of the cone $z^2 = x^2 + y^2$ and above by a circular disc in the plane $z = 1$; take **F** to be the field of Exercise 6B-5; use the divergence theorem.

6C-7 Verify the divergence theorem when S is the closed surface having for its sides a portion of the cylinder $x^2 + y^2 = 1$ and for its top and bottom circular portions of the planes $z = 1$ and $z = 0$; take **F** to be

a) $x^2\,\mathbf{i} + xy\,\mathbf{j}$ b)* $zy\,\mathbf{k}$ c)* $x^2\,\mathbf{i} + xy\,\mathbf{j} + zy\,\mathbf{k}$ (use (a) and (b))

6C-8 Suppose div $\mathbf{F} = 0$ and S_1 and S_2 are the upper and lower hemispheres of the unit sphere centered at the origin. Direct both hemispheres so that the unit normal is "up", i.e., has positive **k**-component.

a) Show that $\displaystyle\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} \mathbf{F}\cdot d\mathbf{S}$, and interpret this physically in terms of flux.

b) State a generalization to an arbitrary closed surface S and a field **F** such that div $\mathbf{F} = 0$.

6C-9* Let $\mathbf{F}$ be the vector field for which all vectors are aimed radially away from the origin, with magnitude $1/\rho^2$.

a) What is the domain of $\mathbf{F}$?

b) Show that div $\mathbf{F} = 0$.

c) Evaluate $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is a sphere of radius a centered at the origin. Does the fact that the answer is not zero contradict the divergence theorem? Explain.

d) Prove using the divergence theorem that $\iint_S \mathbf{F} \cdot d\mathbf{S}$ over a positively oriented closed surface S has the value zero if the surface does not enclose the origin, and the value 4π if it does.

($\mathbf{F}$ is the vector field for the flow arising from a *source of strength* 4π at the origin.)

6C-10 A flow field $\mathbf{F}$ is said to be *incompressible* if $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$ for all closed surfaces S. Assume that $\mathbf{F}$ is continuously differentiable. Show that

$$\mathbf{F} \textit{ is the field of an incompressible flow} \quad \Longleftrightarrow \quad \text{div } \mathbf{F} = 0 \ .$$

6C-11 Show that the flux of the position vector $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ outward through a closed surface S is three times the volume contained in that surface.

6D. Line Integrals in Space

6D-1 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the following fields $\mathbf{F}$ and curves C:

a) $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} - x\,\mathbf{k}$; C is the twisted cubic curve $x = t,\ y = t^2,\ z = t^3$ running from $(0,0,0)$ to $(1,1,1)$.

b) $\mathbf{F}$ is the field of (a); C is the line running from $(0,0,0)$ to $(1,1,1)$

c) $\mathbf{F}$ is the field of (a); C is the path made up of the succession of line segments running from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$.

d) $\mathbf{F} = zx\,\mathbf{i} + zy\,\mathbf{j} + x\,\mathbf{k}$; C is the helix $x = \cos t,\ y = \sin t,\ z = t$, running from $(1,0,0)$ to $(1,0,2\pi)$.

6D-2 Let $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$; show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any curve C lying on a sphere of radius a centered at the origin.

6D-3* a) Let C be the directed line segment running from P to Q, and let $\mathbf{F}$ be a constant vector field. Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = \mathbf{F} \cdot PQ$.

b) Let C be a closed space polygon $P_1 P_2 \ldots P_n P_1$, and let $\mathbf{F}$ be a constant vector field. Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. (Use part (a).)

c) Let C be a closed space curve, $\mathbf{F}$ a constant vector field. Show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$. (Use part (b).)

6D-4 a) Let $f(x,y,z) = x^2 + y^2 + z^2$; calculate $\mathbf{F} = \nabla f$.

b) Let C be the helix of 6D-1d above, but running from $t = 0$ to $t = 2n\pi$. Calculate the work done by $\mathbf{F}$ moving a unit point mass along C; use three methods:

(i) directly

(ii) by using the path-independence of the integral to replace C by a simpler path

(iii) by using the first fundamental theorem for line integrals.

6D-5 Let $\mathbf{F} = \nabla f$, where $f(x,y,z) = \sin(xyz)$. What is the maximum value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ over all possible paths C? Give a path C for which this maximum value is attained.

6D-6* Let $\mathbf{F} = \nabla f$, where $f(x,y,z) = \dfrac{1}{x+y+z+1}$. Find the work done by $\mathbf{F}$ carrying a unit point mass from the origin out to ∞ along a ray in the first octant.
(Take the ray to be $x = at, y = bt, z = ct$, with a, b, c positive and $t \geq 0$.)

6E. Gradient Fields in Space

6E-1 Which of the following differentials are exact? For each one which is, express it in the form df for a suitable function $f(x,y,z)$, using one of the systematic methods.

a) $x^2\,dx + y^2\,dy + z^2\,dz$ b) $y^2z\,dx + 2xyz\,dy + xy^2\,dz$

c) $y(6x^2 + z)\,dx + x(2x^2 + z)\,dy + xy\,dz$

6E-2 Find curl $\mathbf{F}$, if $\mathbf{F} = x^2y\,\mathbf{i} + yz\,\mathbf{j} + xyz^2\,\mathbf{k}$.

6E-3 The fields $\mathbf{F}$ below are defined for all x, y, z. For each,

a) show that curl $\mathbf{F} = \mathbf{0}$;

b) find a potential function $f(x,y,z)$, using either method, or inspection.

(i) $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ (ii) $(2xy + z)\,\mathbf{i} + x^2\,\mathbf{j} + x\,\mathbf{k}$ (iii) $yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$

6E-4 Show that if $f(x,y,z)$ and $g(x,y,z)$ are two functions having the same gradient, then $f = g + c$ for some constant c. (Use the Fundamental Theorem for Line Integrals.)

6E-5 For what values of a and b will $\mathbf{F} = yz^2\,\mathbf{i} + (xz^2 + ayz)\,\mathbf{j} + (bxyz + y^2)\,\mathbf{k}$ be a conservative field? Using these values, find the corresponding potential function $f(x,y,z)$ by one of the systematic methods.

6E-6 a) Define what it means for $M dx + N dy + P dz$ to be an exact differential.

b) Find all values of a, b, c for which

$$(axyz + y^3z^2)dx + (a/2)x^2z + 3xy^2z^2 + byz^3)dy + (3x^2y + cxy^3z + 6y^2z^2)dz$$

will be exact.

c) For those values of a, b, c, express the differential as df for a suitable $f(x,y,z)$.

6F. Stokes' Theorem

6F-1 Verify Stokes' theorem when S is the upper hemisphere of the sphere of radius one centered at the origin and C is its boundary; i.e., calculate both integrals in the theorem and show they are equal. Do this for the vector fields

a) $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$; b) $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$.

6F-2 Verify Stokes' theorem if $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$ and S is the portion of the plane $x + y + z = 0$ cut out by the cylinder $x^2 + y^2 = 1$, and C is its boundary (an ellipse).

6F-3 Verify Stokes' theorem when S is the rectangle with vertices at (0,0,0), (1,1,0), (0,0,1), and (1,1,1), and $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$.

6F-4* Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$, where M, N, P have continuous second partial derivatives.

a) Show by direct calculation that div(curl F) = 0.

b) Using (a), show that $\displaystyle\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS = 0$ for any closed surface S.

6F-5 Let S be the surface formed by the cylinder $x^2 + y^2 = a^2$, $0 \le z \le h$, together with the circular disc forming its top, oriented so the normal vector points up or out. Let $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + x^2\,\mathbf{k}$. Find the flux of $\nabla \times \mathbf{F}$ through S

(a) directly, by calculating two surface integrals;
(b) by using Stokes' theorem.

6G. Topological Questions

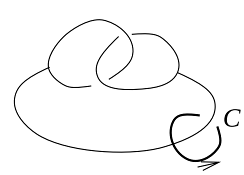
6G-1 Which regions are simply-connected?

a) first octant b) exterior of a torus c) region between two concentric spheres
d) three-space with one of the following removed:
i) a line ii) a point iii) a circle iv) the letter H v) the letter R vi) a ray

6G-2 Show that the fields $\mathbf{F} = \rho^n(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$, where $\rho = \sqrt{x^2 + y^2 + z^2}$, are gradient fields for any value of the integer n. (Use $\rho_x = x/\rho$, etc.)

Then, find the potential function $f(x, y, z)$. (It is easiest to phrase the question in terms of differentials: one wants $df = \rho^n(x\,dx + y\,dy + z\,dz)$; for $n = 0$, you can find f by inspection; from this you can guess the answer for $n \neq 0$ as well. The case $n = -2$ is an exception, and must be handled separately. The printed solutions use this method, somewhat more formally phrased using the fundamental theorem of line integrals.)

6G-3* If D is taken to be the exterior of the wire link shown, then the little closed curve C cannot be shrunk to a point without leaving D, i.e., without crossing the link. Nonetheless, show that C is the boundary of a two-sided surface lying entirely inside D. (So if $\mathbf{F}$ is a field in D such that curl $\mathbf{F} = \mathbf{0}$, the above considerations show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.)



6G-4* In cylindrical coordinates r, θ, z, let $\mathbf{F} = \nabla\varphi$, where $\varphi = \tan^{-1}\dfrac{z}{r-1}$.

a) Interpret φ geometrically. What is the domain of $\mathbf{F}$?

b) From the geometric interpretation what will be the value of $\oint_C \mathbf{F} \cdot d\mathbf{r}$ around a closed path C that links with the unit circle in the xy-plane (for example, take C to be the circle in the yz-plane with radius 1 and center at $(0, 1, 0)$?

6H. Applications to Physics

6H-1 Prove that $\nabla \cdot \nabla \times \mathbf{F} = 0$. What are the appropriate hypotheses about the field $\mathbf{F}$?

6H-2 Show that for any closed surface S, and continuously differentiable vector field $\mathbf{F}$,

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} \;=\; 0\;.$$

Do it two ways: a) using the divergence theorem; b) using Stokes' theorem.

6H-3* Prove each of the following (ϕ is a (scalar) function):

a) $\nabla \cdot (\phi\,\mathbf{F}) \;=\; \phi\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla\phi$

b) $\nabla \times (\phi\mathbf{F}) \;=\; \phi\nabla \times \mathbf{F} + (\nabla\phi) \times \mathbf{F}$

c) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) \;= \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$

6H-4* The *normal derivative.* If S is an oriented surface with unit normal vector $\mathbf{n}$, and ϕ is a function defined and differentiable on some domain containing S, then the **normal derivative** of ϕ on S is defined to be the directional derivative of ϕ in the direction $\mathbf{n}$. In symbols (on the left is the notation for the normal derivative):

$$\frac{\partial\phi}{\partial n} \;=\; \nabla\phi \cdot \mathbf{n}.$$

Prove that if S is closed and D its interior, and if ϕ has continuous second derivatives inside D, then

$$\iint_S \frac{\partial\phi}{\partial n}\,dS \;=\; \iiint_D \nabla^2\phi\,dV\;.$$

(This shows for example that if you are trying to find a *harmonic* function ϕ defined in D and having a prescribed normal derivative on S, you must be sure that $\dfrac{\partial\phi}{\partial n}$ has been prescribed so that $\displaystyle\iint_S \frac{\partial\phi}{\partial n}\,dS \;=\; 0$.

6H-5* Formulate and prove the analogue of the preceding exercise for the plane.

6H-6* Prove that, if S is a closed surface with interior D, and ϕ has continuous second derivatives in D, then

$$\iint_S \phi\frac{\partial\phi}{\partial n}\,dS \;=\; \iiint_D \phi(\nabla^2\phi) + (\nabla\phi)^2\,dV\;.$$

6H-7* Formulate and prove the analogue of the preceding exercise for a plane.

6H-8 A boundary value problem.* Suppose you want to find a function ϕ defined in a domain containing a closed surface S and its interior D, such that (i) ϕ is harmonic in D and (ii) $\phi = 0$ on S.

a) Show that the two conditions imply that $\phi = 0$ on all of D. (Use Exercise 6.)

b) Instead of assuming (ii), assume instead that the values of ϕ on S are prescribed as some continuous function on S. Prove that if a function ϕ exists which is harmonic in D and has these prescribed boundary values, then it is unique — there is only one such function. (In other words, the values of a harmonic function function on the boundary surface S determine its values everywhere inside S.) (Hint: Assume there are two such functions and consider their difference.)

6H-9 Vector potential* In the same way that $\mathbf{F} = \nabla\phi \;\Rightarrow\; \nabla \times \mathbf{F} = \mathbf{0}$ has the partial converse

$$\nabla \times \mathbf{F} = 0 \quad \text{in a simply-connected region} \quad \Rightarrow \quad \mathbf{F} = \nabla f,$$

so the theorem $\quad \mathbf{F} = \nabla \times \mathbf{G} \quad \Rightarrow \quad \nabla \cdot \mathbf{F} = 0 \quad$ has the partial converse

$$(*) \qquad \nabla \cdot \mathbf{F} = 0 \quad \text{in a suitable region} \quad \Rightarrow \quad \mathbf{F} = \nabla \times \mathbf{G}, \quad \text{for some } \mathbf{G}.$$

G is called a **vector potential** for **F**. A suitable region is one with this property: whenever P lies in the region, the whole line segment joining P to the origin lies in the region. (Instead of the origin, one could use some other fixed point.) For instance, a sphere, a cube, or all of 3-space would be suitable regions.

Suppose for instance that $\nabla \cdot \mathbf{F} = 0$ in all of 3-space. Then **G** exists in all of 3-space, and is given by the formula

$$(**) \qquad \mathbf{G} \;=\; \int_0^1 t\,\mathbf{F}(tx, ty, tz) \times \mathbf{R}\, dt, \qquad\qquad \mathbf{R} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

The integral means: integrate separately each component of the vector function occurring in the integrand, and you'll get the corresponding component of **G**.

We shall not prove this formula here; the proof depends on Leibniz' rule for differentiating under an integral sign. We can however try out the formula.

a) Let $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$. Check that div **F**=0, find **G** from the formula (**), and check your answer by verifying that $\mathbf{F} = \text{curl}\ \mathbf{G}$.

b) Show that **G** is unique up to the addition of an arbitrary gradient field; i.e., if **G** is one such field, then all others are of the form

$$(***) \qquad \mathbf{G}' = \mathbf{G} + \nabla f,$$

for an arbitrary function $f(x, y, z)$. (Show that if $\mathbf{G}'$ has the form (***), then $\mathbf{F} = \text{curl}\ \mathbf{G}'$; then show conversely that if $\mathbf{G}'$ is a field such that curl $\mathbf{G}' = \mathbf{F}$, then $\mathbf{G}'$ has the form (***).)

6H-10 Let $\mathbf{B}$ be a magnetic field produced by a moving electric field $\mathbf{E}$. Assume there are no charges in the region. Then one of Maxwell's equations in differential form reads

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} .$$

What is the integrated form of this law? Prove your answer, as in the notes; you can assume that the partial differentiation can be moved outside of the integral sign.

6H-11* In the preceding problem if we also allow for a field $\mathbf{j}$ which gives the current density at each point of space, we get Ampere's law in differential form (as modified by Maxwell):

$$\nabla \times \mathbf{B} = \frac{1}{c}\left(4\pi\,\mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}\right).$$

Give the integrated form of this law, and deduce it from the differential form, as done in the notes.

18.02 Notes and Exercises by A. Mattuck and Bjorn Poonen with the assistance of T.Shifrin and S. LeDuc