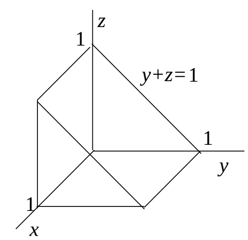
5. Triple Integrals

5A. Triple integrals in rectangular and cylindrical coordinates

5A-1 a) $\displaystyle\int_0^2\int_{-1}^1\int_0^1 (x+y+z)dx\,dy\,dz$ Inner: $\frac{1}{2}x^2 + x(y+z)\Big]_{x=0}^1 = \frac{1}{2} + y + z$

Middle: $\frac{1}{2}y + \frac{1}{2}y^2 + yz\Big]_{y=-1}^1 = 1 + z - (-z) = 1 + 2z$ Outer: $z + z^2\Big]_0^2 = 6$

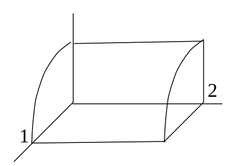
b) $\displaystyle\int_0^2\int_0^{\sqrt{y}}\int_0^{xy} 2xy^2z\,dz\,dx\,dy$ Inner: $xy^2z^2\Big]_0^{xy} = x^3y^4$



Middle: $\frac{1}{4}x^4y^4\Big]_0^{\sqrt{y}} = \frac{1}{4}y^6$ Outer: $\frac{1}{28}y^7\Big]_0^2 = \frac{32}{7}$.

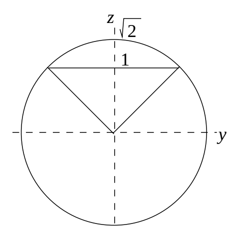
5A-2

a) (i) $\displaystyle\int_0^1\int_0^1\int_0^{1-y} dz\,dy\,dx$ (ii) $\displaystyle\int_0^1\int_0^{1-y}\int_0^1 dx\,dz\,dy$ (iii) $\displaystyle\int_0^1\int_0^1\int_0^{1-z} dy\,dx\,dz$



c) In cylindrical coordinates, with the polar coordinates r and θ in xz-plane, we get

$$\iiint_R r\,dy\,dr\,d\theta = \int_0^{\pi/2}\int_0^1\int_0^2 r\,dy\,dr\,d\theta$$



cross-section view

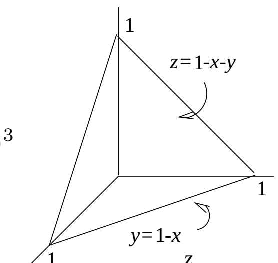
d) The sphere has equation $x^2+y^2+z^2 = 2$, or $r^2+z^2 = 2$ in cylindrical coordinates.

The cone has equation $z^2 = r^2$, or $z = r$. The circle in which they intersect has a radius r found by solving the two equations $z = r$ and $z^2+r^2 = 2$ simultaneously; eliminating z we get $r^2 = 1$, so $r = 1$. Putting it all together, we get

$$\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta.$$

5A-3 By symmetry, $\bar{x} = \bar{y} = \bar{z}$, so it suffices to calculate just one of these, say $\bar{z}$. We have

$$z\text{-moment} = \iiint_D z\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,dz\,dy\,dx$$

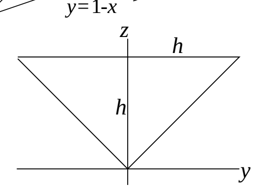


Inner: $\frac{1}{2}z^2\Big]_0^{1-x-y} = \frac{1}{2}(1-x-y)^2$ Middle: $-\frac{1}{6}(1-x-y)^3\Big]_0^{1-x} = \frac{1}{6}(1-x)^3$

Outer: $-\frac{1}{24}(1-x)^4\Big]_0^1 = \frac{1}{24} = \bar{z}$ moment.

mass of D = volume of $D = \frac{1}{3}$(base)(height) $= \frac{1}{3}\cdot\frac{1}{2}\cdot 1 = \frac{1}{6}$.

Therefore $\bar{z} = \frac{1}{24}/\frac{1}{6} = \frac{1}{4}$; this is also $\bar{x}$ and $\bar{y}$, by symmetry.



vertical cross-section

5A-4 Placing the cone as shown, its equation in cylindrical coordinates is $z = r$ and the density is given by $\delta = r$. By the geometry, its projection onto the xy-plane is the interior R of the origin-centered circle of radius h.

a) Mass of solid $D = \iiint_D \delta\, dV = \int_0^{2\pi}\int_0^h\int_r^h r\cdot r\,dz\,dr\,d\theta$

Inner: $(h-r)r^2$; Middle: $\left.\dfrac{hr^3}{3}-\dfrac{r^4}{4}\right]_0^h = \dfrac{h^4}{12}$; Outer: $\dfrac{2\pi h^4}{12}$

b) By symmetry, the center of mass is on the z-axis, so we only have to compute its z-coordinate, $\bar{z}$.

$$z\text{-moment of } D = \iiint_D z\,\delta\,dV = \int_0^{2\pi}\int_0^h\int_r^h zr\cdot r\,dz\,dr\,d\theta$$

Inner: $\left.\tfrac{1}{2}z^2r^2\right]_r^h = \tfrac{1}{2}(h^2r^2-r^4)$ Middle: $\dfrac{1}{2}\left(h^2\dfrac{r^2}{3}-\dfrac{r^5}{5}\right)_0^h = \dfrac{1}{2}h^5\cdot\dfrac{2}{15}$

Outer: $\dfrac{2\pi h^5}{15}$. Therefore, $\bar{z} = \dfrac{\frac{2}{15}\pi h^5}{\frac{2}{12}\pi h^4} = \dfrac{4}{5}h$.

5A-5 Position S so that its base is in the xy-plane and its diagonal D lies along the x-axis (the y-axis would do equally well). The octants divide S into four tetrahedra, which by symmetry have the same moment of inertia about the x-axis; we calculate the one in the first octant. (The picture looks like that for 5A-3, except the height is 2.)

The top of the tetrahedron is a plane intersecting the x- and y-axes at 1, and the z-axis at 2. Its equation is therefore $x+y+\frac{1}{2}z=1$.

The square of the distance of a point (x,y,z) to the axis of rotation (i.e., the x-axis) is given by y^2+z^2. We therefore get:

$$\text{moment of inertia} = 4\int_0^1\int_0^{1-x}\int_0^{2(1-x-y)}(y^2+z^2)\,dz\,dy\,dx.$$

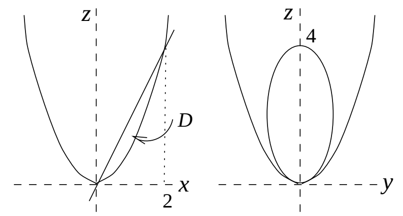
5A-6 Placing D so its axis lies along the positive z-axis and its base is the origin-centered disc of radius a in the xy-plane, the equation of the hemisphere is $z=\sqrt{a^2-x^2-y^2}$, or $z=\sqrt{a^2-r^2}$ in cylindrical coordinates. Doing the inner and outer integrals mentally:

$$z\text{-moment of inertia of } D = \iiint_D r^2\,dV = \int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} r^2\,dz\,r\,dr\,d\theta = 2\pi\int_0^a r^3\sqrt{a^2-r^2}dr.$$

The integral can be done using integration by parts (write the integrand $r^2\cdot r\sqrt{a^2-r^2}$), or by substitution; following the latter course, we substitute $r=a\sin u$, $dr=a\cos u\,du$, and get (using the formulas at the beginning of exercises 3B)

$$\int_0^a r^3\sqrt{a^2-r^2}dr = \int_0^{\pi/2} a^3\sin^3 u\cdot a^2\cos^2 u\,du$$
$$= a^5\int_0^{\pi/2}(\sin^3 u-\sin^5 u)du = a^5\left(\frac{2}{3}-\frac{2\cdot 4}{1\cdot 3\cdot 5}\right) = \frac{2}{15}a^5. \qquad \text{Ans: } \frac{4\pi}{15}a^5.$$

5A-7 The solid D is bounded below by $z = x^2 + y^2$ and above by $z = 2x$. The main problem is determining the projection R of D to the xy-plane, since we need to know this before we can put in the limits on the iterated integral.



cross-section of D view of D along x axis

The outline of R is the projection (i.e., vertical shadow) of the curve in which the paraboloid and plane intersect. This curve is made up of the points in which the graphs of $z = 2x$ and $z = x^2 + y^2$ intersect, i.e., the simultaneous solutions of the two equations. To project the curve, we omit the z-coordinates of the points on it. Algebraically, this amounts to solving the equations simultaneously by eliminating z from the two equations; doing this, we get as the outline of R the curve

$$x^2 + y^2 = 2x \quad \text{or, completing the square,} \quad (x-1)^2 + y^2 = 1.$$

This is a circle of radius 1 and center at $(1, 0)$, whose polar equation is therefore $r = 2\cos\theta$.

We use symmetry to calculate just the right half of D and double the answer:

$$z\text{-moment of inertia of } D = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_{x^2+y^2}^{2x} r^2\, dz\, r\, dr\, d\theta$$

$$= 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_{r^2}^{2r\cos\theta} r^3\, dz\, dr\, d\theta = 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^3(2r\cos\theta - r^2)\, dr\, d\theta$$

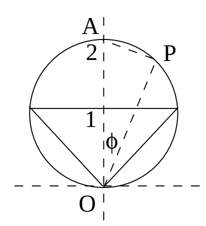
Inner: $\displaystyle \frac{2}{5}r^5\cos\theta - \frac{1}{6}r^6\Big]_0^{2\cos\theta} = \frac{2}{5}\cdot 32\cos^6\theta - \frac{1}{3}\cdot 32\cos^6\theta$

Outer: $\displaystyle \cdot\frac{32}{15}\int_0^{\pi/2}\cos^6\theta\, d\theta = \cdot\frac{32}{15}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2} = \frac{\pi}{3}.$ Ans: $\displaystyle\frac{2\pi}{3}$

5B. Triple Integrals in spherical coordinates

5B-1 a) The angle between the central axis of the cone and any of the lines on the cone is $\pi/4$; the sphere is $\rho = \sqrt{2}$; so the limits are (no integrand given):: $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} d\rho\, d\phi\, d\theta.$

b) The limits are (no integrand is given): $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\infty} d\rho\, d\phi\, d\theta$



cross-section

c) To get the equation of the sphere in spherical coordinates, we note that AOP is always a right triangle, for any position of P on the sphere. Since $AO = 2$ and $OP = \rho$, we get according to the definition of the cosine, $\cos\phi = \rho/2$, or $\rho = 2\cos\phi$. (The picture shows the cross-section of the sphere by the plane containing P and the central axis AO.)

The plane $z = 1$ has in spherical coordinates the equation $\rho\cos\phi = 1$, or $\rho = \sec\phi$. It intersects the sphere in a circle of radius 1; this shows that $\pi/4$ is the maximum value of ϕ for which the ray having angle ϕ intersects the region.. Therefore the limits are (no integrand is given):

$$\int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\phi}^{2\cos\phi} d\rho\, d\phi\, d\theta.$$

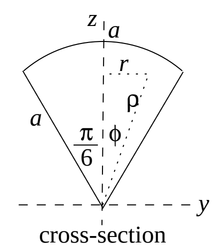
5B-2 Place the solid hemisphere D so that its central axis lies along the positive z-axis and its base is in the xy-plane. By symmetry, $\bar{x} = 0$ and $\bar{y} = 0$, so we only need $\bar{z}$. The integral for it is the product of three separate one-variable integrals, since the integrand is the product of three one-variable functions and the limits of integration are all constants.

$$\bar{z}\text{-moment} = \iiint_D z\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\cdot\left(\frac{1}{4}\rho^4\right)_0^a\cdot\left(\frac{1}{2}\sin^2\phi\right)_0^{\pi/2} = 2\pi\cdot\frac{1}{4}a^4\cdot\frac{1}{2} = \frac{\pi a^4}{4}.$$

Since the mass is $\frac{2}{3}\pi a^3$, we have finally $\bar{z} = \dfrac{\pi a^4/4}{2\pi a^3/3} = \dfrac{3}{8}a.$

5B-3 Place the solid so the vertex is at the origin, and the central axis lies along the positive z-axis. In spherical coordinates, the density is given by $\delta = z = \rho\cos\phi$, and referring to the picture, we have



$$\text{M. of I.} = \iiint_D r^2\cdot z\,dV = \iiint_D (\rho\sin\phi)^2(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/6}\int_0^a \rho^5\sin^3\phi\cos\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\cdot\frac{a^6}{6}\cdot\frac{1}{4}\sin^4\phi\Big]_0^{\pi/6} = 2\pi\cdot\frac{a^6}{6}\cdot\frac{1}{4}\left(\frac{1}{2}\right)^4 = \frac{\pi a^6}{2^6\cdot 3}.$$

5B-4 Place the sphere so its center is at the origin. In each case the iterated integral can be expressed as the product of three one-variable integrals (which are easily calculated), since the integrand is the product of one-variable functions and the limits are constants.

a) $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^a \rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\cdot 2\cdot\frac{1}{4}a^4 = \pi a^4;$ $\qquad$ average $= \dfrac{\pi a^4}{4\pi a^3/3} = \dfrac{3a}{4}.$

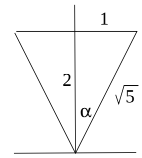
b) Use the z-axis as diameter. The distance of a point from the z-axis is $r = \rho\sin\phi$.

$$\int_0^{2\pi}\int_0^{\pi}\int_0^a \rho\sin\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\cdot\frac{\pi}{2}\cdot\frac{1}{4}a^4 = \frac{\pi^2 a^4}{4}; \qquad \text{average} = \frac{\pi^2 a^4/4}{4\pi a^3/3} = \frac{3\pi a}{16}.$$

c) Use the xy-plane and the upper solid hemisphere. The distance is $z = \rho\cos\phi$.

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^a \rho\cos\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\cdot\frac{1}{2}\cdot\frac{1}{4}a^4 = \frac{\pi a^4}{4}; \qquad \text{average} = \frac{\pi a^4/4}{2\pi a^3/3} = \frac{3a}{8}.$$

5C. Gravitational Attraction



5C-2 The top of the cone is given by $z = 2$; in spherical coordinates: $\rho\cos\phi = 2$. Let α be the angle between the axis of the cone and any of its generators. The density $\delta = 1$. Since the cone is symmetric about its axis, the gravitational attraction has only a k-component, and is

$$G\int_0^{2\pi}\int_0^{\alpha}\int_0^{2/\cos\phi}\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta.$$

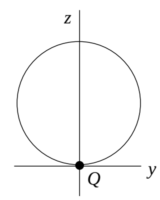
Inner: $\dfrac{2}{\cos\phi}\sin\phi\cos\phi$ Middle: $-2\cos\phi\Big]_0^{\alpha} = -2\cos\alpha + 2$ Outer: $2\pi\cdot 2(1-\cos\alpha)$

Ans: $4\pi G\left(1 - \dfrac{2}{\sqrt{5}}\right)$.

5C-3 Place the sphere as shown so that Q is at the origin. Since it is rotationally symmetric about the z-axis, the force will be in the $\mathbf{k}$-direction.

Equation of sphere: $\rho = 2\cos\phi$ Density: $\delta = \rho^{-1/2}$

$$F_z = G\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\phi}\rho^{-1/2}\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta$$



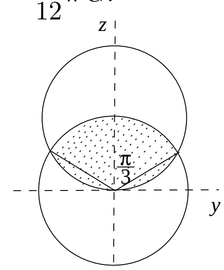
Inner: $\cos\phi\sin\phi\ 2\rho^{1/2}\Big]_0^{2\cos\phi} = 2\sqrt{2}\ \cos^{3/2}\phi\ \sin\phi$

Middle: $2\sqrt{2}\left[-\dfrac{2}{5}\cos^{5/2}\phi\right]_0^{\pi/2} = \dfrac{4\sqrt{2}}{5}$ Outer: $2\pi G\dfrac{4\sqrt{2}}{5} = \dfrac{8\sqrt{2}}{5}\pi G$.

5C-4 Referring to the figure, the total gravitational attraction (which is in the $\mathbf{k}$ direction, by rotational symmetry) is the sum of the two integrals

$$G\int_0^{2\pi}\int_0^{\pi/3}\int_0^{1}\cos\phi\ \sin\phi\,d\rho\,d\phi\,d\theta \quad + \quad G\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2\cos\phi}\cos\phi\ \sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \ 2\pi G\cdot\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 \ + \ 2\pi G\cdot\frac{2}{3}\left(\frac{1}{2}\right)^3 \ = \ \frac{3}{4}\pi G + \frac{1}{6}\pi G = \frac{11}{12}\pi G.$$



The two spheres are shown in cross-section. The spheres intersect at the points where $\phi = \pi/3$.

The first integral respresents the gravitational attraction of the top part of the solid, bounded below by the cone $\phi = \pi/3$ and above by the sphere $\rho = 1$.

The second integral represents the bottom part of the solid, bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone.

18.02 Notes and Exercises by A. Mattuck with the assistance of T.Shifrin and S. LeDuc, and including a section on non-independent variables by Bjorn Poonen.