

3. Double Integrals

3A. Double integrals in rectangular coordinates

3A-1

a) Inner: $6x^2y + y^2 \Big|_{y=-1}^1 = 12x^2$; Outer: $4x^3 \Big|_0^2 = 32$.

b) Inner: $-u \cos t + \frac{1}{2}t^2 \cos u \Big|_{t=0}^\pi = 2u + \frac{1}{2}\pi^2 \cos u$
 Outer: $u^2 + \frac{1}{2}\pi^2 \sin u \Big|_0^{\pi/2} = (\frac{1}{2}\pi)^2 + \frac{1}{2}\pi^2 = \frac{3}{4}\pi^2$.

c) Inner: $x^2y^2 \Big|_{\sqrt{x}}^{x^2} = x^6 - x^3$; Outer: $\frac{1}{7}x^7 - \frac{1}{4}x^4 \Big|_0^1 = \frac{1}{7} - \frac{1}{4} = -\frac{3}{28}$

d) Inner: $v\sqrt{u^2+4} \Big|_0^u = u\sqrt{u^2+4}$; Outer: $\frac{1}{3}(u^2+4)^{3/2} \Big|_0^1 = \frac{1}{3}(5\sqrt{5}-8)$

3A-2

a) (i) $\iint_R dy dx = \int_{-2}^0 \int_{-x}^2 dy dx$ (ii) $\iint_R dx dy = \int_0^2 \int_{-y}^0 dx dy$

b) i) The ends of R are at 0 and 2, since $2x - x^2 = 0$ has 0 and 2 as roots.

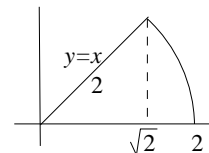
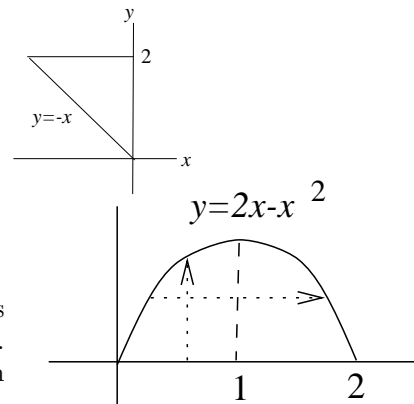
$$\iint_R dy dx = \int_0^2 \int_0^{2x-x^2} dy dx$$

ii) We solve $y = 2x - x^2$ for x in terms of y : write the equation as $x^2 - 2x + y = 0$ and solve for x by the quadratic formula, getting $x = 1 \pm \sqrt{1-y}$. Note also that the maximum point of the graph is (1, 1) (it lies midway between the two roots 0 and 2). We get

$$\iint_R dx dy = \int_0^1 \int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} dx dy,$$

c) (i) $\iint_R dy dx = \int_0^{\sqrt{2}} \int_0^x dy dx + \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-x^2}} dy dx$

(ii) $\iint_R dx dy = \int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} dx dy$



d) Hint: First you have to find the points where the two curves intersect, by solving simultaneously $y^2 = x$ and $y = x - 2$ (eliminate x).

The integral $\iint_R dy dx$ requires two pieces; $\iint_R dx dy$ only one.

3A-3 a) $\iint_R x dA = \int_0^2 \int_0^{1-x/2} x dy dx$;

Inner: $x(1 - \frac{1}{2}x)$ Outer: $\frac{1}{2}x^2 - \frac{1}{6}x^3 \Big|_0^2 = \frac{4}{2} - \frac{8}{6} = \frac{2}{3}$.

$$\text{b) } \iint_R (2x + y^2) dA = \int_0^1 \int_0^{1-y^2} (2x + y^2) dx dy$$

$$\text{Inner: } x^2 + y^2 x \Big|_0^{1-y^2} = 1 - y^2; \quad \text{Outer: } y - \frac{1}{3}y^3 \Big|_0^1 = \frac{2}{3}.$$

$$\text{c) } \iint_R y dA = \int_0^1 \int_{y-1}^{1-y} y dx dy$$

$$\text{Inner: } xy \Big|_{y-1}^{1-y} = y[(1-y) - (y-1)] = 2y - 2y^2 \quad \text{Outer: } y^2 - \frac{2}{3}y^3 \Big|_0^1 = \frac{1}{3}.$$

$$\mathbf{3A-4} \text{ a) } \iint_R \sin^2 x dA = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} \sin^2 x dy dx$$

$$\text{Inner: } y \sin^2 x \Big|_0^{\cos x} = \cos x \sin^2 x \quad \text{Outer: } \frac{1}{3} \sin^3 x \Big|_{-\pi/2}^{\pi/2} = \frac{1}{3}(1 - (-1)) = \frac{2}{3}.$$

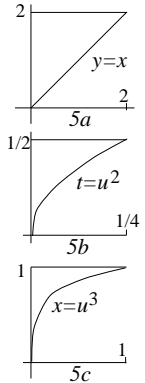
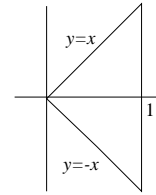
$$\text{b) } \iint_R xy dA = \int_0^1 \int_{x^2}^x (xy) dy dx.$$

$$\text{Inner: } \frac{1}{2}xy^2 \Big|_{x^2}^x = \frac{1}{2}(x^3 - x^5) \quad \text{Outer: } \frac{1}{2} \left(\frac{x^4}{4} - \frac{x^6}{6} \right) \Big|_0^1 = \frac{1}{2} \cdot \frac{1}{12} = \frac{1}{24}.$$

c) The function $x^2 - y^2$ is zero on the lines $y = x$ and $y = -x$, and positive on the region R shown, lying between $x = 0$ and $x = 1$. Therefore

$$\text{Volume} = \iint_R (x^2 - y^2) dA = \int_0^1 \int_{-x}^x (x^2 - y^2) dy dx.$$

$$\text{Inner: } x^2y - \frac{1}{3}y^3 \Big|_{-x}^x = \frac{4}{3}x^3; \quad \text{Outer: } \frac{1}{3}x^4 \Big|_0^1 = \frac{1}{3}.$$



$$\mathbf{3A-5} \text{ a) } \int_0^2 \int_x^2 e^{-y^2} dy dx = \int_0^2 \int_0^y e^{-y^2} dx dy = \int_0^2 e^{-y^2} y dy = -\frac{1}{2}e^{-y^2} \Big|_0^2 = \frac{1}{2}(1 - e^{-4})$$

$$\text{b) } \int_0^{\frac{1}{4}} \int_{\sqrt{t}}^{\frac{1}{2}} \frac{e^u}{u} du dt = \int_0^{\frac{1}{2}} \int_0^{u^2} \frac{e^u}{u} dt du = \int_0^{\frac{1}{2}} u e^u du = (u-1)e^u \Big|_0^{\frac{1}{2}} = 1 - \frac{1}{2}\sqrt{e}$$

$$\text{c) } \int_0^1 \int_{x^{1/3}}^1 \frac{1}{1+u^4} du dx = \int_0^1 \int_0^{u^3} \frac{1}{1+u^4} dx du = \int_0^1 \frac{u^3}{1+u^4} du = \frac{1}{4} \ln(1+u^4) \Big|_0^1 = \frac{\ln 2}{4}.$$

$$\mathbf{3A-6} \quad 0; \quad 2 \iint_S e^x dA, \quad S = \text{upper half of } R; \quad 4 \iint_Q x^2 dA, \quad Q = \text{first quadrant}$$

$$0; \quad 4 \iint_Q x^2 dA; \quad 0$$

$$\mathbf{3A-7} \text{ a) } x^4 + y^4 \geq 0 \Rightarrow \frac{1}{1+x^4+y^4} \leq 1$$

$$\text{b) } \iint_R \frac{x dA}{1+x^2+y^2} \leq \int_0^1 \int_0^1 \frac{x}{1+x^2} dx dy = \frac{1}{2} \ln(1+x^2) \Big|_0^1 = \frac{\ln 2}{2} < \frac{.7}{2}.$$

3B. Double Integrals in polar coordinates

3B-1

a) In polar coordinates, the line $x = -1$ becomes $r \cos \theta = -1$, or $r = -\sec \theta$. We also need the polar angle of the intersection points; since the right triangle is a 30-60-90 triangle (it has one leg 1 and hypotenuse 2), the limits are (no integrand is given):

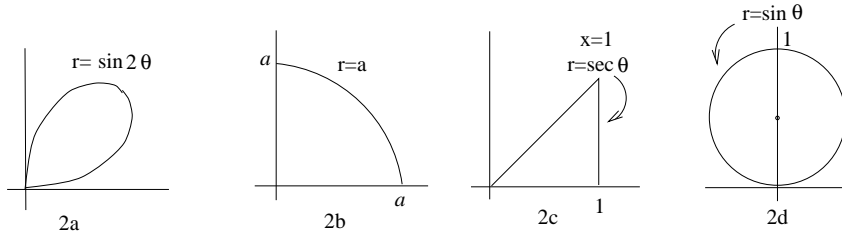
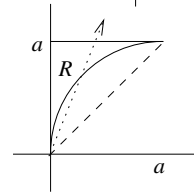
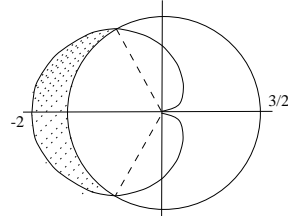
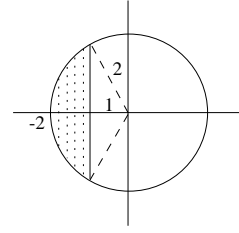
$$\iint_R dr d\theta = \int_{2\pi/3}^{4\pi/3} \int_{-\sec \theta}^2 dr d\theta.$$

c) We need the polar angle of the intersection points. To find it, we solve the two equations $r = \frac{3}{2}$ and $r = 1 - \cos \theta$ simultaneously. Eliminating r , we get $\frac{3}{2} = 1 - \cos \theta$, from which $\theta = 2\pi/3$ and $4\pi/3$. Thus the limits are (no integrand is given):

$$\iint_R dr d\theta = \int_{2\pi/3}^{4\pi/3} \int_{3/2}^{1-\cos \theta} dr d\theta.$$

d) The circle has polar equation $r = 2a \cos \theta$. The line $y = a$ has polar equation $r \sin \theta = a$, or $r = a \csc \theta$. Thus the limits are (no integrand):

$$\iint_R dr d\theta = \int_{\pi/4}^{\pi/2} \int_{2a \cos \theta}^{a \csc \theta} dr d\theta.$$



3B-2 a) $\int_0^{\pi/2} \int_0^{\sin 2\theta} \frac{r dr d\theta}{r} = \int_0^{\pi/2} \sin 2\theta d\theta = -\frac{1}{2} \cos 2\theta \Big|_0^{\pi/2} = -\frac{1}{2}(-1 - 1) = 1.$

b) $\int_0^{\pi/2} \int_0^a \frac{r}{1+r^2} dr d\theta = \frac{\pi}{2} \cdot \frac{1}{2} \ln(1+r^2) \Big|_0^a = \frac{\pi}{4} \ln(1+a^2).$

c) $\int_0^{\pi/4} \int_0^{\sec \theta} \tan^2 \theta \cdot r dr d\theta = \frac{1}{2} \int_0^{\pi/4} \tan^2 \theta \sec^2 \theta d\theta = \frac{1}{6} \tan^3 \theta \Big|_0^{\pi/4} = \frac{1}{6}.$

d) $\int_0^{\pi/2} \int_0^{\sin \theta} \frac{r}{\sqrt{1-r^2}} dr d\theta$

Inner: $-\sqrt{1-r^2} \Big|_0^{\sin \theta} = 1 - \cos \theta$ Outer: $\theta - \sin \theta \Big|_0^{\pi/2} = \pi/2 - 1.$

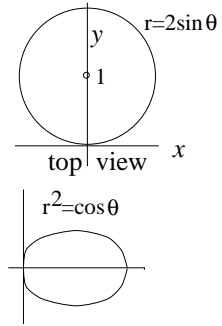
3B-3 a) the hemisphere is the graph of $z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - r^2}$, so we get

$$\iint_R \sqrt{a^2 - r^2} dA = \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2} r dr d\theta = 2\pi \cdot \left[-\frac{1}{3}(a^2 - r^2)^{3/2} \right]_0^a = 2\pi \cdot \frac{1}{3}a^3 = \frac{2}{3}\pi a^3.$$

$$\text{b) } \int_0^{\pi/2} \int_0^a (r \cos \theta)(r \sin \theta)r dr d\theta = \int_0^a r^3 dr \int_0^{\pi/2} \sin \theta \cos \theta d\theta = \frac{a^4}{4} \cdot \frac{1}{2} = \frac{a^4}{8}.$$

c) In order to be able to use the integral formulas at the beginning of 3B, we use symmetry about the y -axis to compute the volume of just the right side, and double the answer.

$$\begin{aligned} \iint_R \sqrt{x^2 + y^2} dA &= 2 \int_0^{\pi/2} \int_0^{2\sin\theta} r r dr d\theta = 2 \int_0^{\pi/2} \frac{1}{3}(2\sin\theta)^3 d\theta \\ &= 2 \cdot \frac{8}{3} \cdot \frac{2}{3} = \frac{32}{9}, \text{ by the integral formula at the beginning of } \mathbf{3B}. \end{aligned}$$



$$\text{d) } 2 \int_0^{\pi/2} \int_0^{\sqrt{\cos\theta}} r^2 r dr d\theta = 2 \int_0^{\pi/2} \frac{1}{4} \cos^2 \theta d\theta = 2 \cdot \frac{1}{4} \cdot \frac{\pi}{4} = \frac{\pi}{8}.$$

3C. Applications of Double Integration

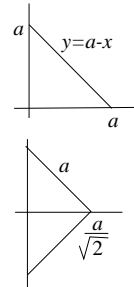
3C-1 Placing the figure so its legs are on the positive x - and y -axes,

$$\text{a) M.I.} = \int_0^a \int_0^{a-x} x^2 dy dx \quad \text{Inner: } x^2 y \Big|_0^{a-x} = x^2(a-x); \quad \text{Outer: } \left[\frac{1}{3}x^3 a - \frac{1}{4}x^4 \right]_0^a = \frac{1}{12}a^4.$$

$$\text{b) } \iint_R (x^2 + y^2) dA = \iint_R x^2 dA + \iint_R y^2 dA = \frac{1}{12}a^4 + \frac{1}{12}a^4 = \frac{1}{6}a^4.$$

c) Divide the triangle symmetrically into two smaller triangles, their legs are $\frac{a}{\sqrt{2}}$;

$$\text{Using the result of part (a), M.I. of } R \text{ about hypotenuse} = 2 \cdot \frac{1}{12} \left(\frac{a}{\sqrt{2}} \right)^4 = \frac{a^4}{24}$$



3C-2 In both cases, \bar{x} is clear by symmetry; we only need \bar{y} .

$$\text{a) Mass is } \iint_R dA = \int_0^{\pi} \sin x dx = 2$$

$$y\text{-moment is } \iint_R y dA = \int_0^{\pi} \int_0^{\sin x} y dy dx = \frac{1}{2} \int_0^{\pi} \sin^2 x dx = \frac{\pi}{4}; \text{ therefore } \bar{y} = \frac{\pi}{8}.$$

b) Mass is $\iint_R y dA = \frac{\pi}{4}$, by part (a). Using the formulas at the beginning of **3B**,

$$y\text{-moment is } \iint_R y^2 dA = \int_0^{\pi} \int_0^{\sin x} y^2 dy dx = 2 \int_0^{\pi/2} \frac{\sin^3 x}{3} dx = 2 \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9},$$

$$\text{Therefore } \bar{y} = \frac{4}{9} \cdot \frac{4}{\pi} = \frac{16}{9\pi}.$$

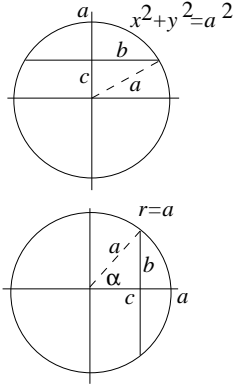
3C-3 Place the segment either horizontally or vertically, so the diameter is respectively on the x or y axis. Find the moment of half the segment and double the answer.

(a) (Horizontally, using rectangular coordinates) Note that $a^2 - c^2 = b^2$.

$$\int_0^b \int_c^{\sqrt{a^2-x^2}} y \, dy \, dx = \int_0^b \frac{1}{2}(a^2 - x^2 - c^2) \, dx = \frac{1}{2} \left[b^2 x - \frac{x^3}{3} \right]_0^b = \frac{1}{3} b^3; \quad \text{ans: } \frac{2}{3} b^3.$$

(b) (Vertically, using polar coordinates). Note that $x = c$ becomes $r = c \sec \theta$.

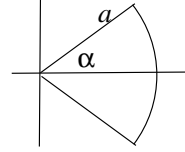
$$\begin{aligned} \text{Moment} &= \int_0^\alpha \int_{c \sec \theta}^a (r \cos \theta) r \, dr \, d\theta \quad \text{Inner: } \frac{1}{3} r^3 \cos \theta \Big|_{c \sec \theta}^a = \frac{1}{3} (a^3 \cos \theta - c^3 \sec^2 \theta) \\ \text{Outer: } &\frac{1}{3} \left[a^3 \sin \theta - c^3 \tan \theta \right]_0^\alpha = \frac{1}{3} (a^2 b - c^2 b) = \frac{1}{3} b^3; \quad \text{ans: } \frac{2}{3} b^3. \end{aligned}$$



3C-4 Place the sector so its vertex is at the origin and its axis of symmetry lies along the positive x -axis. By symmetry, the center of mass lies on the x -axis, so we only need find \bar{x} .

Since $\delta = 1$, the area and mass of the disc are the same: $\pi a^2 \cdot \frac{2\alpha}{2\pi} = a^2 \alpha$.

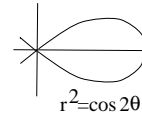
$$\begin{aligned} x\text{-moment: } &2 \int_0^\alpha \int_0^a r \cos \theta \cdot r \, dr \, d\theta \quad \text{Inner: } \frac{2}{3} r^3 \cos \theta \Big|_0^a; \\ \text{Outer: } &\frac{2}{3} a^3 \sin \theta \Big|_0^\alpha = \frac{2}{3} a^3 \sin \alpha \quad \bar{x} = \frac{\frac{2}{3} a^3 \sin \alpha}{a^2 \alpha} = \frac{2}{3} \cdot a \cdot \frac{\sin \alpha}{\alpha}. \end{aligned}$$



3C-5 By symmetry, we use just the upper half of the loop and double the answer. The upper half lies between $\theta = 0$ and $\theta = \pi/4$.

$$2 \int_0^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} r^2 r \, dr \, d\theta = 2 \int_0^{\pi/4} \frac{1}{4} a^4 \cos^2 2\theta \, d\theta$$

$$\text{Putting } u = 2\theta, \text{ the above} = \frac{a^4}{2 \cdot 2} \int_0^{\pi/2} \cos^2 u \, du = \frac{a^4}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi a^4}{16}.$$

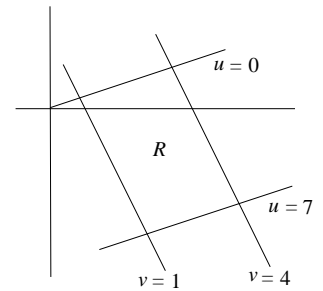


3D. Changing Variables

3D-1 Let $u = x - 3y$, $v = 2x + y$; $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = 7$; $\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{7}$.

$$\iint_R \frac{x - 3y}{2x + y} \, dx \, dy = \frac{1}{7} \int_0^7 \int_1^4 \frac{u}{v} \, dv \, du$$

$$\text{Inner: } u \ln v \Big|_1^4 = u \ln 4; \quad \text{Outer: } \frac{1}{2} u^2 \ln 4 \Big|_0^7 = \frac{49 \ln 4}{2}; \quad \text{Ans: } \frac{1}{7} \frac{49 \ln 4}{2} = 7 \ln 2$$



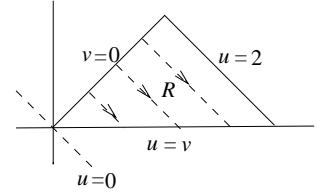
3D-2 Let $u = x + y$, $v = x - y$. Then $\frac{\partial(u, v)}{\partial(x, y)} = 2$; $\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{2}$.

To get the uv -equation of the bottom of the triangular region:

$$y = 0 \Rightarrow u = x, v = x \Rightarrow u = v.$$

$$\iint_R \cos\left(\frac{x-y}{x+y}\right) dx dy = \frac{1}{2} \int_0^2 \int_0^u \cos \frac{v}{u} dv du$$

Inner: $u \sin \frac{v}{u} \Big|_0^u = u \sin 1$ Outer: $\frac{1}{2} u^2 \sin 1 \Big|_0^2 = 2 \sin 1$ Ans: $\sin 1$



3D-3 Let $u = x$, $v = 2y$; $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$

Letting R be the elliptical region whose boundary is $x^2 + 4y^2 = 16$ in xy -coordinates, and $u^2 + v^2 = 16$ in uv -coordinates (a circular disc), we have

$$\begin{aligned} \iint_R (16 - x^2 - 4y^2) dy dx &= \frac{1}{2} \iint_R (16 - u^2 - v^2) dv du \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^4 (16 - r^2) r dr d\theta = \pi \left(16 \frac{r^2}{2} - \frac{r^4}{4} \right)_0^4 = 64\pi. \end{aligned}$$

3D-4 Let $u = x + y$, $v = 2x - 3y$; then $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5$; $\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{5}$.

We next express the boundary of the region R in uv -coordinates.

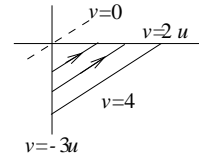
For the x -axis, we have $y = 0$, so $u = x$, $v = 2x$, giving $v = 2u$.

For the y -axis, we have $x = 0$, so $u = y$, $v = -3y$, giving $v = -3u$.

It is best to integrate first over the lines shown, $v = c$; this means v is held constant, i.e., we are integrating first with respect to u . This gives

$$\iint_R (2x - 3y)^2 (x + y)^2 dx dy = \int_0^4 \int_{-v/3}^{v/2} v^2 u^2 \frac{du dv}{5}.$$

Inner: $\frac{v^2}{15} u^3 \Big|_{-v/3}^{v/2} = \frac{v^2}{15} v^3 \left(\frac{1}{8} - \frac{-1}{27} \right)$ Outer: $\frac{v^6}{6 \cdot 15} \left(\frac{1}{8} + \frac{1}{27} \right)_0^4 = \frac{4^6}{6 \cdot 15} \left(\frac{1}{8} + \frac{1}{27} \right)$.

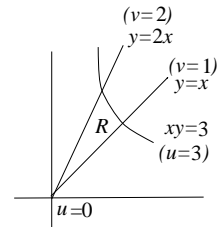


3D-5 Let $u = xy$, $v = y/x$; in the other direction this gives $y^2 = uv$, $x^2 = u/v$.

We have $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = \frac{2y}{x} = 2v$; $\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{2v}$; this gives

$$\iint_R (x^2 + y^2) dx dy = \int_0^3 \int_1^2 \left(\frac{u}{v} + uv \right) \frac{1}{2v} dv du.$$

Inner: $\frac{-u}{2v} + \frac{u}{2} v \Big|_1^2 = u \left(-\frac{1}{4} + 1 + \frac{1}{2} - \frac{1}{2} \right) = \frac{3u}{4}$; Outer: $\frac{3}{8} u^2 \Big|_0^3 = \frac{27}{8}$.



3D-8 a) $y = x^2$; therefore $u = x^3$, $v = x$, which gives $u = v^3$.

b) We get $\frac{u}{v} + uv = 1$, or $u = \frac{v}{v^2 + 1}$; (cf. 3D-5)

**18.02 Notes and Exercises by A. Mattuck and
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