2. Partial Differentiation

2A. Functions and Partial Derivatives

2A-1 Sketch five level curves for each of the following functions. Also, for a-d, sketch the portion of the graph of the function lying in the first octant; include in your sketch the traces of the graph in the three coordinate planes, if possible.

a) $1-x-y$ b) $\sqrt{x^2+y^2}$ c) x^2+y^2 d) $1-x^2-y^2$ e) x^2-y^2

2A-2 Calculate the first partial derivatives of each of the following functions:

a) $w=x^3y-3xy^2+2y^2$ b) $z=\dfrac{x}{y}$ c) $\sin(3x+2y)$ d) e^{x^2y}
e) $z=x\ln(2x+y)$ f) x^2z-2yz^3

2A-3 Verify that $f_{xy}=f_{yx}$ for each of the following:

a) x^my^n, (m,n positive integers) b) $\dfrac{x}{x+y}$ c) $\cos(x^2+y)$
d) $f(x)g(y)$, for any differentiable f and g

2A-4 By using $f_{xy}=f_{yx}$, tell for what value of the constant a there exists a function $f(x,y)$ for which $f_x=axy+3y^2$, $f_y=x^2+6xy$, and then using this value, find such a function by inspection.

2A-5 Show the following functions $w=f(x,y)$ satisfy the equation $w_{xx}+w_{yy}=0$ (called the two-dimensional *Laplace equation*):

a) $w=e^{ax}\sin ay$ (a constant) b) $w=\ln(x^2+y^2)$

2B. Tangent Plane; Linear Approximation

2B-1 Give the equation of the tangent plane to each of these surfaces at the point indicated.

a) $z=xy^2$, $(1,1,1)$ b) $w=y^2/x$, $(1,2,4)$

2B-2 a) Find the equation of the tangent plane to the cone $z=\sqrt{x^2+y^2}$ at the point $P_0:(x_0,y_0,z_0)$ on the cone.

b) Write parametric equations for the ray from the origin passing through P_0, and using them, show the ray lies on both the cone and the tangent plane at P_0.

2B-3 Using the approximation formula, find the approximate change in the hypotenuse of a right triangle, if the legs, initially of length 3 and 4, are each increased by .010 .

2B-4 The combined resistance R of two wires in parallel, having resistances R_1 and R_2 respectively, is given by

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}.$$

If the resistance in the wires are initially 1 and 2 ohms, with a possible error in each of $\pm.1$ ohm, what is the value of R, and by how much might this be in error? (Use the approximation formula.)

2B-5 Give the linearizations of each of the following functions at the indicated points:

a) $(x+y+2)^2$ at $(0,0)$; at $(1,2)$ b) $e^x\cos y$ at $(0,0)$; at $(0,\pi/2)$

2B-6 To determine the volume of a cylinder of radius around 2 and height around 3, about how accurately should the radius and height be measured for the error in the calculated volume not to exceed .1 ?

2B-7 a) If x and y are known to within .01, with what accuracy can the polar coordinates r and θ be calculated? Assume $x = 3,\ y = 4$.

b) At this point, are r and θ more sensitive to small changes in x or in y? Draw a picture showing x, y, r, θ and confirm your results by using geometric intuition.

2B-8* Two sides of a triangle are a and b, and θ is the included angle. The third side is c.

a) Give the approximation for Δc in terms of a, b, c, θ, and $\Delta a, \Delta b, \Delta\theta$.

b) If $a = 1,\ b = 2,\ \theta = \pi/3$, is c more sensitive to small changes in a or b?

2B-9 a) Around the point $(1, 0)$, is $w = x^2(y + 1)$ more sensitive to changes in x or in y?

b) What should the ratio of Δy to Δx be in order that small changes with this ratio produce *no* change in w, i.e., no first-order change — of course w will change a little, but like $(\Delta x)^2$, not like Δx.

2B-10* a) If $|a|$ is much larger than $|b|, |c|$, and $|d|$, to which entry is the value of $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ most sensitive?

b) Given a 3×3 determinant, how would you determine to which entry the value of the determinant is most sensitive? (Consider the various Laplace expansions by the cofactors of a given row or column.)

2C. Differentials; Approximations

2C-1 Find the differential (dw or dz). Make the answer look as neat as possible.

a) $w = \ln(xyz)$ b) $w = x^3y^2z$ c) $z = \dfrac{x - y}{x + y}$ d) $w = \sin^{-1}\dfrac{u}{t}$ (use $\sqrt{t^2 - u^2}$)

2C-2 The dimensions of a rectangular box are 5, 10, and 20 cm., with a possible measurement error in each side of $\pm.1$ cm. Use differentials to find what possible error should be attached to its volume.

2C-3 Two sides of a triangle have lengths respectively a and b, with θ the included angle. Let A be the area of the triangle.

a) Express dA in terms of the variables and their differentials.

b) If $a = 1,\ b = 2,\ \theta = \pi/6$, to which variable is A most sensitive? least sensitive?

c) Using the values in (b), if the possible error in each value is .02, what is the possible error in A, to two decimal places?

2C-4 The pressure, volume, and temperature of an ideal gas confined to a container are related by the equation $PV = kT$, where k is a constant depending on the amount of gas and the units. Calculate dP two ways:

a) Express P in terms of V and T, and calculate dP as usual.

b) Calculate the differential of both sides of the equation, getting a "differential equation", and then solve it algebraically for dP.

c) Show the two answers agree.

2C-5 The following equations define w implicitly as a function of the other variables. Find dw in terms of all the variables by taking the differential of both sides and solving algebraically for dw.

a) $\dfrac{1}{w} = \dfrac{1}{t} + \dfrac{1}{u} + \dfrac{1}{v}$ b) $u^2 + 2v^2 + 3w^2 = 10$

2D. Gradient and Directional Derivative

2D-1 In each of the following, a function f, a point P, and a vector $\mathbf{A}$ are given. Calculate the gradient of f at the point, and the directional derivative $\left.\dfrac{df}{ds}\right|_{\mathbf{u}}$ at the point, in the direction $\mathbf{u}$ of the given vector $\mathbf{A}$.

a) $x^3 + 2y^3$; $(1,1)$, $\mathbf{i} - \mathbf{j}$ b) $w = \dfrac{xy}{z}$; $(2,-1,1)$, $\mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}$

c) $z = x\sin y + y\cos x$; $(0,\pi/2)$, $-3\,\mathbf{i} + 4\,\mathbf{j}$ d) $w = \ln(2t+3u)$; $(-1,1)$, $4\,\mathbf{i} - 3\,\mathbf{j}$

e) $f(u,v,w) = (u + 2v + 3w)^2$; $(1,-1,1)$, $-2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$

2D-2 For the following functions, each with a given point P,

(i) find the maximum and minimum values of $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}}$, as $\mathbf{u}$ varies;

(ii) tell for which directions the maximum and minimum occur;

(iii) find the direction(s) $\mathbf{u}$ for which $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = 0$.

a) $w = \ln(4x - 3y)$, $(1,1)$ b) $w = xy + yz + xz$, $(1,-1,2)$

c) $w = \sin^2(t-u)$, $(\pi/4, 0)$

2D-3 By viewing the following surfaces as a contour surface of a function $f(x,y,z)$, find its tangent plane at the given point.

a) $xy^2z^3 = 12$, $(3,2,1)$; b) the ellipsoid $x^2 + 4y^2 + 9z^2 = 14$, $(1,1,1)$

c) the cone $x^2 + y^2 - z^2 = 0$, (x_0, y_0, z_0) (simplify your answer)

2D-4 The function $T = \ln(x^2+y^2)$ gives the temperature at each point in the plane (except $(0,0)$).

a) At the point $P : (1,2)$, in which direction should you go to get the most rapid increase in T?

b) At P, about how far should you go in the direction found in part (a) to get an increase of .20 in T?

c) At P, approximately how far should you go in the direction of $\mathbf{i} + \mathbf{j}$ to get an increase of about .12?

d) At P, in which direction(s) will the rate of change of temperature be 0?

2D-5 The function $T = x^2 + 2y^2 + 2z^2$ gives the temperature at each point in space.

a) What shape are the isotherms?.

b) At the point $P : (1,1,1)$, in which direction should you go to get the most rapid decrease in T?

c) At P, about how far should you go in the direction of part (b) to get a decrease of 1.2 in T?

d) At P, approximately how far should you go in the direction of $\mathbf{i} - 2\,\mathbf{j} + 2\,\mathbf{k}$ to get an increase of .10?

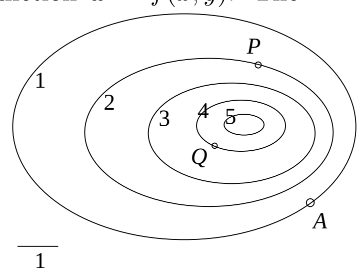
2D-6 Show that $\nabla(uv) = u\nabla v + v\nabla u$, and deduce that $\left.\dfrac{d(uv)}{ds}\right|_{\mathbf{u}} = u\left.\dfrac{dv}{ds}\right|_{\mathbf{u}} + v\left.\dfrac{du}{ds}\right|_{\mathbf{u}}$. (Assume that u and v are functions of two variables.)

2D-7 Suppose $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = 2$, $\left.\dfrac{dw}{ds}\right|_{\mathbf{v}} = 1$ at P, where $\mathbf{u} = \dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$, $\mathbf{v} = \dfrac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$. Find $(\nabla w)_P$.

(This illustrates that the gradient can be calculated knowing the directional derivatives in any two non-parallel directions, not just the two standard directions $\mathbf{i}$ and $\mathbf{j}$.)

2D-8 The atmospheric pressure in a region of space near the origin is given by the formula $P = 30 + (x+1)(y+2)e^z$. Approximately where is the point closest to the origin at which the pressure is 31.1?

2D-9 The accompanying picture shows the level curves of a function $w = f(x, y)$. The value of w on each curve is marked. A unit distance is given.



a) Draw in the gradient vector at A.
b) Find a point B where $w = 3$ and $\partial w/\partial x = 0$.
c) Find a point C where $w = 3$ and $\partial w/\partial y = 0$.
d) At the point P estimate the value of $\partial w/\partial x$ and $\partial w/\partial y$.
e) At the point Q, estimate dw/ds in the direction of $\mathbf{i}+\mathbf{j}$
f) At the point Q, estimate dw/ds in the direction of $\mathbf{i}-\mathbf{j}$.
g) Approximately where is the gradient $\mathbf{0}$?

2E. Chain Rule

2E-1 In the following, find $\dfrac{dw}{dt}$ for the composite function $w = f(x(t), y(t), z(t))$ in two ways:

(i) use the chain rule, then express your answer in terms of t by using $x = x(t)$, etc.;
(ii) express the composite function f in terms of t, and differentiate.

a) $w = xyz, \quad x = t,\ y = t^2,\ z = t^3$ b) $w = x^2 - y^2, \quad x = \cos t,\ y = \sin t$
c) $w = \ln(u^2 + v^2), \quad u = 2\cos t,\ v = 2\sin t$

2E-2 In each of these, information about the gradient of an unknown function $f(x, y)$ is given; x and y are in turn functions of t. Use the chain rule to find out additional information about the composite function $w = f\big(x(t), y(t)\big)$, without trying to determine f explicitly.

a) $\nabla w = 2\,\mathbf{i} + 3\,\mathbf{j}$ at $P : (1, 0)$; $x = \cos t,\ y = \sin t$. Find the value of $\dfrac{dw}{dt}$ at $t = 0$.

b) $\nabla w = y\,\mathbf{i} + x\,\mathbf{j}$; $x = \cos t,\ y = \sin t$. Find $\dfrac{dw}{dt}$ and tell for what t-values it is zero.

c) $\nabla f = \langle 1, -1, 2\rangle$ at $(1, 1, 1)$. Let $x = t,\ y = t^2,\ z = t^3$; find $\dfrac{df}{dt}$ at $t = 1$.

d) $\nabla f = \langle 3x^2y, x^3 + z, y\rangle$; $x = t,\ y = t^2,\ z = t^3$. Find $\dfrac{df}{dt}$.

2E-3 a) Use the chain rule for $f(u, v)$, where $u = u(t),\ v = v(t)$, to prove the product rule

$$D(uv) = v\,Du + u\,Dv, \qquad \text{where } D = \frac{d}{dt}.$$

b) Using the chain rule for $f(u, v, w)$, derive a similar product rule for $\dfrac{d}{dt}(uvw)$, and use it to differentiate $te^{2t}\sin t$.

c)* Derive similarly a rule for the derivative $\dfrac{d}{dt}u^v$, and use it to differentiate $(\ln t)^t$.

2E-4 Let $w = f(x,y)$, and assume that $\nabla w = 2\,\mathbf{i} + 3\,\mathbf{j}$ at $(0,1)$. If $x = u^2 - v^2,\ y = uv$, find $\dfrac{\partial w}{\partial u},\ \dfrac{\partial w}{\partial v}$ at $u = 1,\ v = 1$.

2E-5 Let $w = f(x,y)$, and suppose we change from rectangular to polar coordinates: $x = r\cos\theta,\ y = r\sin\theta$.

a) Show that $(w_x)^2 + (w_y)^2 = (w_r)^2 + \dfrac{1}{r^2}(w_\theta)^2$.

b) Suppose $\nabla w = 2\,\mathbf{i} - \mathbf{j}$ at the point $x = 1,\ y = 1$. Find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial \theta}$ when $r = \sqrt{2},\ \theta = \pi/4$, and verify the relation in part (a), at the point.

2E-6 Let $w = f(x,y)$, and make the change of variables $x = u^2 - v^2,\ y = 2uv$. Show

$$(w_x)^2 + (w_y)^2 = \frac{(w_u)^2 + (w_v)^2}{4(u^2 + v^2)}$$

2E-7 The Jacobian matrix for the change of variables $x = x(u,v),\ y = y(u,v)$ is defined to be $J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$. Let $\nabla f(x,y)$ be represented as the row vector $\langle f_x\ f_y\rangle$.
Show that

$$\nabla f\big(x(u,v), y(u,v)\big) \;=\; \nabla f(x,y)\cdot J \qquad \text{(matrix multiplication).}$$

2E-8 a) Let $w = f(y/x)$; i.e., w is the composite of the functions $w = f(u),\ u = y/x$. Show that w satisfies the PDE (partial differential equation) $\quad x\,\dfrac{\partial w}{\partial x} + y\,\dfrac{\partial w}{\partial y} = 0.$

b)* Let $w = f(x^2 - y^2)$; show that w satisfies the PDE $\quad y\,\dfrac{\partial w}{\partial x} + x\,\dfrac{\partial w}{\partial y} = 0.$

c)* Let $w = f(ax + by)$; show that w satisfies the PDE $\quad b\,\dfrac{\partial w}{\partial x} - a\,\dfrac{\partial w}{\partial y} = 0.$

2F. Maximum-minimum Problems

2F-1 Find the point(s) on each of the following surfaces which is closest to the origin. (Hint: it's easier to minimize the square of the distance, rather than the distance itself.)

a) $xyz^2 = 1$ b) $x^2 - yz = 1$

2F-2 A rectangular produce box is to be made of cardboard; the sides of single thickness, the front and back of double thickness, and the bottom of triple thickness, with the top left open. Its volume is to be 1 cubic foot; what proportions for the sides will use the least cardboard?

2F-3* Consider all planes passing through the point (2,1,1) and such that the intercepts on the three coordinate axes are all positive. For which of these planes is the product of the three intercepts smallest? (Hint: take the plane in the form $z = ax + by + c$, where a and b are the independent variables.)

2F-4* Find the extremal point of $x^2 + 2xy + 4y^2 + 6$ and show it is a minimum point by completing the square.

2F-5 A drawer in a chest has an open top; the bottom and back are made of cheap wood costing \$1/sq. ft; the sides have to be thicker, and cost \$2/sq.ft., while the front costs \$4/sq.ft. for the better quality wood and finishing. The volume is to be 2.5 cu. ft. What dimensions will produce the drawer costing the least to manufacture?

2G. Least-squares Interpolation

2G-1 Find by the method of least squares the line which best fits the three data points given. Do it from scratch, using (2) in Notes LS and differentiation (use the chain rule). Sketch the line and the three points as a check.

a)* $(0,0),(0,2),(1,3)$ b)* $(0,0),(1,2),(2,1)$ c) $(1,1),(2,3),(3,2)$

2G-2* Show that the equations (4) for the method of least squares have a unique solution, unless all the x_i are equal. Explain geometrically why this exception occurs.

Hint: use the fact that for all values of u, we have $\sum_1^n (x_i - u)^2 \geq 0$, since squares are always non-negative. Write the left side as a quadratic polynomial in u. Usually it has no roots. What does this imply about the coefficients? When does it have a root? (Answer these two questions by using the quadratic formula.)

2G-3* Use least squares to fit a second degree polynomial exactly through the points $(-1,-1),(0,0),(1,3)$ (see (6) and (7) in Notes LS).

2G-4 What linear equations in a,b,c does the method of least squares lead to, when you use it to fit a linear function $z = a+bx+cy$ to a set of data points (x_i, y_i, z_i), $i = 1,\dots,n$?

2G-5* What equations are you led to for determining a when you try to fit the exponential curve $y = e^{ax}$ to two data points $(1,y_1),(2,y_2)$ by the method of least squares?

The moral is: don't do it this way. In general to fit an exponential $y = ce^{ax}$ to a set of data points (x_i, y_i), take the log of both sides:

$$\ln y \;=\; a\,x + \ln c$$

This gives a linear function in the variables x and $\ln y$, whose coefficients a and $\ln c$ can be determined by applying the method of least squares to fit the data points $(x_i, \ln y_i)$.

2H. Max-min: 2nd Derivative Criterion; Boundary Curves

2H-1 For each of the following functions, find the critical points, and classify them using the 2nd-derivative criterion.

a) $x^2 - xy - 2y^2 - 3x - 3y + 1$ b) $3x^2 + xy + y^2 - x - 2y + 4$ c) $2x^4 + y^2 - xy + 1$

d) $x^3 - 3xy + y^3$ e) $(x^3+1)(y^3+1)$

2H-2* In Notes LS, use the 2nd-derivative criterion to verify that the critical point (m_0, b_0) determining the regression (= least-squares) line $y = m_0 x + b_0$ really minimizes the function $D(m,b)$ giving the sum of the squares of the deviations. (You will need the inequality in 1B-15, for n-vectors $\mathbf{A} = \langle a_1, a_2, \dots, a_n\rangle$, defining $|A| = \sqrt{\sum a_i^2}$ and $\mathbf{A}\cdot\mathbf{B} = \sum a_i b_i$.)

2H-3 Find the maximum and minimum of the function $f(x,y) = x^2 + y^2 + 2x + 4y - 1$ in the right half-plane R bounded by the diagonal line $y = -x$.

2H-4 Find the maximum and minimum points of the function $xy - x - y + 2$ on

a) the first quadrant b) the square $R: \ 0 \le x \le 2, \ 0 \le y \le 2$;

(study its values at the unique critical point and on the boundary lines) point.

c) use the data to guess the critical point type, and confirm it by the second derivative test.

2H-5 Find the maximum and minimum points of the function $f(x, y) = x + \sqrt{3}\, y + 2$ on the unit disc $R: x^2 + y^2 \le 1$.

2H-6 a) Two wires of length 4 are cut in the same way into three pieces, of length x, y and z; the four x, y pieces are used as the four sides of a rectangle; the two z pieces are bent at the middle and joined at the ends to make a square of side $z/2$. Find the rectangle and square made this way which together have the largest and the smallest total area.

Using the answer, tell what type the critical point is.

b) Confirm the critical point type by using the second derivative test.

2H-7 a) Find the maximum and minimum points of the function $2x^2 - 2xy + y^2 - 2x$ on the rectangle $R: \ 0 \le x \le 2; \ -1 \le y \le 2$; using this information, determine the type of the critical point.

b) Confirm the critical point type by using the second derivative test.

2I. Lagrange Multipliers

2I-1 A rectangular box is placed in the first octant so that one corner Q is at the origin and the three sides adjacent to Q lie in the coordinate planes. The corner P diagonally opposite Q lies on the surface $f(x, y, z) = c$. Using Lagrange multipliers, tell for which point P the box will have the largest volume, and tell how you know it gives a maximum point, if the surface is

a) the plane $x + 2y + 3z = 18$ b) the ellipsoid $x^2 + 2y^2 + 4z^2 = 12$.

2I-2 Using Lagrange multipliers, tell which point P in the first octant and on the surface $x^3y^2z = 6\sqrt{3}$ is closest to the origin. (As usual, it is easier algebraically to minimize $|OP|^2$ rather than $|OP|$.)

2I-3 (Repeat of 2F-2, but this time use Lagrange multipliers.) A rectangular produce box is to be made of cardboard; the sides of single thickness, the ends of double thickness, and the bottom of triple thickness, with the top left open. Its volume is to be 1 cubic foot; what should be its proportions in order to use the least cardboard?

2I-4 In an open-top wooden drawer, the two sides and back cost \$2/sq.ft., the bottom \$1/sq.ft. and the front \$4/sq.ft. Using Lagrange multipliers, show that the following problems lead to the same set of three equations in λ, plus a different fourth equation, and they have the same solution.

a) Find the dimensions of the drawer with largest capacity that can be made for a total wood cost of \$72.

b) Find the dimensions of the most economical drawer having volume 24 cu. ft.

2J. Non-independent Variables

2J-1 Let $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$.

Calculate by direct substitution: a) $\left(\dfrac{\partial w}{\partial y}\right)_z$ b) $\left(\dfrac{\partial w}{\partial z}\right)_y$.

2J-2 Calculate the two derivatives in 2J-1 by using
(i) the chain rule (differentiate $z = x^2 + y^2$ implicitly) (ii) differentials

2J-3 Let $w = x^3y - z^2t$ and $xy = zt$.

Using the chain rule calculate, in terms of x, y, z, t, the derivatives

a) $\left(\dfrac{\partial w}{\partial t}\right)_{x,z}$ b) $\left(\dfrac{\partial w}{\partial z}\right)_{x,y}$

2J-4 Repeat 2J-3, doing the calculation using differentials.

2J-5 Let $S = S(p, v, T)$ be the entropy of a gas, assumed to obey the ideal gas law: $pv = nRT$ (n, R constants). Give expressions in terms of the formal partial derivatives S_p, S_v, and S_T for

a) $\left(\dfrac{\partial S}{\partial p}\right)_v$ b) $\left(\dfrac{\partial S}{\partial T}\right)_v$

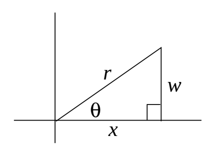
2J-6 If $w = u^3 - uv^2$, $u = xy$, $v = u + x$, find $\left(\dfrac{\partial w}{\partial u}\right)_x$ and $\left(\dfrac{\partial w}{\partial x}\right)_u$ using
a) the chain rule b) differentials .

2J-7 Let P be the point $(1, -1, 1)$, and assume $z = x^2 + y + 1$, and that $f(x, y, z)$ is a differentiable function for which $\nabla f(x, y, z) = 2\,\mathbf{i} + \mathbf{j} - 3\,\mathbf{k}$ at P.

Let $g(x, z) = f(x, y(x, z), z)$; find ∇g at the point $(1, 1)$, i.e., $x = 1$, $z = 1$.

2J-8 Interpreting r, θ as polar coordinates, let $w = \sqrt{r^2 - x^2}$.

a) Calculate $\left(\dfrac{\partial w}{\partial r}\right)_\theta$, by first writing w in terms of r and θ.



b)* Obtain the answer by intuitive geometrical reasoning (see picture).

2J-9* Prove the two-Jacobian rule in notes N section 10 ; use differentials.

2J-10* One of the laws of thermodynamics is expressed by the equation

$$\left(\frac{\partial U}{\partial p}\right)_T + T\left(\frac{\partial V}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial p}\right)_T = 0.$$

Show that the law takes the following form when the independent variables are changed to U and V:

$$\left(\frac{\partial T}{\partial V}\right)_U + T\left(\frac{\partial p}{\partial U}\right)_V - p\left(\frac{\partial T}{\partial U}\right)_V = 0.$$

2J-11* For the law in 2J-10, show the change of variables gives the law the following forms:

a) $$T - p\left(\frac{\partial T}{\partial p}\right)_V + \frac{\partial(T,U)}{\partial(V,p)} = 0, \qquad \text{(independent variables } V,p)$$

b) $$\left(\frac{\partial T}{\partial p}\right)_U - T\left(\frac{\partial V}{\partial U}\right)_p + p\,\frac{\partial(V,T)}{\partial(U,p)} = 0, \qquad \text{(independent variables } U,p)$$

2K. Partial Differential Equations

2K-1. Show that $w = \ln r$, where $r = \sqrt{x^2+y^2}$ is the usual polar coordinate, satisfies the two-dimensional Laplace equation (Notes P (1), without z), if $(x,y) \neq (0,0)$. What's wrong with $(0,0)$?

(The calculation will go faster if you remember that $\ln\sqrt{a} = \frac{1}{2}\ln a$.)

2K-2. For what value(s) of n will $w = (x^2+y^2+z^2)^n$ solve the 3-dimensional Laplace equation (Notes P, (1))? Where have you seen this function in physics?

2K-3. The solutions in exercises 2K-1 and 2K-2 have circular and spherical symmetry, respectively. But there are many other solutions. For example

a) find all solutions of the two-dimensional Laplace equation (see 2K-1) of the form

$$w = ax^2 + bxy + cy^2$$

and show they can all be written in the form $c_1 f_1(x,y) + c_2 f_2(x,y)$, where c_1, c_2 are arbitrary constants, and f_1, f_2 are two particular polynomials — that is, all such solutions are linear combinations of two particular polynomial solutions.

b)* Find and derive the analogue of part (a) for all of the cubic polynomial solutions $ax^3 + bx^2y + cxy^2 + dy^3$ to the two-dimensional Laplace equation.

2K-4. Show that the one-dimensional wave equation (Notes P, (4), first equation) is satisfied by any function of the form

$$w = f(x+ct) + g(x-ct) ,$$

where $f(u)$ and $g(u)$ are arbitrary twice-differentiable functions of one variable.

Take $g(u) = 0$, and interpret physically the solution $w = f(x+ct)$. What does $f(x)$ represent? What is the relation of $f(x+ct)$ to it?

> Note how this exercise shows that a solution to the wave equation can involve completely arbitrary functions; this is also clear from the remarks about the Laplace equation being solved by any gravitational or electrostatic potential function in a mass- or charge-free region of space.

2K-5. Find solutions to the one-dimensional heat equation (Notes P, (5), first equation) having the form

$$w = e^{rt}\sin kx \qquad k, r \text{ constants}$$

satisfying the additional conditions for all t:

$$w(0,t) = 0, \qquad w(1,t) = 0 .$$

Interpret your solutions physically. What happens to the temperature as $t \to \infty$?

18.02 Notes and Exercises by A. Mattuck with the assistance of T.Shifrin and S. LeDuc, and including a section on non-independent variables by Bjorn Poonen.