Topic 9 Notes

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9 Definite integrals using the residue theorem

9.1 Introduction

In this topic we'll use the residue theorem to compute some real definite integrals.

$$\int_a^b f(x)\,dx$$

The general approach is always the same

1. Find a complex analytic function $g(z)$ which either equals f on the real axis or which is closely connected to f, e.g. $f(x) = \cos(x)$, $g(z) = \mathrm{e}^{iz}$.
2. Pick a closed contour C that includes the part of the real axis in the integral.
3. The contour will be made up of pieces. It should be such that we can compute $\int g(z)\,dz$ over each of the pieces except the part on the real axis.
4. Use the residue theorem to compute $\int_C g(z)\,dz$.
5. Combine the previous steps to deduce the value of the integral we want.

9.2 Integrals of functions that decay

The theorems in this section will guide us in choosing the closed contour C described in the introduction.

The first theorem is for functions that decay faster than $1/z$.

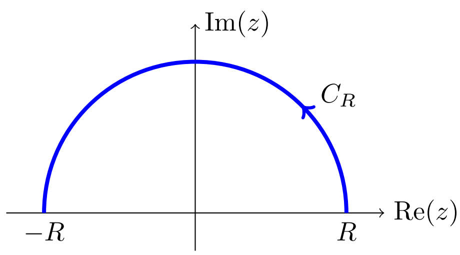
Theorem 9.1. (a) Suppose $f(z)$ is defined in the upper half-plane. If there is an $a > 1$ and $M > 0$ such that

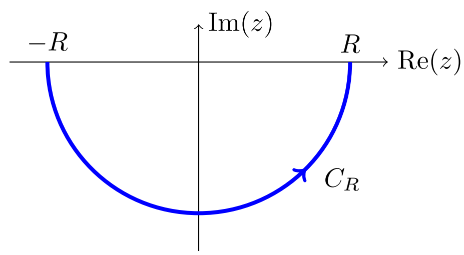
$$|f(z)| < \frac{M}{|z|^a}$$

for $|z|$ large then

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0,$$

where C_R is the semicircle shown below on the left.





Semicircles: left: $Re^{i\theta}$, $0 < \theta < \pi$ right: $Re^{i\theta}$, $\pi < \theta < 2\pi$.

(b) If $f(z)$ is defined in the lower half-plane and

$$|f(z)| < \frac{M}{|z|^a},$$

where $a > 1$ then

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0,$$

where C_R is the semicircle shown above on the right.

Proof. We prove (a), (b) is essentially the same. We use the triangle inequality for integrals and the estimate given in the hypothesis. For R large

$$\left| \int_{C_R} f(z)\, dz \right| \le \int_{C_R} |f(z)|\, |dz| \le \int_{C_R} \frac{M}{|z|^a}\, |dz| = \int_0^\pi \frac{M}{R^a} R\, d\theta = \frac{M\pi}{R^{a-1}}.$$

Since $a > 1$ this clearly goes to 0 as $R \to \infty$. QED

The next theorem is for functions that decay like $1/z$. It requires some more care to state and prove.

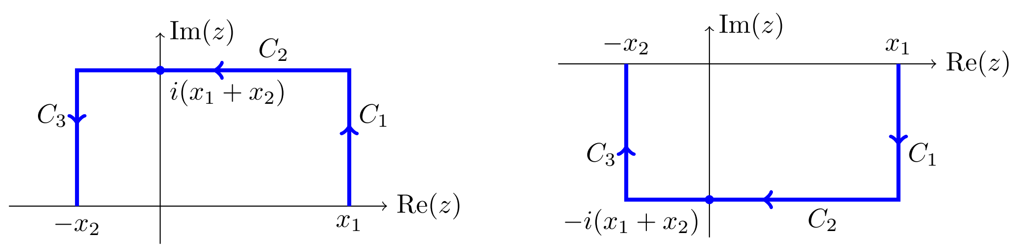
Theorem 9.2. **(a)** Suppose $f(z)$ is defined in the upper half-plane. If there is an $M > 0$ such that

$$|f(z)| < \frac{M}{|z|}$$

for $|z|$ large then for $a > 0$

$$\lim_{x_1\to\infty,\, x_2\to\infty} \int_{C_1+C_2+C_3} f(z) e^{iaz}\, dz = 0,$$

where $C_1 + C_2 + C_3$ is the rectangular path shown below on the left.



Rectangular paths of height and width $x_1 + x_2$.

(b) Similarly, if $a < 0$ then

$$\lim_{x_1\to\infty,\, x_2\to\infty} \int_{C_1+C_2+C_3} f(z) e^{iaz}\, dz = 0,$$

where $C_1 + C_2 + C_3$ is the rectangular path shown above on the right.

Note. In contrast to Theorem 9.1 this theorem needs to include the factor e^{iaz}.

Proof. **(a)** We start by parametrizing C_1, C_2, C_3.

C_1: $\gamma_1(t) = x_1 + it$, t from 0 to $x_1 + x_2$

C_2: $\gamma_2(t) = t + i(x_1 + x_2)$, t from x_1 to $-x_2$

C_3: $\gamma_3(t) = -x_2 + it$, t from $x_1 + x_2$ to 0.

Next we look at each integral in turn. We assume x_1 and x_2 are large enough that

$$|f(z)| < \frac{M}{|z|}$$

on each of the curves C_j.

$$\begin{aligned}\left|\int_{C_1} f(z)\mathrm{e}^{iaz}\,dz\right| &\le \int_{C_1} |f(z)\mathrm{e}^{iaz}|\,|dz| \le \int_{C_1} \frac{M}{|z|}|\mathrm{e}^{iaz}|\,|dz| \\ &= \int_0^{x_1+x_2} \frac{M}{\sqrt{x_1^2+t^2}}|\mathrm{e}^{iax_1-at}|\,dt \\ &\le \frac{M}{x_1}\int_0^{x_1+x_2} \mathrm{e}^{-at}\,dt \\ &= \frac{M}{x_1}(1-\mathrm{e}^{-a(x_1+x_2)})/a.\end{aligned}$$

Since $a > 0$, it is clear that this last expression goes to 0 as x_1 and x_2 go to ∞.

$$\begin{aligned}\left|\int_{C_2} f(z)\mathrm{e}^{iaz}\,dz\right| &\le \int_{C_2} |f(z)\mathrm{e}^{iaz}|\,|dz| \le \int_{C_2} \frac{M}{|z|}|\mathrm{e}^{iaz}|\,|dz| \\ &= \int_{-x_2}^{x_1} \frac{M}{\sqrt{t^2+(x_1+x_2)^2}}|\mathrm{e}^{iat-a(x_1+x_2)}|\,dt \\ &\le \frac{M\mathrm{e}^{-a(x_1+x_2)}}{x_1+x_2}\int_0^{x_1+x_2} dt \\ &\le M\mathrm{e}^{-a(x_1+x_2)}\end{aligned}$$

Again, clearly this last expression goes to 0 as x_1 and x_2 go to ∞.

The argument for C_3 is essentially the same as for C_1, so we leave it to the reader.

The proof for part (b) is the same. You need to keep track of the sign in the exponentials and make sure it is negative.

Example. See Example 9.16 below for an example using Theorem 9.2.

9.3 Integrals $\int_{-\infty}^{\infty}$ and $\int_0^{\infty}$

Example 9.3. Compute

$$I = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\,dx.$$

Solution: Let

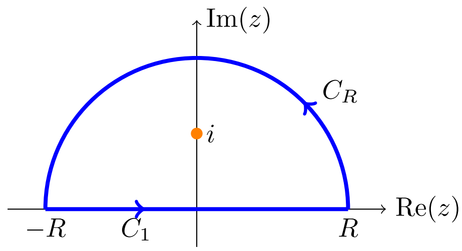
$$f(z) = 1/(1+z^2)^2.$$

It is clear that for z large

$$f(z) \approx 1/z^4.$$

In particular, the hypothesis of Theorem 9.1 is satisfied. Using the contour shown below we have, by the residue theorem,

$$\int_{C_1+C_R} f(z)\,dz = 2\pi i \sum \text{ residues of } f \text{ inside the contour.} \tag{1}$$



We examine each of the pieces in the above equation.

$\displaystyle\int_{C_R} f(z)\,dz$: By Theorem 9.1(a),

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0.$$

$\displaystyle\int_{C_1} f(z)\,dz$: Directly, we see that

$$\lim_{R\to\infty} \int_{C_1} f(z)\,dz = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = I.$$

So letting $R \to \infty$, Equation 1 becomes

$$I = \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \text{ residues of } f \text{ inside the contour.}$$

Finally, we compute the needed residues: $f(z)$ has poles of order 2 at $\pm i$. Only $z = i$ is inside the contour, so we compute the residue there. Let

$$g(z) = (z-i)^2 f(z) = \frac{1}{(z+i)^2}.$$

Then

$$\text{Res}(f, i) = g'(i) = -\frac{2}{(2i)^3} = \frac{1}{4i}$$

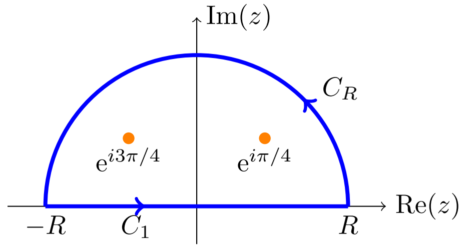
So,

$$I = 2\pi i \,\text{Res}(f, i) = \boxed{\frac{\pi}{2}}.$$

Example 9.4. Compute

$$I = \int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx.$$

Solution: Let $f(z) = 1/(1+z^4)$. We use the same contour as in the previous example



As in the previous example,

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0$$

and

$$\lim_{R\to\infty} \int_{C_1} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx = I.$$

So, by the residue theorem

$$I = \lim_{R\to\infty} \int_{C_1+C_R} f(z)\, dz = 2\pi i \sum \text{ residues of } f \text{ inside the contour.}$$

The poles of f are all simple and at

$$e^{i\pi/4},\ e^{i3\pi/4},\ e^{i5\pi/4},\ e^{i7\pi/4}.$$

Only $e^{i\pi/4}$ and $e^{i3\pi/4}$ are inside the contour. We compute their residues as limits using L'Hospital's rule. For $z_1 = e^{i\pi/4}$:

$$\text{Res}(f, z_1) = \lim_{z\to z_1} (z - z_1) f(z) = \lim_{z\to z_1} \frac{z - z_1}{1 + z^4} = \lim_{z\to z_1} \frac{1}{4z^3} = \frac{1}{4e^{i3\pi/4}} = \frac{e^{-i3\pi/4}}{4}$$

and for $z_2 = e^{i3\pi/4}$:

$$\text{Res}(f, z_2) = \lim_{z\to z_2} (z - z_2) f(z) = \lim_{z\to z_2} \frac{z - z_2}{1 + z^4} = \lim_{z\to z_2} \frac{1}{4z^3} = \frac{1}{4e^{i9\pi/4}} = \frac{e^{-i\pi/4}}{4}$$

So,

$$I = 2\pi i(\text{Res}(f, z_1) + \text{Res}(f, z_2)) = 2\pi i \left(\frac{-1-i}{4\sqrt{2}} + \frac{1-i}{4\sqrt{2}} \right) = 2\pi i \left(-\frac{2i}{4\sqrt{2}} \right) = \boxed{\pi \frac{\sqrt{2}}{2}}$$

Example 9.5. Suppose $b > 0$. Show

$$\int_0^{\infty} \frac{\cos(x)}{x^2 + b^2}\, dx = \frac{\pi e^{-b}}{2b}.$$

Solution: The first thing to note is that the integrand is even, so

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + b^2}.$$

Also note that the square in the denominator tells us the integral is absolutely convergent.

We have to be careful because $\cos(z)$ goes to infinity in either half-plane, so the hypotheses of Theorem 9.1 are not satisfied. The trick is to replace $\cos(x)$ by e^{ix}, so

$$\tilde{I} = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{x^2+b^2}\,dx, \qquad \text{with} \qquad I = \frac{1}{2}\,\mathrm{Re}(\tilde{I}).$$

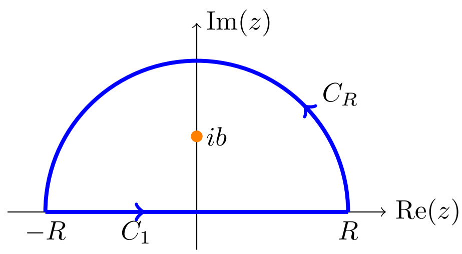
Now let

$$f(z) = \frac{\mathrm{e}^{iz}}{z^2+b^2}.$$

For $z = x + iy$ with $y > 0$ we have

$$|f(z)| = \frac{|\mathrm{e}^{i(x+iy)}|}{|z^2+b^2|} = \frac{\mathrm{e}^{-y}}{|z^2+b^2|}.$$

Since $\mathrm{e}^{-y} < 1$, $f(z)$ satisfies the hypotheses of Theorem 9.1 in the upper half-plane. Now we can use the same contour as in the previous examples



We have

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$$

and

$$\lim_{R\to\infty} \int_{C_1} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = \tilde{I}.$$

So, by the residue theorem

$$\tilde{I} = \lim_{R\to\infty} \int_{C_1+C_R} f(z)\,dz = 2\pi i \sum \text{ residues of } f \text{ inside the contour.}$$

The poles of f are at $\pm bi$ and both are simple. Only bi is inside the contour. We compute the residue as a limit using L'Hospital's rule

$$\mathrm{Res}(f, bi) = \lim_{z\to bi} (z - bi)\frac{\mathrm{e}^{iz}}{z^2+b^2} = \frac{\mathrm{e}^{-b}}{2bi}.$$

So,

$$\tilde{I} = 2\pi i\,\mathrm{Res}(f, bi) = \frac{\pi \mathrm{e}^{-b}}{b}.$$

Finally,

$$I = \frac{1}{2}\,\mathrm{Re}(\tilde{I}) = \frac{\pi \mathrm{e}^{-b}}{2b},$$

as claimed.

Warning: Be careful when replacing $\cos(z)$ by e^{iz} that it is appropriate. A key point in the above example was that $I = \frac{1}{2}\,\mathrm{Re}(\tilde{I})$. This is needed to make the replacement useful.

9.4 Trigonometric integrals

The trick here is to put together some elementary properties of $z = \mathrm{e}^{i\theta}$ on the unit circle.

1. $\mathrm{e}^{-i\theta} = 1/z$.
2. $\cos(\theta) = \dfrac{\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}}{2} = \dfrac{z + 1/z}{2}$.
3. $\sin(\theta) = \dfrac{\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}}{2i} = \dfrac{z - 1/z}{2i}$.

We start with an example. After that we'll state a more general theorem.

Example 9.6. Compute

$$\int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos(\theta)}.$$

Assume that $|a| \neq 1$.

Solution: Notice that $[0, 2\pi]$ is the interval used to parametrize the unit circle as $z = \mathrm{e}^{i\theta}$. We need to make two substitutions:

$$\cos(\theta) = \frac{z + 1/z}{2}$$

$$dz = i\mathrm{e}^{i\theta}\, d\theta \quad \Leftrightarrow \quad d\theta = \frac{dz}{iz}$$

Making these substitutions we get

$$\begin{aligned} I &= \int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos(\theta)} \\ &= \int_{|z|=1} \frac{1}{1 + a^2 - 2a(z + 1/z)/2} \cdot \frac{dz}{iz} \\ &= \int_{|z|=1} \frac{1}{i((1 + a^2)z - a(z^2 + 1))}\, dz. \end{aligned}$$

So, let

$$f(z) = \frac{1}{i((1 + a^2)z - a(z^2 + 1))}.$$

The residue theorem implies

$$I = 2\pi i \sum \text{ residues of } f \text{ inside the unit circle.}$$

We can factor the denominator:

$$f(z) = \frac{-1}{ia(z - a)(z - 1/a)}.$$

The poles are at a, $1/a$. One is inside the unit circle and one is outside.

$$\text{If } |a| > 1 \text{ then } 1/a \text{ is inside the unit circle and } \operatorname{Res}(f, 1/a) = \frac{1}{i(a^2 - 1)}$$

$$\text{If } |a| < 1 \text{ then } a \text{ is inside the unit circle and } \operatorname{Res}(f, a) = \frac{1}{i(1 - a^2)}$$

We have

$$I = \begin{cases} \frac{2\pi}{a^2-1} & \text{if } |a| > 1 \\ \frac{2\pi}{1-a^2} & \text{if } |a| < 1 \end{cases}$$

The example illustrates a general technique which we state now.

Theorem 9.7. Suppose $R(x, y)$ is a rational function with no poles on the circle

$$x^2 + y^2 = 1$$

then for

$$f(z) = \frac{1}{iz} R\left(\frac{z + 1/z}{2}, \frac{z - 1/z}{2i}\right)$$

we have

$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta))\, d\theta = 2\pi i \sum \text{ residues of } f \text{ inside } |z| = 1.$$

Proof. We make the same substitutions as in Example 9.6. So,

$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta))\, d\theta = \int_{|z|=1} R\left(\frac{z + 1/z}{2}, \frac{z - 1/z}{2i}\right) \frac{dz}{iz}$$

The assumption about poles means that f has no poles on the contour $|z| = 1$. The residue theorem now implies the theorem.

9.5 Integrands with branch cuts

Example 9.8. Compute

$$I = \int_0^{\infty} \frac{x^{1/3}}{1 + x^2}\, dx.$$

Solution: Let

$$f(x) = \frac{x^{1/3}}{1 + x^2}.$$

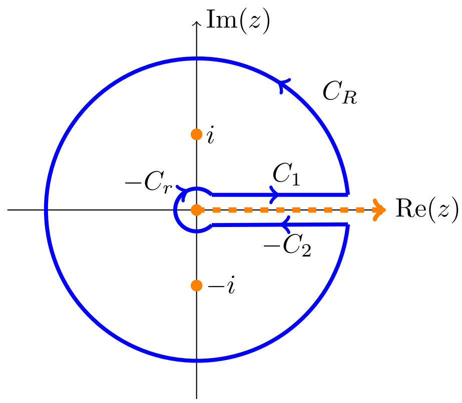
Since this is asymptotically comparable to $x^{-5/3}$, the integral is absolutely convergent. As a complex function

$$f(z) = \frac{z^{1/3}}{1 + z^2}$$

needs a branch cut to be analytic (or even continuous), so we will need to take that into account with our choice of contour.

First, choose the following branch cut along the positive real axis. That is, for $z = re^{i\theta}$ not on the axis, we have $0 < \theta < 2\pi$.

Next, we use the contour $C_1 + C_R - C_2 - C_r$ shown below.



Contour around branch cut: inner circle of radius r, outer of radius R.

We put convenient signs on the pieces so that the integrals are parametrized in a natural way. You should read this contour as having r so small that C_1 and C_2 are essentially on the x-axis. Note well, that, since C_1 and C_2 are on opposite sides of the branch cut, the integral

$$\int_{C_1 - C_2} f(z)\, dz \neq 0.$$

First we analyze the integral over each piece of the curve.

On C_R: Theorem 9.1 says that

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0.$$

On C_r: For concreteness, assume $r < 1/2$. We have $|z| = r$, so

$$|f(z)| = \frac{|z^{1/3}|}{|1+z^2|} \leq \frac{r^{1/3}}{1-r^2} \leq \frac{(1/2)^{1/3}}{3/4}.$$

Call the last number in the above equation M. We have shown that, for small r, $|f(z)| < M$. So,

$$\left| \int_{C_r} f(z)\, dz \right| \leq \int_0^{2\pi} |f(r\mathrm{e}^{i\theta})||ir\mathrm{e}^{i\theta}|\, d\theta \leq \int_0^{2\pi} Mr\, d\theta = 2\pi M r.$$

Clearly this goes to zero as $r \to 0$.

On C_1:

$$\lim_{r\to 0,\, R\to\infty} \int_{C_1} f(z)\, dz = \int_0^\infty f(x)\, dx = I.$$

On C_2: We have (essentially) $\theta = 2\pi$, so $z^{1/3} = \mathrm{e}^{i2\pi/3}|z|^{1/3}$. Thus,

$$\lim_{r\to 0,\, R\to\infty} \int_{C_2} f(z)\, dz = \mathrm{e}^{i2\pi/3} \int_0^\infty f(x)\, dx = \mathrm{e}^{i2\pi/3} I.$$

The poles of $f(z)$ are at $\pm i$. Since f is meromorphic inside our contour the residue theorem says

$$\int_{C_1 + C_R - C_2 - C_r} f(z)\, dz = 2\pi i(\mathrm{Res}(f, i) + \mathrm{Res}(f, -i)).$$

Letting $r \to 0$ and $R \to \infty$ the analysis above shows

$$(1 - e^{i2\pi/3})I = 2\pi i(\text{Res}(f, i) + \text{Res}(f, -i))$$

All that's left is to compute the residues using the chosen branch of $z^{1/3}$

$$\text{Res}(f, -i) = \frac{(-i)^{1/3}}{-2i} = \frac{(e^{i3\pi/2})^{1/3}}{2e^{i3\pi/2}} = \frac{e^{-i\pi}}{2} = -\frac{1}{2}$$
$$\text{Res}(f, i) = \frac{i^{1/3}}{2i} = \frac{e^{i\pi/6}}{2e^{i\pi/2}} = \frac{e^{-i\pi/3}}{2}$$

A little more algebra gives

$$(1 - e^{i2\pi/3})I = 2\pi i \cdot \frac{-1 + e^{-i\pi/3}}{2} = \pi i(-1 + 1/2 - i\sqrt{3}/2) = -\pi i e^{i\pi/3}.$$

Continuing

$$I = \frac{-\pi i e^{i\pi/3}}{1 - e^{i2\pi/3}} = \frac{\pi i}{e^{i\pi/3} - e^{-\pi i/3}} = \frac{\pi/2}{(e^{i\pi/3} - e^{-i\pi/3})/2i} = \frac{\pi/2}{\sin(\pi/3)} = \frac{\pi}{\sqrt{3}}.$$

Whew! (Note: a sanity check is that the result is real, which it had to be.)

Example 9.9. Compute

$$I = \int_1^\infty \frac{dx}{x\sqrt{x^2 - 1}}.$$

Solution: Let

$$f(z) = \frac{1}{z\sqrt{z^2 - 1}}.$$

The first thing we'll show is that the integral

$$\int_1^\infty f(x)\, dx$$

is absolutely convergent. To do this we split it into two integrals

$$\int_1^\infty \frac{dx}{x\sqrt{x^2 - 1}} = \int_1^2 \frac{dx}{x\sqrt{x^2 - 1}} + \int_2^\infty \frac{dx}{x\sqrt{x^2 - 1}}.$$

The first integral on the right can be rewritten as

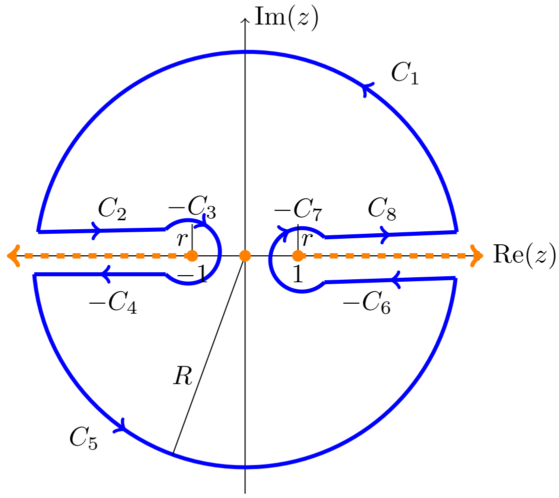
$$\int_1^2 \frac{1}{x\sqrt{x + 1}} \cdot \frac{1}{\sqrt{x - 1}}\, dx \le \int_1^2 \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{x - 1}}\, dx = \frac{2}{\sqrt{2}}\sqrt{x - 1}\Big|_1^2.$$

This shows the first integral is absolutely convergent.

The function $f(x)$ is asymptotically comparable to $1/x^2$, so the integral from 2 to ∞ is also absolutely convergent.

We can conclude that the original integral is absolutely convergent.

Next, we use the following contour. Here we assume the big circles have radius R and the small ones have radius r.



We use the branch cut for square root that removes the positive real axis. In this branch

$$0 < \arg(z) < 2\pi \quad \text{and} \quad 0 < \arg(\sqrt{w}) < \pi.$$

For $f(z)$, this necessitates the branch cut that removes the rays $[1, \infty)$ and $(-\infty, -1]$ from the complex plane.

The pole at $z = 0$ is the only singularity of $f(z)$ inside the contour. It is easy to compute that

$$\text{Res}(f, 0) = \frac{1}{\sqrt{-1}} = \frac{1}{i} = -i.$$

So, the residue theorem gives us

$$\int_{C_1+C_2-C_3-C_4+C_5-C_6-C_7+C_8} f(z)\, dz = 2\pi i \, \text{Res}(f, 0) = 2\pi. \tag{2}$$

In a moment we will show the following limits

$$\lim_{R\to\infty} \int_{C_1} f(z)\, dz = \lim_{R\to\infty} \int_{C_5} f(z)\, dz = 0$$
$$\lim_{r\to 0} \int_{C_3} f(z)\, dz = \lim_{r\to 0} \int_{C_7} f(z)\, dz = 0.$$

We will also show

$$\begin{aligned} \lim_{R\to\infty,\, r\to 0} \int_{C_2} f(z)\, dz &= \lim_{R\to\infty,\, r\to 0} \int_{-C_4} f(z)\, dz \\ &= \lim_{R\to\infty,\, r\to 0} \int_{-C_6} f(z)\, dz = \lim_{R\to\infty,\, r\to 0} \int_{C_8} f(z)\, dz = I. \end{aligned}$$

Using these limits, Equation 2 implies $4I = 2\pi$, i.e.

$$\boxed{I = \pi/2.}$$

All that's left is to prove the limits asserted above.

The limits for C_1 and C_5 follow from Theorem 9.1 because

$$|f(z)| \approx 1/|z|^{3/2}$$

for large z.

We get the limit for C_3 as follows. Suppose r is small, say much less than 1. If

$$z = -1 + re^{i\theta}$$

is on C_3 then,

$$|f(z)| = \frac{1}{|z\sqrt{z-1}\sqrt{z+1}|} = \frac{1}{|-1+re^{i\theta}|\sqrt{|-2+re^{i\theta}|}\sqrt{r}} \leq \frac{M}{\sqrt{r}}.$$

where M is chosen to be bigger than

$$\frac{1}{|-1+re^{i\theta}|\sqrt{|-2+re^{i\theta}|}}$$

for all small r.

Thus,

$$\left|\int_{C_3} f(z)\,dz\right| \leq \int_{C_3} \frac{M}{\sqrt{r}}\,|dz| \leq \frac{M}{\sqrt{r}} \cdot 2\pi r = 2\pi M\sqrt{r}.$$

This last expression clearly goes to 0 as $r \to 0$.

The limit for the integral over C_7 is similar.

We can parameterize the straight line C_8 by

$$z = x + i\epsilon,$$

where ϵ is a small positive number and x goes from (approximately) 1 to ∞. Thus, on C_8, we have

$$\arg(z^2-1) \approx 0 \quad \text{and} \quad f(z) \approx f(x).$$

All these approximations become exact as $r \to 0$. Thus,

$$\lim_{R\to\infty,\, r\to 0} \int_{C_8} f(z)\,dz = \int_1^\infty f(x)\,dx = I.$$

We can parameterize $-C_6$ by

$$z = x - i\epsilon$$

where x goes from ∞ to 1. Thus, on C_6, we have

$$\arg(z^2-1) \approx 2\pi,$$

so

$$\sqrt{z^2-1} \approx -\sqrt{x^2-1}.$$

This implies

$$f(z) \approx -\frac{1}{x\sqrt{x^2-1}} = -f(x).$$

Thus,

$$\lim_{R\to\infty,\, r\to 0} \int_{-C_6} f(z)\, dz = \int_{\infty}^{1} -f(x)\, dx = \int_{1}^{\infty} f(x)\, dx = I.$$

We can parameterize C_2 by $z = -x + i\epsilon$ where x goes from ∞ to 1. Thus, on C_2, we have

$$\arg(z^2 - 1) \approx 2\pi,$$

so

$$\sqrt{z^2 - 1} \approx -\sqrt{x^2 - 1}.$$

This implies

$$f(z) \approx \frac{1}{(-x)(-\sqrt{x^2-1})} = f(x).$$

Thus,

$$\lim_{R\to\infty,\, r\to 0} \int_{C_2} f(z)\, dz = \int_{\infty}^{1} f(x)\,(-dx) = \int_{1}^{\infty} f(x)\, dx = I.$$

The last curve $-C_4$ is handled similarly.

9.6 Cauchy principal value

First an example to motivate defining the principal value of an integral. We'll actually compute the integral in the next section.

Example 9.10. Let

$$I = \int_0^\infty \frac{\sin(x)}{x}\, dx.$$

This integral is not absolutely convergent, but it is conditionally convergent. Formally, of course, we mean

$$I = \lim_{R\to\infty} \int_0^R \frac{\sin(x)}{x}\, dx.$$

We can proceed as in Example 9.5. First note that $\sin(x)/x$ is even, so

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx.$$

Next, to avoid the problem that $\sin(z)$ goes to infinity in both the upper and lower half-planes we replace the integrand by $\frac{\mathrm{e}^{ix}}{x}$.

We've changed the problem to computing

$$\tilde{I} = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{x}\, dx.$$

The problems with this integral are caused by the pole at 0. The biggest problem is that the integral doesn't converge! The other problem is that when we try to use our usual strategy of choosing a closed contour we can't use one that includes $z = 0$ on the real axis. This is our motivation for defining principal value. We will come back to this example below.

Definition. Suppose we have a function $f(x)$ that is continuous on the real line except at the point x_1, then we define the Cauchy principal value as

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty,\, r_1\to 0} \int_{-R}^{x_1-r_1} f(x)\,dx + \int_{x_1+r_1}^{R} f(x)\,dx.$$

Provided the limit converges. You should notice that the intervals around x_1 and around ∞ are symmetric. Of course, if the integral

$$\int_{-\infty}^{\infty} f(x)\,dx$$

converges, then so does the principal value and they give the same value. We can make the definition more flexible by including the following cases.

1. If $f(x)$ is continuous on the entire real line then we define the principal value as

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

2. If we have multiple points of discontinuity, $x_1 < x_2 < x_3 < \ldots < x_n$, then

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,dx = \lim \int_{-R}^{x_1-r_1} f(x)\,dx + \int_{x_1+r_1}^{x_2-r_2} + \int_{x_2+r_2}^{x_3-r_3} + \ldots \int_{x_n+r_n}^{R} f(x)\,dx.$$

Here the limit is taken as $R \to \infty$ and each of the $r_k \to 0$.

Intervals of integration for principal value are symmetric around x_k and ∞

The next example shows that sometimes the principal value converges when the integral itself does not. The opposite is never true. That is, we have the following theorem.

Theorem 9.11. If $f(x)$ has discontinuities at $x_1 < x_2 < \ldots < x_n$ and $\int_{-\infty}^{\infty} f(x)\,dx$ converges then so does p.v. $\int_{-\infty}^{\infty} f(x)\,dx$.

Proof. The proof amounts to understanding the definition of convergence of integrals as limits. The integral converges means that each of the limits

$$\begin{aligned} &\lim_{R_1\to\infty,\, a_1\to 0} \int_{-R_1}^{x_1-a_1} f(x)\,dx \\ &\lim_{b_1\to 0,\, a_2\to 0} \int_{x_1+b_1}^{x_2-a_2} f(x)\,dx \\ &\ldots \\ &\lim_{R_2\to\infty,\, b_n\to 0} \int_{x_n+b_n}^{R_2} f(x)\,dx. \end{aligned} \tag{3}$$

converges. There is no symmetry requirement, i.e. R_1 and R_2 are completely independent, as are a_1 and b_1 etc.

The principal value converges means

$$\lim \int_{-R}^{x_1-r_1} + \int_{x_1+r_1}^{x_2-r_2} + \int_{x_2+r_2}^{x_3-r_3} + \ldots \int_{x_n+r_n}^{R} f(x)\,dx \tag{4}$$

converges. Here the limit is taken over all the parameter $R \to \infty$, $r_k \to 0$. This limit has symmetry, e.g. we replaced both a_1 and b_1 in Equation 3 by r_1 etc. Certainly if the limits in Equation 3 converge then so do the limits in Equation 4. QED

Example 9.12. Consider both

$$\int_{-\infty}^{\infty} \frac{1}{x}\,dx \quad \text{and} \quad \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{x}\,dx.$$

The first integral diverges since

$$\int_{-R_1}^{-r_1} \frac{1}{x}\,dx + \int_{r_2}^{R_2} \frac{1}{x}\,dx = \ln(r_1) - \ln(R_1) + \ln(R_2) - \ln(r_2).$$

This clearly diverges as R_1, $R_2 \to \infty$ and r_1, $r_2 \to 0$.

On the other hand the symmetric integral

$$\int_{-R}^{-r} \frac{1}{x}\,dx + \int_{r}^{R} \frac{1}{x}\,dx = \ln(r) - \ln(R) + \ln(R) - \ln(r) = 0.$$

This clearly converges to 0.

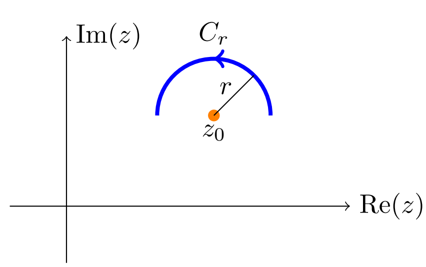
We will see that the principal value occurs naturally when we integrate on semicircles around points. We prepare for this in the next section.

9.7 Integrals over portions of circles

We will need the following theorem in order to combine principal value and the residue theorem.

Theorem 9.13. Suppose $f(z)$ has a **simple** pole at z_0. Let C_r be the semicircle $\gamma(\theta) = z_0 + re^{i\theta}$, with $0 \le \theta \le \pi$. Then

$$\lim_{r\to 0} \int_{C_r} f(z)\,dz = \pi i \,\text{Res}(f, z_0) \tag{5}$$



Small semicircle of radius r around z_0

Proof. Since we take the limit as r goes to 0, we can assume r is small enough that $f(z)$ has a Laurent expansion of the punctured disk of radius r centered at z_0. That is, since the pole is simple,

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \ldots \qquad \text{for } 0 < |z - z_0| \le r.$$

Thus,

$$\int_{C_r} f(z)\, dz = \int_0^\pi f(z_0 + r\mathrm{e}^{i\theta})\, ri\mathrm{e}^{i\theta}\, d\theta = \int_0^\pi \left(b_1 i + a_0 ir\mathrm{e}^{i\theta} + a_1 ir^2\mathrm{e}^{i2\theta} + \ldots\right) d\theta$$

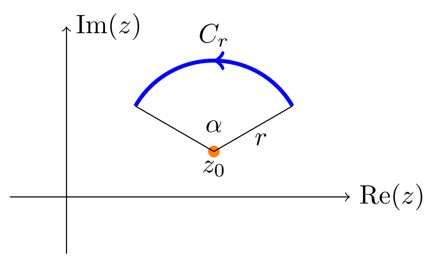
The b_1 term gives $\pi i b_1$. Clearly all the other terms go to 0 as $r \to 0$. QED.

If the pole is not simple the theorem doesn't hold and, in fact, the limit does not exist.

The same proof gives a slightly more general theorem.

Theorem 9.14. Suppose $f(z)$ has a **simple** pole at z_0. Let C_r be the circular arc $\gamma(\theta) = z_0 + r\mathrm{e}^{i\theta}$, with $\theta_0 \le \theta \le \theta_0 + \alpha$. Then

$$\lim_{r\to 0} \int_{C_r} f(z)\, dz = \alpha i \,\mathrm{Res}(f, z_0)$$



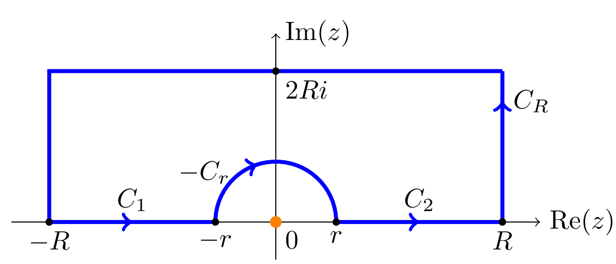
Small circular arc of radius r around z_0

Example 9.15. (Return to Example 9.10.) A long time ago we left off Example 9.10 to define principal value. Let's now use the principal value to compute

$$\tilde{I} = \text{p.v.} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{x}\, dx.$$

Solution: We use the indented contour shown below. The indentation is the little semicircle the goes around $z = 0$. There are no poles inside the contour so the residue theorem implies

$$\int_{C_1 - C_r + C_2 + C_R} \frac{\mathrm{e}^{iz}}{z}\, dz = 0.$$



Next we break the contour into pieces.

$$\lim_{R\to\infty,\, r\to 0} \int_{C_1+C_2} \frac{\mathrm{e}^{iz}}{z}\, dz = \tilde{I}.$$

Theorem 9.2(a) implies

$$\lim_{R\to\infty} \int_{C_R} \frac{\mathrm{e}^{iz}}{z}\, dz = 0.$$

Equation 5 in Theorem 9.13 tells us that

$$\lim_{r\to 0} \int_{C_r} \frac{\mathrm{e}^{iz}}{z}\, dz = \pi i \operatorname{Res}\left(\frac{\mathrm{e}^{iz}}{z}, 0\right) = \pi i$$

Combining all this together we have

$$\lim_{R\to\infty,\, r\to 0} \int_{C_1 - C_r + C_2 + C_R} \frac{\mathrm{e}^{iz}}{z}\, dz = \tilde{I} - \pi i = 0,$$

so $\tilde{I} = \pi i$. Thus, looking back at Example 5, where $I = \displaystyle\int_0^\infty \frac{\sin(x)}{x}\, dx$, we have

$$I = \frac{1}{2}\operatorname{Im}(\tilde{I}) = \frac{\pi}{2}.$$

There is a subtlety about convergence we alluded to above. That is, I is a genuine (conditionally) convergent integral, but $\tilde{I}$ only exists as a principal value. However since I is a convergent integral we know that computing the principle value as we just did is sufficient to give the value of the convergent integral.

9.8 Fourier transform

Definition. The Fourier transform of a function $f(x)$ is defined by

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-ix\omega}\, dx$$

This is often read as 'f-hat'.

Theorem. (Fourier inversion formula.) We can recover the original function $f(x)$ with the Fourier inversion formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)\mathrm{e}^{ix\omega}\, d\omega.$$

So, the Fourier transform converts a function of x to a function of ω and the Fourier inversion converts it back. Of course, everything above is dependent on the convergence of the various integrals.

Proof. We will not give the proof here. (We may get to it later in the course.)

Example 9.16. Let

$$f(t) = \begin{cases} \mathrm{e}^{-at} & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases},$$

where $a > 0$. Compute $\widehat{f}(\omega)$ and verify the Fourier inversion formula in this case.

Solution: Computing $\widehat{f}$ is easy: For $a > 0$

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,dt = \int_0^{\infty} \mathrm{e}^{-at}\mathrm{e}^{-i\omega t}\,dt = \frac{1}{a+i\omega} \text{ (recall } a > 0).$$

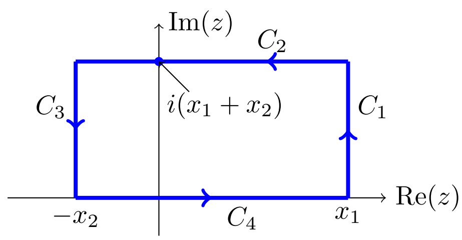
We should first note that the inversion integral converges. To avoid distraction we show this at the end of this example.

Now, let

$$g(z) = \frac{1}{a+iz}$$

Note that $\widehat{f}(\omega) = g(\omega)$ and $|g(z)| < \dfrac{M}{|z|}$ for large $|z|$.

To verify the inversion formula we consider the cases $t > 0$ and $t < 0$ separately. For $t > 0$ we use the standard contour.



Theorem 9.2(a) implies that

$$\lim_{x_1\to\infty,\, x_2\to\infty} \int_{C_1+C_2+C_3} g(z)\mathrm{e}^{izt}\,dz = 0 \tag{6}$$

Clearly

$$\lim_{x_1\to\infty,\, x_2\to\infty} \int_{C_4} g(z)\mathrm{e}^{izt}\,dz = \int_{-\infty}^{\infty} \widehat{f}(\omega)\,d\omega \tag{7}$$

The only pole of $g(z)\mathrm{e}^{izt}$ is at $z = ia$, which is in the upper half-plane. So, applying the residue theorem to the entire closed contour, we get for large x_1, x_2:

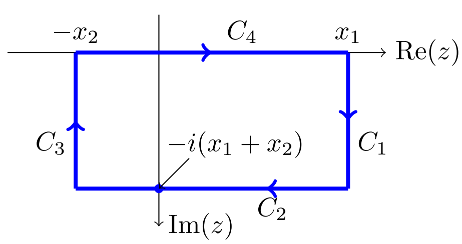
$$\int_{C_1+C_2+C_3+C_4} g(z)\mathrm{e}^{izt}\,dz = 2\pi i \operatorname{Res}\left(\frac{\mathrm{e}^{izt}}{a+iz}, ia\right) = \frac{\mathrm{e}^{-at}}{i}. \tag{8}$$

Combining the three equations 6, 7 and 8, we have

$$\int_{-\infty}^{\infty} \widehat{f}(\omega)\,d\omega = 2\pi\mathrm{e}^{-at} \qquad \text{for } t > 0$$

This shows the inversion formula holds for $t > 0$.

For $t < 0$ we use the contour



Theorem 9.2(b) implies that

$$\lim_{x_1\to\infty,\, x_2\to\infty} \int_{C_1+C_2+C_3} g(z)\mathrm{e}^{izt}\, dz = 0$$

Clearly

$$\lim_{x_1\to\infty,\, x_2\to\infty} \frac{1}{2\pi} \int_{C_4} g(z)\mathrm{e}^{izt}\, dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega)\, d\omega$$

Since, there are no poles of $g(z)\mathrm{e}^{izt}$ in the lower half-plane, applying the residue theorem to the entire closed contour, we get for large x_1, x_2:

$$\int_{C_1+C_2+C_3+C_4} g(z)\mathrm{e}^{izt}\, dz = -2\pi i \operatorname{Res}\left(\frac{\mathrm{e}^{izt}}{a+iz}, ia\right) = 0.$$

Thus,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega)\, d\omega = 0 \qquad \text{for } t < 0$$

This shows the inversion formula holds for $t < 0$.

Finally, we give the promised argument that the inversion integral converges. By definition

$$\begin{aligned}
\int_{-\infty}^{\infty} \widehat{f}(\omega)\mathrm{e}^{i\omega t}\, d\omega &= \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\omega t}}{a+i\omega}\, d\omega \\
&= \int_{-\infty}^{\infty} \frac{a\cos(\omega t) + \omega\sin(\omega t) - i\omega\cos(\omega t) + ia\sin(\omega t)}{a^2+\omega^2}\, d\omega
\end{aligned}$$

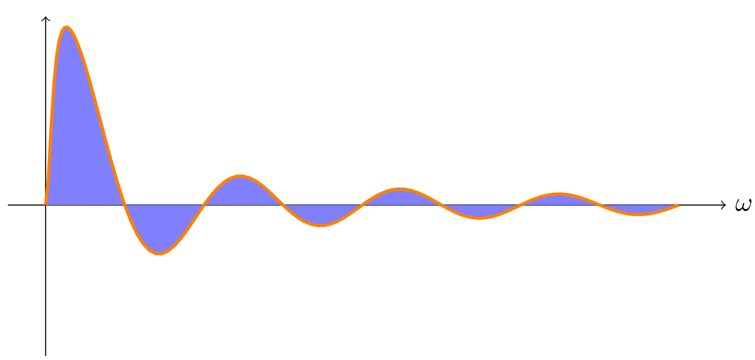
The terms without a factor of ω in the numerator converge absolutely because of the ω^2 in the denominator. The terms with a factor of ω in the numerator do not converge absolutely. For example, since

$$\frac{\omega\sin(\omega t)}{a^2+\omega^2}$$

decays like $1/\omega$, its integral is not absolutely convergent. However, we claim that the integral does converge conditionally. That is, both limits

$$\lim_{R_2\to\infty} \int_0^{R_2} \frac{\omega\sin(\omega t)}{a^2+\omega^2}\, d\omega \quad \text{and} \quad \lim_{R_1\to\infty} \int_{-R_1}^{0} \frac{\omega\sin(\omega t)}{a^2+\omega^2}\, d\omega$$

exist and are finite. The key is that, as $\sin(\omega t)$ alternates between positive and negative arches, the function $\dfrac{\omega}{a^2+\omega^2}$ is decaying monotonically. So, in the integral, the area under each arch adds or subtracts less than the arch before. This means that as R_1 (or R_2) grows the total area under the curve oscillates with a decaying amplitude around some limiting value.



Total area oscillates with a decaying amplitude.

9.9 Solving DEs using the Fourier transform

Let

$$D = \frac{d}{dt}.$$

Our goal is to see how to use the Fourier transform to solve differential equations like

$$P(D)y = f(t).$$

Here $P(D)$ is a polynomial operator, e.g.

$$D^2 + 8D + 7I.$$

We first note the following formula:

$$\widehat{Df}(\omega) = i\omega\widehat{f}. \tag{9}$$

Proof. This is just integration by parts:

$$\begin{aligned}
\widehat{Df}(\omega) &= \int_{-\infty}^{\infty} f'(t)\mathrm{e}^{-i\omega t}\,dt \\
&= f(t)\mathrm{e}^{-i\omega t}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)(-i\omega\mathrm{e}^{-i\omega t}\,dt \\
&= i\omega\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,dt \\
&= i\omega\widehat{f}(\omega) \quad \text{QED}
\end{aligned}$$

In the third line we assumed that f decays so that $f(\infty) = f(-\infty) = 0$.

It is a simple extension of Equation 9 to see

$$(\widehat{P(D)f})(\omega) = P(i\omega)\widehat{f}.$$

We can now use this to solve some differential equations.

Example 9.17. Solve the equation

$$y''(t) + 8y'(t) + 7y(t) = f(t) = \begin{cases} \mathrm{e}^{-at} & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Solution: In this case, we have

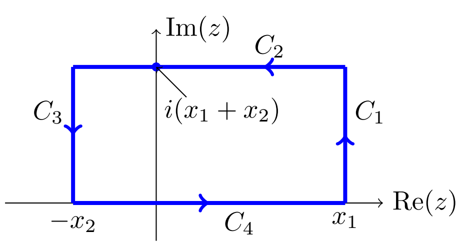
$$P(D) = D^2 + 8D + 7I,$$

so

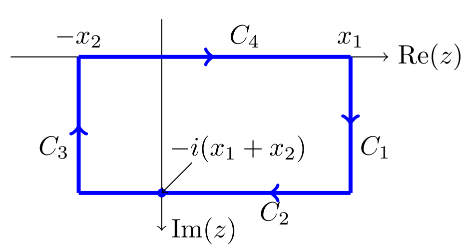
$$P(s) = s^2 + 8s + 7 = (s + 7)(s + 1).$$

The DE
$$P(D)y = f(t)$$
transforms to
$$P(iw)\widehat{y} = \widehat{f}.$$
Using the Fourier transform of f found in Example 9.16 we have
$$\widehat{y}(\omega) = \frac{\widehat{f}}{P(i\omega)} = \frac{1}{(a+i\omega)(7+i\omega)(1+i\omega)}.$$
Fourier inversion says that
$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{y}(\omega)\mathrm{e}^{i\omega t}\,d\omega$$
As always, we want to extend $\widehat{y}$ to be function of a complex variable z. Let's call it $g(z)$:
$$g(z) = \frac{1}{(a+iz)(7+iz)(1+iz)}.$$
Now we can proceed exactly as in Example 9.16. We know $|g(z)| < M/|z|^3$ for some constant M. Thus, the conditions of Theorem 9.2 are easily met. So, just as in Example 9.16, we have:

For $t > 0$, e^{izt} is bounded in the upper half-plane, so we use the contour below on the left.
$$\begin{aligned} y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{y}(\omega)\mathrm{e}^{i\omega t}\,d\omega &= \frac{1}{2\pi}\lim_{x_1\to\infty,\, x_2\to\infty}\int_{C_4} g(z)\mathrm{e}^{izt}\,dz \\ &= \frac{1}{2\pi}\lim_{x_1\to\infty,\, x_2\to\infty}\int_{C_1+C_2+C_3+C_4} g(z)\mathrm{e}^{izt}\,dz \\ &= i\sum \text{residues of } \mathrm{e}^{izt}g(z) \text{ in the upper half-plane} \end{aligned}$$
The poles of $\mathrm{e}^{izt}g(z)$ are at
$$ia, \quad 7i, \quad i.$$
These are all in the upper half-plane. The residues are respectively,
$$\frac{\mathrm{e}^{-at}}{i(7-a)(1-a)}, \quad \frac{\mathrm{e}^{-7t}}{i(a-7)(-6)}, \quad \frac{\mathrm{e}^{-t}}{i(a-1)(6)}$$
Thus, for $t > 0$ we have
$$y(t) = \frac{\mathrm{e}^{-at}}{(7-a)(1-a)} - \frac{\mathrm{e}^{-7t}}{(a-7)(6)} + \frac{\mathrm{e}^{-t}}{(a-1)(6)}.$$



Contour for $t > 0$



Contour for $t < 0$

More briefly, when $t < 0$ we use the contour above on the right. We get the exact same string of equalities except the sum is over the residues of $e^{izt} g(z)$ in the lower half-plane. Since there are no poles in the lower half-plane, we find that

$$\widehat{y}(t) = 0$$

when $t < 0$.

Conclusion (reorganizing the signs and order of the terms):

$$y(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{e^{-at}}{(7-a)(1-a)} + \frac{e^{-7t}}{(7-a)(6)} - \frac{e^{-t}}{(1-a)(6)} & \text{for } t > 0. \end{cases}$$

Note. Because $|g(z)| < M/|z|^3$, we could replace the rectangular contours by semicircles to compute the Fourier inversion integral.

Example 9.18. Consider

$$y'' + y = f(t) = \begin{cases} e^{-at} & \text{if } t > 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Find a solution for $t > 0$.

Solution: We work a little more quickly than in the previous example.

Taking the Fourier transform we get

$$\widehat{y}(\omega) = \frac{\widehat{f}(\omega)}{P(i\omega)} = \frac{\widehat{f}(\omega)}{1 - \omega^2} = \frac{1}{(a + i\omega)(1 - \omega^2)}.$$

(In the last expression, we used the known Fourier transform of f.)

As usual, we extend $\widehat{y}(\omega)$ to a function of z:

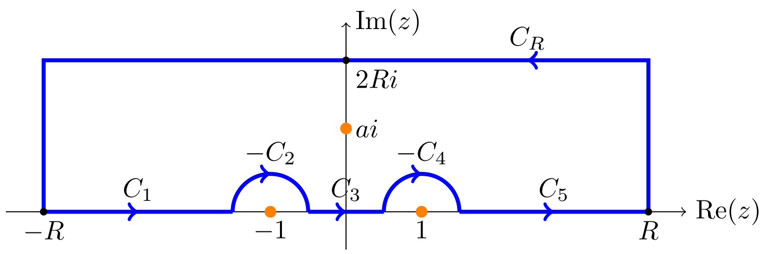
$$g(z) = \frac{1}{(a + iz)(1 - z^2)}.$$

This has simple poles at

$$-1, \quad 1, \quad ai.$$

Since some of the poles are on the real axis, we will need to use an indented contour along the real axis and use principal value to compute the integral.

The contour is shown below. We assume each of the small indents is a semicircle with radius r. The big rectangular path from $(R, 0)$ to $(-R, 0)$ is called C_R.



For $t > 0$ the function $\mathrm{e}^{izt}g(z) < M/|z|^3$ in the upper half-plane. Thus, we get the following limits:

$$\begin{aligned}
\lim_{R\to\infty} \int_{C_R} \mathrm{e}^{izt} g(z)\,dz &= 0 && \text{(Theorem 9.2(b))} \\
\lim_{R\to\infty,\, r\to 0} \int_{C_2} \mathrm{e}^{izt} g(z)\,dz &= \pi i \operatorname{Res}(\mathrm{e}^{izt}g(z), -1) && \text{(Theorem 9.14)} \\
\lim_{R\to\infty,\, r\to 0} \int_{C_4} \mathrm{e}^{izt} g(z)\,dz &= \pi i \operatorname{Res}(\mathrm{e}^{izt}g(z), 1) && \text{(Theorem 9.14)} \\
\lim_{R\to\infty,\, r\to 0} \int_{C_1+C_3+C_5} \mathrm{e}^{izt} g(z)\,dz &= \text{p.v.} \int_{-\infty}^{\infty} \widehat{y}(t)\mathrm{e}^{i\omega t}\,dt
\end{aligned}$$

Putting this together with the residue theorem we have

$$\begin{aligned}
\lim_{R\to\infty,\, r\to 0} \int_{C_1-C_2+C_3-C_4+C_5+C_R} \mathrm{e}^{izt} g(z)\,dz &= \text{p.v.} \int_{-\infty}^{\infty} \widehat{y}(t)\mathrm{e}^{i\omega t}\,dt - \pi i \operatorname{Res}(\mathrm{e}^{izt}g(z), -1) - \pi i \operatorname{Res}(\mathrm{e}^{izt}g(z), 1) \\
&= 2\pi i \operatorname{Res}(\mathrm{e}^{izt}, ai).
\end{aligned}$$

All that's left is to compute the residues and do some arithmetic. We don't show the calculations, but give the results

$$\begin{aligned}
\operatorname{Res}(\mathrm{e}^{izt}g(z), -1) &= \frac{\mathrm{e}^{-it}}{2(a-i)} \\
\operatorname{Res}(\mathrm{e}^{izt}g(z), 1) &= -\frac{\mathrm{e}^{it}}{2(a+i)} \\
\operatorname{Res}(\mathrm{e}^{izt}g(z), ai) &= -\frac{\mathrm{e}^{-at}}{i(1+a^2)}
\end{aligned}$$

We get, for $t > 0$,

$$\begin{aligned}
y(t) &= \frac{1}{2\pi}\text{p.v.} \int_{-\infty}^{\infty} \widehat{y}(t)\mathrm{e}^{i\omega t}\,dt \\
&= \frac{i}{2} \operatorname{Res}(\mathrm{e}^{izt}g(z), -1) + \frac{i}{2} \operatorname{Res}(\mathrm{e}^{izt}g(z), 1) + i \operatorname{Res}(\mathrm{e}^{izt}g(z), ai) \\
&= \frac{e^{-at}}{1+a^2} + \frac{a}{1+a^2}\sin(t) - \frac{1}{1+a^2}\cos(t).
\end{aligned}$$