Topic 8 Notes

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8 Residue Theorem

8.1 Poles and zeros

We remind you of the following terminology: Suppose $f(z)$ is analytic at z_0 and

$$f(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \ldots,$$

with $a_n \neq 0$. Then we say f has a zero of order n at z_0. If $n = 1$ we say z_0 is a simple zero.

Suppose f has an *isolated* singularity at z_0 and Laurent series

$$f(z) = \frac{b_n}{(z - z_0)^n} + \frac{b_{n-1}}{(z - z_0)^{n-1}} + \ldots + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \ldots$$

which converges on $0 < |z - z_0| < R$ and with $b_n \neq 0$. Then we say f has a pole of order n at z_0. If $n = 1$ we say z_0 is a simple pole.

There are several examples in the Topic 7 notes. Here is one more

Example 8.1.

$$f(z) = \frac{z + 1}{z^3(z^2 + 1)}$$

has isolated singularities at $z = 0, \pm i$ and a zero at $z = -1$. We will show that $z = 0$ is a pole of order 3, $z = \pm i$ are poles of order 1 and $z = -1$ is a zero of order 1. The style of argument is the same in each case.

At $z = 0$:

$$f(z) = \frac{1}{z^3} \cdot \frac{z + 1}{z^2 + 1}.$$

Call the second factor $g(z)$. Since $g(z)$ is analytic at $z = 0$ and $g(0) = 1$, it has a Taylor series

$$g(z) = \frac{z + 1}{z^2 + 1} = 1 + a_1 z + a_2 z^2 + \ldots$$

Therefore

$$f(z) = \frac{1}{z^3} + \frac{a_1}{z^2} + \frac{a_2}{z} + \ldots.$$

This shows $z = 0$ is a pole of order 3.

At $z = i$: $f(z) = \frac{1}{z-i} \cdot \frac{z+1}{z^3(z+i)}$. Call the second factor $g(z)$. Since $g(z)$ is analytic at $z = i$, it has a Taylor series

$$g(z) = \frac{z + 1}{z^3(z + i)} = a_0 + a_1(z - i) + a_2(z - i)^2 + \ldots$$

where $a_0 = g(i) \neq 0$. Therefore

$$f(z) = \frac{a_0}{z - i} + a_1 + a_2(z - i) + \ldots.$$

This shows $z = i$ is a pole of order 1.

The arguments for $z = -i$ and $z = -1$ are similar.

8.2 Words: Holomorphic and meromorphic

Definition. A function that is analytic on a region A is called holomorphic on A.

A function that is analytic on A except for a set of poles of finite order is called meromorphic on A.

Example 8.2. Let

$$f(z) = \frac{z + z^2 + z^3}{(z-2)(z-3)(z-4)(z-5)}.$$

This is meromorphic on $\mathbf{C}$ with (simple) poles at $z = 2, 3, 4, 5$.

8.3 Behavior of functions near zeros and poles

The basic idea is that near a zero of order n, a function behaves like $(z - z_0)^n$ and near a pole of order n, a function behaves like $1/(z - z_0)^n$. The following make this a little more precise.

Behavior near a zero. If f has a zero of order n at z_0 then near z_0,

$$f(z) \approx a_n (z - z_0)^n,$$

for some constant a_n.

Proof. By definition f has a Taylor series around z_0 of the form

$$\begin{aligned} f(z) &= a_n(z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \ldots \\ &= a_n(z-z_0)^n \left(1 + \frac{a_{n+1}}{a_n}(z-z_0) + \frac{a_{n+2}}{a_n}(z-z_0)^2 + \ldots\right) \end{aligned}$$

Since the second factor equals 1 at z_0, the claim follows.

Behavior near a finite pole. If f has a pole of order n at z_0 then near z_0,

$$f(z) \approx \frac{b_n}{(z - z_0)^n},$$

for some constant b_n.

Proof. This is nearly identical to the previous argument. By definition f has a Laurent series around z_0 of the form

$$\begin{aligned} f(z) &= \frac{b_n}{(z-z_0)^n} + \frac{b_{n-1}}{(z-z_0)^{n-1}} + \ldots + \frac{b_1}{z-z_0} + a_0 + \ldots \\ &= \frac{b_n}{(z-z_0)^n}\left(1 + \frac{b_{n-1}}{b_n}(z-z_0) + \frac{b_{n-2}}{b_n}(z-z_0)^2 + \ldots\right) \end{aligned}$$

Since the second factor equals 1 at z_0, the claim follows.

8.3.1 Picard's theorem and essential singularities

Near an essential singularity we have Picard's theorem. We won't prove or make use of this theorem in 18.04. Still, we feel it is pretty enough to warrant showing to you.

Picard's theorem. If $f(z)$ has an essential singularity at z_0 then in every neighborhood of z_0, $f(z)$ takes on all possible values infinitely many times, with the possible exception of one value.

Example 8.3. It is easy to see that in any neighborhood of $z = 0$ the function $w = e^{1/z}$ takes every value except $w = 0$.

8.3.2 Quotients of functions

We have the following statement about quotients of functions. We could make similar statements if one or both functions has a pole instead of a zero.

Theorem. Suppose f has a zero of order m at z_0 and g has a zero of order n at z_0. Let

$$h(z) = \frac{f(z)}{g(z)}.$$

Then

- If $n > m$ then $h(z)$ has a pole of order $n - m$ at z_0.
- If $n < m$ then $h(z)$ has a zero of order $m - n$ at z_0.
- If $n = m$ then $h(z)$ is analytic and nonzero at z_0.

We can paraphrase this as $h(z)$ has 'pole' of order $n - m$ at z_0. If $n - m$ is negative then the 'pole' is actually a zero.

Proof. You should be able to supply the proof. It is nearly identical to the proofs above: express f and g as Taylor series and take the quotient.

Example 8.4. Let

$$h(z) = \frac{\sin(z)}{z^2}.$$

We know $\sin(z)$ has a zero of order 1 at $z = 0$ and z^2 has a zero of order 2. So, $h(z)$ has a pole of order 1 at $z = 0$. Of course, we can see this easily using Taylor series

$$h(z) = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \ldots\right)$$

8.4 Residues

In this section we'll explore calculating residues. We've seen enough already to know that this will be useful. We will see that even more clearly when we look at the residue theorem in the next section.

We introduced residues in the previous topic. We repeat the definition here for completeness.

Definition. Consider the function $f(z)$ with an isolated singularity at z_0, i.e. defined on the region $0 < |z - z_0| < r$ and with Laurent series (on that region)

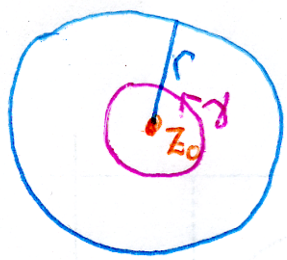
$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The residue of f at z_0 is b_1. This is denoted

$$\text{Res}(f, z_0) = b_1 \quad \text{or} \quad \underset{z=z_0}{\text{Res}}\, f = b_1.$$

What is the importance of the residue? If γ is a small, simple closed curve that goes counterclockwise around b_1 then

$$\int_\gamma f(z) = 2\pi i b_1.$$



γ small enough to be inside $|z - z_0| < r$, surround z_0 and contain no other singularity of f.

This is easy to see by integrating the Laurent series term by term. The only nonzero integral comes from the term b_1/z.

Example 8.5.

$$f(z) = e^{1/2z} = 1 + \frac{1}{2z} + \frac{1}{2(2z)^2} + \ldots$$

has an isolated singularity at 0. From the Laurent series we see that $\text{Res}(f, 0) = 1/2$.

Example 8.6.

(i) Let

$$f(z) = \frac{1}{z^3} + \frac{2}{z^2} + \frac{4}{z} + 5 + 6z.$$

f has a pole of order 3 at $z = 0$ and $\text{Res}(f, 0) = 4$.

(ii) Suppose

$$f(z) = \frac{2}{z} + g(z),$$

where g is analytic at $z = 0$. Then, f has a simple pole at 0 and $\text{Res}(f, 0) = 2$.

(iii) Let

$$f(z) = \cos(z) = 1 - z^2/2! + \ldots.$$

Then f is analytic at $z = 0$ and $\text{Res}(f, 0) = 0$.

(iv) Let

$$f(z) = \frac{\sin(z)}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \ldots\right) = 1 - \frac{z^2}{3!} + \ldots$$

So, f has a removable singularity at $z = 0$ and $\text{Res}(f, 0) = 0$.

Example 8.7. Using partial fractions. Let

$$f(z) = \frac{z}{z^2+1}.$$

Find the poles and residues of f.

Solution: Using partial fractions we write

$$f(z) = \frac{z}{(z-i)(z+i)} = \frac{1}{2}\cdot\frac{1}{z-i} + \frac{1}{2}\cdot\frac{1}{z+i}.$$

The poles are at $z = \pm i$. We compute the residues at each pole:

At $z = i$:

$$f(z) = \frac{1}{2}\cdot\frac{1}{z-i} + \text{ something analytic at } i.$$

Therefore the pole is simple and $\text{Res}(f, i) = 1/2$.

At $z = -i$:

$$f(z) = \frac{1}{2}\cdot\frac{1}{z+i} + \text{ something analytic at } -i.$$

Therefore the pole is simple and $\text{Res}(f, -i) = 1/2$.

Example 8.8. Mild warning! Let

$$f(z) = -\frac{1}{z(1-z)}$$

then we have the following Laurent expansions for f around $z = 0$.

On $0 < |z| < 1$:

$$f(z) = -\frac{1}{z}\cdot\frac{1}{1-z} = -\frac{1}{z}(1 + z + z^2 + \ldots).$$

Therefore the pole at $z = 0$ is simple and $\text{Res}(f, 0) = -1$.

On $1 < |z| < \infty$:

$$f(z) = \frac{1}{z^2}\cdot\frac{1}{1-1/z} = \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right).$$

Even though this is a valid Laurent expansion you must not use it to compute the residue at 0. This is because the definition of residue requires that we use the Laurent series on the region $0 < |z - z_0| < r$.

Example 8.9. Let

$$f(z) = \log(1 + z).$$

This has a singularity at $z = -1$, but it is not isolated, so not a pole and therefore there is no residue at $z = -1$.

8.4.1 Residues at simple poles

Simple poles occur frequently enough that we'll study computing their residues in some detail. Here are a number of ways to spot a simple pole and compute its residue. The justification for all of them goes back to Laurent series.

Suppose $f(z)$ has an isolated singularity at $z = z_0$. Then we have the following properties.

Property 1. If the Laurent series for $f(z)$ has the form

$$\frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \ldots$$

then f has a simple pole at z_0 and $\text{Res}(f, z_0) = b_1$.

Property 2 If

$$g(z) = (z - z_0)f(z)$$

is analytic at z_0 then z_0 is either a simple pole or a removable singularity. In either case $\text{Res}(f, z_0) = g(z_0)$. (In the removable singularity case the residue is 0.)

Proof. Directly from the Laurent series for f around z_0.

Property 3. If f has a simple pole at z_0 then

$$\lim_{z \to z_0} (z - z_0)f(z) = \text{Res}(f, z_0)$$

This says that the limit exists and equals the residue. Conversely, if the limit exists then either the pole is simple, or f is analytic at z_0. In both cases the limit equals the residue.

Proof. Directly from the Laurent series for f around z_0.

Property 4. If f has a simple pole at z_0 and $g(z)$ is analytic at z_0 then

$$\text{Res}(fg, z_0) = g(z_0)\,\text{Res}(f, z_0).$$

If $g(z_0) \neq 0$ then

$$\text{Res}(f/g, z_0) = \frac{1}{g(z_0)}\,\text{Res}(f, z_0).$$

Proof. Since z_0 is a simple pole,

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0)$$

Since g is analytic,

$$g(z) = c_0 + c_1(z - z_0) + \ldots,$$

where $c_0 = g(z_0)$. Multiplying these series together it is clear that

$$\text{Res}(fg, z_0) = c_0 b_1 = g(z_0)\,\text{Res}(f, z_0). \qquad \text{QED}$$

The statement about quotients f/g follows from the proof for products because $1/g$ is analytic at z_0.

Property 5. If $g(z)$ has a simple zero at z_0 then $1/g(z)$ has a simple pole at z_0 and

$$\text{Res}(1/g, z_0) = \frac{1}{g'(z_0)}.$$

Proof. The algebra for this is similar to what we've done several times above. The Taylor expansion for g is

$$g(z) = a_1(z - z_0) + a_2(z - z_0)^2 + \ldots,$$

where $a_1 = g'(z_0)$. So

$$\frac{1}{g(z)} = \frac{1}{a_1(z-z_0)}\left(\frac{1}{1+\frac{a_2}{a_1}(z-z_0)+\ldots}\right)$$

The second factor on the right is analytic at z_0 and equals 1 at z_0. Therefore we know the Laurent expansion of $1/g$ is

$$\frac{1}{g(z)} = \frac{1}{a_1(z-z_0)}\left(1+c_1(z-z_0)+\ldots\right)$$

Clearly the residue is $1/a_1 = 1/g'(z_0)$. QED.

Example 8.10. Let

$$f(z) = \frac{2+z+z^2}{(z-2)(z-3)(z-4)(z-5)}.$$

Show all the poles are simple and compute their residues.

Solution: The poles are at $z = 2, 3, 4, 5$. They are all isolated. We'll look at $z = 2$ the others are similar. Multiplying by $z-2$ we get

$$g(z) = (z-2)f(z) = \frac{2+z+z^2}{(z-3)(z-4)(z-5)}.$$

This is analytic at $z = 2$ and

$$g(2) = \frac{8}{-6} = -\frac{4}{3}.$$

So the pole is simple and $\text{Res}(f, 2) = -4/3$.

Example 8.11. Let

$$f(z) = \frac{1}{\sin(z)}.$$

Find all the poles and their residues.

Solution: The poles of $f(z)$ are the zeros of $\sin(z)$, i.e. $n\pi$ for n an integer. Since the derivative

$$\sin'(n\pi) = \cos(n\pi) \neq 0,$$

the zeros are simple and by Property 5 above

$$\text{Res}(f, n\pi) = \frac{1}{\cos(n\pi)} = (-1)^n.$$

Example 8.12. Let

$$f(z) = \frac{1}{z(z^2+1)(z-2)^2}.$$

Identify all the poles and say which ones are simple.

Solution: Clearly the poles are at $z = 0, \pm i, 2$.

At $z = 0$:

$$g(z) = zf(z)$$

is analytic at 0 and $g(0) = 1/4$. So the pole is simple and the residue is $g(0) = 1/4$.

At $z = i$:

$$g(z) = (z - i)f(z) = \frac{1}{z(z+i)(z-2)^2}$$

is analytic at i, the pole is simple and the residue is $g(i)$.

At $z = -i$: This is similar to the case $z = i$. The pole is simple.

At $z = 2$:

$$g(z) = (z-2)f(z) = \frac{1}{z(z^2+1)(z-2)}$$

is not analytic at 2, so the pole is not simple. (It should be obvious that it's a pole of order 2.)

Example 8.13. Let $p(z)$, $q(z)$ be analytic at $z = z_0$. Assume $p(z_0) \neq 0$, $q(z_0) = 0$, $q'(z_0) \neq 0$. Find

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)}.$$

Solution: Since $q'(z_0) \neq 0$, q has a simple zero at z_0. So $1/q$ has a simple pole at z_0 and

$$\text{Res}(1/q, z_0) = \frac{1}{q'(z_0)}$$

Since $p(z_0) \neq 0$ we know

$$\text{Res}(p/q, z_0) = p(z_0)\,\text{Res}(1/q, z_0) = \frac{p(z_0)}{q'(z_0)}.$$

8.4.2 Residues at finite poles

For higher-order poles we can make statements similar to those for simple poles, but the formulas and computations are more involved. The general principle is the following

Higher order poles. If $f(z)$ has a pole of order k at z_0 then

$$g(z) = (z - z_0)^k f(z)$$

is analytic at z_0 and if

$$g(z) = a_0 + a_1(z - z_0) + \ldots$$

then

$$\text{Res}(f, z_0) = a_{k-1} = \frac{g^{(k-1)}(z_0)}{(k-1)!}.$$

Proof. This is clear using Taylor and Laurent series for g and f.

Example 8.14. Let

$$f(z) = \frac{\sinh(z)}{z^5}$$

and find the residue at $z = 0$.

Solution: We know the Taylor series for

$$\sinh(z) = z + z^3/3! + z^5/5! + \ldots$$

(You can find this using $\sinh(z) = (e^z - e^{-z})/2$ and the Taylor series for e^z.) Therefore,

$$f(z) = \frac{1}{z^4} + \frac{1}{3!z^2} + \frac{1}{5!} + \ldots$$

We see $\text{Res}(f, 0) = 0$.

Note, we could have seen this by realizing that $f(z)$ is an even function.

Example 8.15. Let

$$f(z) = \frac{\sinh(z)e^z}{z^5}.$$

Find the residue at $z = 0$.

Solution: It is clear that $\text{Res}(f, 0)$ equals the coefficient of z^4 in the Taylor expansion of $\sinh(z)e^z$. We compute this directly as

$$\sinh(z)e^z = \left(z + \frac{z^3}{3!} + \ldots\right)\left(1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \ldots\right) = \ldots + \left(\frac{1}{4!} + \frac{1}{3!}\right)z^4 + \ldots$$

So

$$\text{Res}(f, 0) = \frac{1}{3!} + \frac{1}{4!} = \frac{5}{24}.$$

Example 8.16. Find the residue of

$$f(z) = \frac{1}{z(z^2+1)(z-2)^2}$$

at $z = 2$.

Solution: $g(z) = (z-2)^2 f(z) = \frac{1}{z(z^2+1)}$ is analytic at $z = 2$. So, the residue we want is the a_1 term in its Taylor series, i.e. $g'(2)$. This is easy, if dull, to compute

$$\text{Res}(f, 2) = g'(2) = -\frac{13}{100}$$

8.4.3 cot(z)

The function $\cot(z)$ turns out to be very useful in applications. This stems largely from the fact that it has simple poles at all multiples of π and the residue is 1 at each pole. We show that first.

Fact. $f(z) = \cot(z)$ has simple poles at $n\pi$ for n an integer and $\text{Res}(f, n\pi) = 1$.

Proof.

$$f(z) = \frac{\cos(z)}{\sin(z)}.$$

This has poles at the zeros of sin, i.e. at $z = n\pi$. At poles f is of the form p/q where q has a simple zero at z_0 and $p(z_0) \neq 0$. Thus we can use the formula

$$\text{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}.$$

In our case, we have

$$\text{Res}(f, n\pi) = \frac{\cos(n\pi)}{\cos(n\pi)} = 1,$$

as claimed.

Sometimes we need more terms in the Laurent expansion of $\cot(z)$. There is no known easy formula for the terms, but we can easily compute as many as we need using the following technique.

Example 8.17. Compute the first several terms of the Laurent expansion of $\cot(z)$ around $z = 0$.

Solution: Since $\cot(z)$ has a simple pole at 0 we know

$$\cot(z) = \frac{b_1}{z} + a_0 + a_1 z + a_2 z^2 + \ldots$$

We also know

$$\cot(z) = \frac{\cos(z)}{\sin(z)} = \frac{1 - z^2/2 + z^4/4! - \ldots}{z - z^3/3! + z^5/5! - \ldots}$$

Cross multiplying the two expressions we get

$$\left(\frac{b_1}{z} + a_0 + a_1 z + a_2 z^2 + \ldots\right)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right) = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \ldots$$

We can do the multiplication and equate the coefficients of like powers of z.

$$b_1 + a_0 z + \left(-\frac{b_1}{3!} + a_1\right) z^2 + \left(-\frac{a_0}{3!} + a_2\right) z^3 + \left(\frac{b_1}{5!} - \frac{a_1}{3!} + a_3\right) z^4 = 1 - \frac{z^2}{2!} + \frac{z^4}{4!}$$

So, starting from $b_1 = 1$ and $a_0 = 0$, we get

$$\begin{aligned} -b_1/3! + a_1 &= -1/2! & \Rightarrow \quad & a_1 = -1/3 \\ -a_0/3! + a_2 &= 0 & \Rightarrow \quad & a_2 = 0 \\ b_1/5! - a_1/3! + a_3 &= 1/4! & \Rightarrow \quad & a_3 = -1/45. \end{aligned}$$

As noted above, all the even terms are 0 as they should be. We have

$$\cot(z) = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \ldots$$

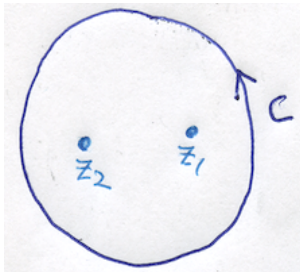
8.5 Cauchy Residue Theorem

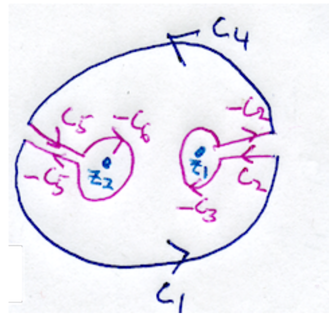
This is one of the major theorems in 18.04. It will allow us to make systematic our previous somewhat ad hoc approach to computing integrals on contours that surround singularities.

Theorem. (**Cauchy's residue theorem**) Suppose $f(z)$ is analytic in the region A except for a set of isolated singularities. Also suppose C is a simple closed curve in A that doesn't go through any of the singularities of f and is oriented counterclockwise. Then

$$\int_C f(z)\, dz = 2\pi i \sum \text{ residues of } f \text{ inside } C$$

Proof. The proof is based of the following figures. They only show a curve with two singularities inside it, but the generalization to any number of signularities is straightforward. In what follows we are going to abuse language and say pole when we mean isolated singularity, i.e. a finite order pole or an essential singularity ('infinite order pole').





The left figure shows the curve C surrounding two poles z_1 and z_2 of f. The right figure shows the same curve with some cuts and small circles added. It is chosen so that there are no poles of f inside it and so that the little circles around each of the poles are so small that there are no other poles inside them. The right hand curve is

$$\tilde{C} = C_1 + C_2 - C_3 - C_2 + C_4 + C_5 - C_6 - C_5$$

The left hand curve is $C = C_1 + C_4$. Since there are no poles inside $\tilde{C}$ we have, by Cauchy's theorem,

$$\int_{\tilde{C}} f(z)\, dz = \int_{C_1+C_2-C_3-C_2+C_4+C_5-C_6-C_5} f(z)\, dz = 0$$

Dropping C_2 and C_5, which are both added and subtracted, this becomes

$$\int_{C_1+C_4} f(z)\, dz = \int_{C_3+C_6} f(z)\, dz \tag{1}$$

If

$$f(z) = \ldots + \frac{b_2}{(z-z_1)^2} + \frac{b_1}{z-z_1} + a_0 + a_1(z-z_1) + \ldots$$

is the Laurent expansion of f around z_1 then

$$\begin{aligned}\int_{C_3} f(z)\, dz &= \int_{C_3} \ldots + \frac{b_2}{(z-z_1)^2} + \frac{b_1}{z-z_1} + a_0 + a_1(z-z_1) + \ldots\, dz \\ &= 2\pi i b_1 \\ &= 2\pi i \operatorname{Res}(f, z_1)\end{aligned}$$

Likewise

$$\int_{C_6} f(z)\, dz = 2\pi i \operatorname{Res}(f, z_2).$$

Using these residues and the fact that $C = C_1 + C_4$, Equation 1 becomes

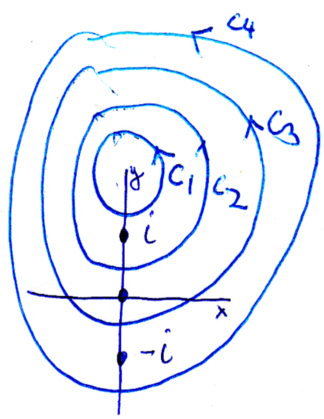
$$\int_C f(z)\, dz = 2\pi i \left[\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2)\right].$$

That proves the residue theorem for the case of two poles. As we said, generalizing to any number of poles is straightforward.

Example 8.18. Let

$$f(z) = \frac{1}{z(z^2+1)}.$$

Compute $\int f(z)\,dz$ over each of the contours C_1, C_2, C_3, C_4 shown.



Solution: The poles of $f(z)$ are at $z = 0, \pm i$. Using the residue theorem we just need to compute the residues of each of these poles.

At $z = 0$:

$$g(z) = zf(z) = \frac{1}{z^2+1}$$

is analytic at 0 so the pole is simple and

$$\text{Res}(f, 0) = g(0) = 1.$$

At $z = i$:

$$g(z) = (z-i)f(z) = \frac{1}{z(z+i)}$$

is analytic at i so the pole is simple and

$$\text{Res}(f, i) = g(i) = -1/2.$$

At $z = -i$:

$$g(z) = (z+i)f(z) = \frac{1}{z(z-i)}$$

is analytic at $-i$ so the pole is simple and

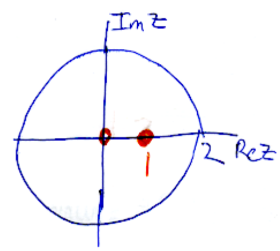
$$\text{Res}(f, -i) = g(-i) = -1/2.$$

Using the residue theorem we have

$$\int_{C_1} f(z)\,dz = 0 \;\; (\text{since } f \text{ is analytic inside } C_1)$$
$$\int_{C_2} f(z)\,dz = 2\pi i\,\text{Res}(f, i) = -\pi i$$
$$\int_{C_3} f(z)\,dz = 2\pi i\,[\text{Res}(f, i) + \text{Res}(f, 0)] = \pi i$$
$$\int_{C_4} f(z)\,dz = 2\pi i\,[\text{Res}(f, i) + \text{Res}(f, 0) + \text{Res}(f, -i)] = 0.$$

Example 8.19. Compute
$$\int_{|z|=2} \frac{5z-2}{z(z-1)}\, dz.$$

Solution: Let
$$f(z) = \frac{5z-2}{z(z-1)}.$$
The poles of f are at $z = 0, 1$ and the contour encloses them both.



At $z = 0$:
$$g(z) = zf(z) = \frac{5z-2}{(z-1)}$$
is analytic at 0 so the pole is simple and
$$\mathrm{Res}(f, 0) = g(0) = 2.$$

At $z = 1$:
$$g(z) = (z-1)f(z) = \frac{5z-2}{z}$$
is analytic at 1 so the pole is simple and
$$\mathrm{Res}(f, 1) = g(1) = 3.$$

Finally
$$\int_C \frac{5z-2}{z(z-1)}\, dz = 2\pi i\, [\mathrm{Res}(f, 0) + \mathrm{Res}(f, 1)] = 10\pi i.$$

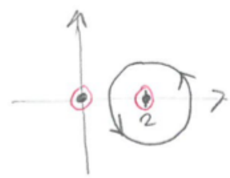
Example 8.20. Compute
$$\int_{|z|=1} z^2 \sin(1/z)\, dz.$$

Solution: Let
$$f(z) = z^2 \sin(1/z).$$
f has an isolated singularity at $z = 0$. Using the Taylor series for $\sin(w)$ we get
$$z^2 \sin(1/z) = z^2 \left(\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \ldots \right) = z - \frac{1/6}{z} + \ldots$$
So, $\mathrm{Res}(f, 0) = b_1 = -1/6$. Thus the residue theorem gives
$$\int_{|z|=1} z^2 \sin(1/z)\, dz = 2\pi i\, \mathrm{Res}(f, 0) = -\frac{i\pi}{3}.$$

Example 8.21. Compute

$$\int_C \frac{dz}{z(z-2)^4}\,dz,$$

where, $C: |z-2| = 1$.



Solution: Let

$$f(z) = \frac{1}{z(z-2)^4}.$$

The singularity at $z = 0$ is outside the contour of integration so it doesn't contribute to the integral.

To use the residue theorem we need to find the residue of f at $z = 2$. There are a number of ways to do this. Here's one:

$$\begin{aligned}\frac{1}{z} &= \frac{1}{2+(z-2)} \\ &= \frac{1}{2}\cdot\frac{1}{1+(z-2)/2} \\ &= \frac{1}{2}\left(1 - \frac{z-2}{2} + \frac{(z-2)^2}{4} - \frac{(z-2)^3}{8} + \ldots\right)\end{aligned}$$

This is valid on $0 < |z-2| < 2$. So,

$$f(z) = \frac{1}{(z-2)^4}\cdot\frac{1}{z} = \frac{1}{2(z-2)^4} - \frac{1}{4(z-2)^3} + \frac{1}{8(z-2)^2} - \frac{1}{16(z-2)} + \ldots$$

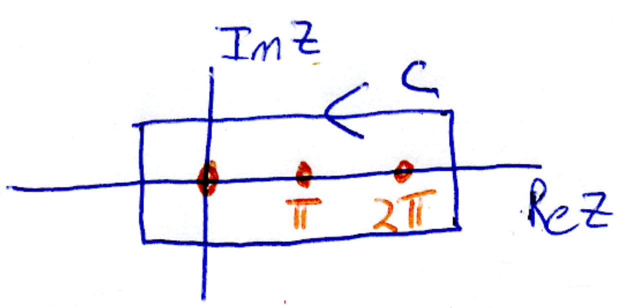
Thus, $\text{Res}(f, 2) = -1/16$ and

$$\int_C f(z)\,dz = 2\pi i\,\text{Res}(f,2) = -\frac{\pi i}{8}.$$

Example 8.22. Compute

$$\int_C \frac{1}{\sin(z)}\,dz$$

over the contour C shown.



Solution: Let

$$f(z) = 1/\sin(z).$$

There are 3 poles of f inside C at 0, π and 2π. We can find the residues by taking the limit of $(z - z_0)f(z)$. Each of the limits is computed using L'Hospital's rule. (This is valid, since the rule is just a statement about power series. We could also have used Property 5 from the section on residues of simple poles above.)

At $z = 0$:

$$\lim_{z\to 0} \frac{z}{\sin(z)} = \lim_{z\to 0} \frac{1}{\cos(z)} = 1.$$

Since the limit exists, $z = 0$ is a simple pole and

$$\text{Res}(f, 0) = 1.$$

At $z = \pi$:

$$\lim_{z\to \pi} \frac{z - \pi}{\sin(z)} = \lim_{z\to \pi} \frac{1}{\cos(z)} = -1.$$

Since the limit exists, $z = \pi$ is a simple pole and

$$\text{Res}(f, \pi) = -1.$$

At $z = 2\pi$: The same argument shows

$$\text{Res}(f, 2\pi) = 1.$$

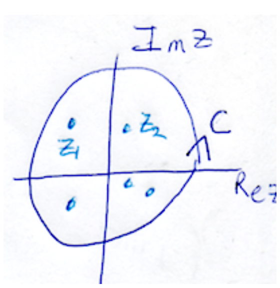
Now, by the residue theorem

$$\int_C f(z)\, dz = 2\pi i\, [\text{Res}(f, 0) + \text{Res}(f, \pi) + \text{Res}(f, 2\pi)] = 2\pi i.$$

8.6 Residue at ∞

The residue at ∞ is a clever device that can sometimes allow us to replace the computation of many residues with the computation of a single residue.

Suppose that f is analytic in $\mathbf{C}$ except for a finite number of singularities. Let C be a positively oriented curve that is large enough to contain all the singularities.



All the poles of f are inside C

Definition. We define the residue of f at infinity by

$$\text{Res}(f, \infty) = -\frac{1}{2\pi i} \int_C f(z)\, dz.$$

We should first explain the idea here. The interior of a simple closed curve is everything to left as you traverse the curve. The curve C is oriented counterclockwise, so its interior contains all the poles of f. The residue theorem says the integral over C is determined by the residues of these poles.

On the other hand, the interior of the curve $-C$ is everything outside of C. There are no poles of f in that region. If we want the residue theorem to hold (which we do –it's that important) then the only option is to have a residue at ∞ and define it as we did.

The definition of the residue at infinity assumes all the poles of f are inside C. Therefore the residue theorem implies

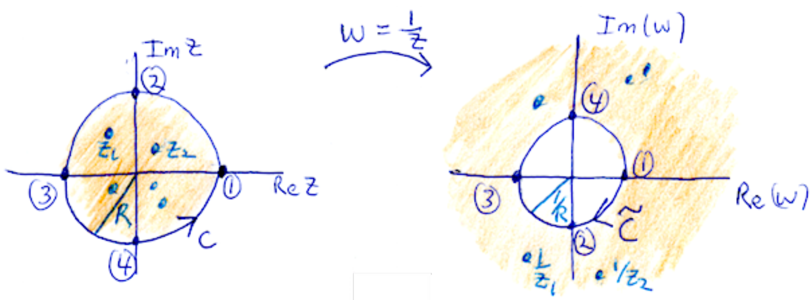
$$\text{Res}(f, \infty) = -\sum \text{ the residues of } f.$$

To make this useful we need a way to compute the residue directly. This is given by the following theorem.

Theorem. If f is analytic in $\mathbf{C}$ except for a finite number of singularities then

$$\text{Res}(f, \infty) = -\text{Res}\left(\frac{1}{w^2} f(1/w), 0\right).$$

Proof. The proof is just a change of variables: $w = 1/z$.



Change of variable: $w = 1/z$

First note that $z = 1/w$ and

$$dz = -(1/w^2)\, dw.$$

Next, note that the map $w = 1/z$ carries the positively oriented z-circle of radius R to the negatively oriented w-circle of radius $1/R$. (To see the orientiation, follow the circled points 1, 2, 3, 4 on C in the z-plane as they are mapped to points on $\tilde{C}$ in the w-plane.) Thus,

$$\text{Res}(f, \infty) = -\frac{1}{2\pi i}\int_C f(z)\, dz = \frac{1}{2\pi i}\int_{\tilde{C}} f(1/w)\frac{1}{w^2}\, dw$$

Finally, note that $z = 1/w$ maps all the poles inside the circle C to points outside the circle $\tilde{C}$. So the only possible pole of $(1/w^2)f(1/w)$ that is inside $\tilde{C}$ is at $w = 0$. Now, since $\tilde{C}$ is oriented clockwise, the residue theorem says

$$\frac{1}{2\pi i}\int_{\tilde{C}} f(1/w)\frac{1}{w^2}\, dw = -\text{Res}\left(\frac{1}{w^2} f(1/w), 0\right)$$

Comparing this with the equation just above finishes the proof.

Example 8.23. Let

$$f(z) = \frac{5z - 2}{z(z-1)}.$$

Earlier we computed

$$\int_{|z|=2} f(z)\,dz = 10\pi i$$

by computing residues at $z = 0$ and $z = 1$. Recompute this integral by computing a single residue at infinity.

Solution:

$$\frac{1}{w^2} f(1/w) = \frac{1}{w^2} \frac{5/w - 2}{(1/w)(1/w - 1)} = \frac{5 - 2w}{w(1 - w)}.$$

We easily compute that

$$\text{Res}(f, \infty) = -\,\text{Res}\left(\frac{1}{w^2} f(1/w), 0\right) = -5.$$

Since $|z| = 2$ contains all the singularities of f we have

$$\int_{|z|=2} f(z)\,dz = -2\pi i\,\text{Res}(f, \infty) = 10\pi i.$$

This is the same answer we got before!
