# Topic 7 Notes

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# 7 Taylor and Laurent series

## 7.1 Introduction

We originally defined an analytic function as one where the derivative, defined as a limit of ratios, existed. We went on to prove Cauchy's theorem and Cauchy's integral formula. These revealed some deep properties of analytic functions, e.g. the existence of derivatives of all orders.

Our goal in this topic is to express analytic functions as infinite power series. This will lead us to Taylor series. When a complex function has an isolated singularity at a point we will replace Taylor series by Laurent series. Not surprisingly we will derive these series from Cauchy's integral formula.

Although we come to power series representations after exploring other properties of analytic functions, they will be one of our main tools in understanding and computing with analytic functions.

## 7.2 Geometric series

Having a detailed understanding of geometric series will enable us to use Cauchy's integral formula to understand power series representations of analytic functions. We start with the definition:

**Definition.** A finite geometric series has one of the following (all equivalent) forms.

$$\begin{aligned} S_n &= a(1 + r + r^2 + r^3 + \ldots + r^n) \\ &= a + ar + ar^2 + ar^3 + \ldots + ar^n \\ &= \sum_{j=0}^{n} ar^j \\ &= a\sum_{j=0}^{n} r^j \end{aligned}$$

The number $r$ is called the ratio of the geometric series because it is the ratio of consecutive terms of the series.

**Theorem.** The sum of a finite geometric series is given by

$$S_n = a(1 + r + r^2 + r^3 + \ldots + r^n) = \frac{a(1 - r^{n+1})}{1 - r}. \qquad (1)$$

*Proof.* This is a standard trick that you've probably seen before.

$$\begin{array}{llll} S_n & = a + & ar + ar^2 + \ldots + ar^n & \\ rS_n & = & ar + ar^2 + \ldots + ar^n & + ar^{n+1} \end{array}$$

When we subtract these two equations most terms cancel and we get

$$S_n - rS_n = a - ar^{n+1}$$

Some simple algebra now gives us the formula in Equation 1.

**Definition.** An infinite geometric series has the same form as the finite geometric series except there is no last term:

$$S = a + ar + ar^2 + \ldots = a\sum_{j=0}^{\infty} r^j.$$

**Note.** We will usually simply say 'geometric series' instead of 'infinite geometric series'.

**Theorem.** If $|r| < 1$ then the infinite geometric series converges to

$$S = a\sum_{j=0}^{\infty} r^j = \frac{a}{1-r} \tag{2}$$

If $|r| \geq 1$ then the series does not converge.

*Proof.* This is an easy consequence of the formula for the sum of a finite geometric series. Simply let $n \to \infty$ in Equation 1.

**Note.** We have assumed a familiarity with convergence of infinite series. We will go over this in more detail in the appendix to this topic.

### 7.2.1 Connection to Cauchy's integral formula

Cauchy's integral formula says

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\, dw.$$

Inside the integral we have the expression

$$\frac{1}{w-z}$$

which looks a lot like the sum of a geometric series. We will make frequent use of the following manipulations of this expression.

$$\frac{1}{w-z} = \frac{1}{w}\cdot\frac{1}{1-z/w} = \frac{1}{w}\left(1 + (z/w) + (z/w)^2 + \ldots\right) \tag{3}$$

The geometric series in this equation has ratio $z/w$. Therefore, the series converges, i.e. the formula is valid, whenever $|z/w| < 1$, or equivalently when

$$|z| < |w|.$$

Similarly,

$$\frac{1}{w-z} = -\frac{1}{z}\cdot\frac{1}{1-w/z} = -\frac{1}{z}\left(1 + (w/z) + (w/z)^2 + \ldots\right) \tag{4}$$

The series converges, i.e. the formula is valid, whenever $|w/z| < 1$, or equivalently when

$$|z| > |w|.$$

## 7.3 Convergence of power series

When we include powers of the variable $z$ in the series we will call it a power series. In this section we'll state the main theorem we need about the convergence of power series. Technical details will be pushed to the appendix for the interested reader.

**Theorem 7.1.** Consider the power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

There is a number $R \geq 0$ such that:

1. If $R > 0$ then the series converges absolutely to an analytic function for $|z - z_0| < R$.
2. The series diverges for $|z - z_0| > R$. $R$ is called the radius of convergence. The disk $|z - z_0| < R$ is called the disk of convergence.
3. The derivative is given by term-by-term differentiation

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}$$

   The series for $f'$ also has radius of convergence $R$.

4. If $\gamma$ is a bounded curve inside the disk of convergence then the integral is given by term-by-term integration

$$\int_\gamma f(z)\, dz = \sum_{n=0}^{\infty} \int_\gamma a_n (z - z_0)^n$$

**Notes.**

- The theorem doesn't say what happens when $|z - z_0| = R$.
- If $R = \infty$ the function $f(z)$ is entire.
- If $R = 0$ the series only converges at the point $z = z_0$. In this case, the series does not represent an analytic function on any disk around $z_0$.
- Often (not always) we can find $R$ using the ratio test.

The proof of this theorem is in the appendix.

### 7.3.1 Ratio test and root test

Here are two standard tests from calculus on the convergence of infinite series.

**Ratio test.** Consider the series $\sum_0^\infty c_n$. If $L = \lim_{n\to\infty} |c_{n+1}/c_n|$ exists, then:

1. If $L < 1$ then the series converges absolutely.

2. If $L > 1$ then the series diverges.
3. If $L = 1$ then the test gives no information.

**Note.** In words, $L$ is the limit of the absolute ratios of consecutive terms.

Again the proof will be in the appendix. (It boils down to comparison with a geometric series.)

**Example 7.2.** Consider the geometric series $1 + z + z^2 + z^3 + \ldots$. The limit of the absolute ratios of consecutive terms is

$$L = \lim_{n\to\infty} \frac{|z^{n+1}|}{|z^n|} = |z|$$

Thus, the ratio test agrees that the geometric series converges when $|z| < 1$. We know this converges to $1/(1-z)$. Note, the disk of convergence ends exactly at the singularity $z = 1$.

**Example 7.3.** Consider the series $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$. The limit from the ratio test is

$$L = \lim_{n\to\infty} \frac{|z^{n+1}|/(n+1)!}{|z^n|/n!} = \lim \frac{|z|}{n+1} = 0.$$

Since $L < 1$ this series converges for every $z$. Thus, by Theorem 7.1, the radius of convergence for this series is $\infty$. That is, $f(z)$ is entire. Of course we know that $f(z) = \mathrm{e}^z$.

**Root test.** Consider the series $\sum_0^\infty c_n$. If $L = \lim_{n\to\infty} |c_n|^{1/n}$ exists, then:

1. If $L < 1$ then the series converges absolutely.
2. If $L > 1$ then the series diverges.
3. If $L = 1$ then the test gives no information .

**Note.** In words, $L$ is the limit of the $n$th roots of the (absolute value) of the terms.

The geometric series is so fundamental that we should check the root test on it.

**Example 7.4.** Consider the geometric series $1 + z + z^2 + z^3 + \ldots$. The limit of the $n$th roots of the terms is

$$L = \lim_{n\to\infty} |z^n|^{1/n} = \lim |z| = |z|$$

Happily, the root test agrees that the geometric series converges when $|z| < 1$.

## 7.4 Taylor series

The previous section showed that a power series converges to an analytic function inside its disk of convergence. Taylor's theorem completes the story by giving the converse: around each point of analyticity an analytic function equals a convergent power series.

**Theorem 7.5.** (Taylor's theorem) Suppose $f(z)$ is an analytic function in a region $A$. Let $z_0 \in A$. Then,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the series converges on any disk $|z - z_0| < r$ contained in $A$. Furthermore, we have formulas for the coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} \, dz. \tag{5}$$

(Where $\gamma$ is any simple closed curve in $A$ around $z_0$, with its interior entirely in $A$.)

We call the series the power series representing $f$ around $z_0$.

The proof will be given below. First we look at some consequences of Taylor's theorem.

**Corollary.** The power series representing an analytic function around a point $z_0$ is unique. That is, the coefficients are uniquely determined by the function $f(z)$.

*Proof.* Taylor's theorem gives a formula for the coefficients.

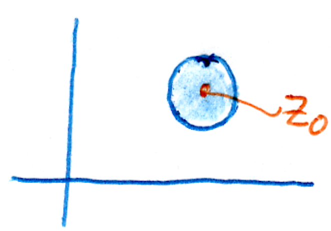
### 7.4.1 Order of a zero

**Theorem.** Suppose $f(z)$ is analytic on the disk $|z - z_0| < r$ and $f$ is not identically 0. Then there is an integer $k \geq 0$ such that $a_k \neq 0$ and $f$ has Taylor series around $z_0$ given by

$$f(z) = (z - z_0)^k (a_k + a_{k+1}(z - z_0) + \ldots) = (z - z_0)^k \sum_{n=k}^{\infty} a_n (z - z_0)^{n-k}. \tag{6}$$

*Proof.* Since $f(z)$ is not identically 0, not all the Taylor coefficients are zero. So, we take $k$ to be the index of the first nonzero coefficient.

**Theorem 7.6.** Zeros are isolated. If $f(z)$ is analytic and not identically zero then the zeros of $f$ are isolated. (By isolated we mean that we can draw a small disk around any zeros that doesn't contain any other zeros.)




Isolated zero at $z_0$: $f(z_0) = 0$, $f(z) \neq 0$ elsewhere in the disk.

*Proof.* Suppose $f(z_0) = 0$. Write $f$ as in Equation 6. There are two factors:

$$(z - z_0)^k$$

and

$$g(z) = a_k + a_{k+1}(z - z_0) + \ldots$$

Clearly $(z - z_0)^k \neq 0$ if $z \neq z_0$. We have $g(z_0) = a_k \neq 0$, so $g(z)$ is not 0 on some small neighborhood of $z_0$. We conclude that on this neighborhood the product is only zero when $z = z_0$, i.e. $z_0$ is an isolated 0.

**Definition.** The integer $k$ in Theorem 7.6 is called the order of the zero of $f$ at $z_0$.

Note, if $f(z_0) \neq 0$ then $z_0$ is a zero of order 0.

## 7.5 Taylor series examples

The uniqueness of Taylor series along with the fact that they converge on any disk around $z_0$ where the function is analytic allows us to use lots of computational tricks to find the series and be sure that it converges.

**Example 7.7.** Use the formula for the coefficients in terms of derivatives to give the Taylor series of $f(z) = e^z$ around $z = 0$.

*Solution:* Since $f'(z) = e^z$, we have $f^{(n)}(0) = e^0 = 1$. So,

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

**Example 7.8.** Expand $f(z) = z^8 e^{3z}$ in a Taylor series around $z = 0$.

*Solution:* Let $w = 3z$. So,

$$e^{3z} = e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!} = \sum_{k=0}^{\infty} \frac{3^n}{n!} z^n$$

Thus,

$$f(z) = \sum_{n=0}^{\infty} \frac{3^n}{n!} z^{n+8}.$$

**Example 7.9.** Find the Taylor series of $\sin(z)$ around $z = 0$ (Sometimes the Taylor series around 0 is called the Maclaurin series.)

*Solution:* We give two methods for doing this.

**Method 1.**

$$f^{(n)}(0) = \frac{d^n \sin(z)}{dz^n} = \begin{cases} (-1)^m & \text{for } n = 2m+1 = \text{odd}, \ m = 0, 1, 2, \ldots \\ 0 & \text{for } n \text{ even} \end{cases}$$

**Method 2.** Using

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i},$$

we have

$$\sin(z) = \frac{1}{2i}\left[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right] = \frac{1}{2i}\sum_{n=0}^{\infty}[(1-(-1)^n)]\frac{i^n z^n}{n!}$$

(We need absolute convergence to add series like this.)

Conclusion:

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

which converges for $|z| < \infty$.

**Example 7.10.** Expand the rational function

$$f(z) = \frac{1 + 2z^2}{z^3 + z^5}$$

around $z = 0$.

*Solution:* Note that $f$ has a singularity at 0, so we can't expect a convergent Taylor series expansion. We'll aim for the next best thing using the following shortcut.

$$f(z) = \frac{1}{z^3}\frac{2(1+z^2)-1}{1+z^2} = \frac{1}{z^3}\left[2 - \frac{1}{1+z^2}\right].$$

Using the geometric series we have

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty}(-z^2)^n = 1 - z^2 + z^4 - z^6 + \ldots$$

Putting it all together

$$f(z) = \frac{1}{z^3}\left(2 - 1 + z^2 - z^4 + \ldots\right) = \left(\frac{1}{z^3} + \frac{1}{z}\right) - \sum_{n=0}^{\infty}(-1)^n z^{2n+1}$$

**Note.** The first terms are called the singular part, i.e. those with negative powers of $z$. The summation is called the regular or analytic part. Since the geometric series for $1/(1+z^2)$ converges for $|z| < 1$, the entire series is valid in $0 < |z| < 1$

**Example 7.11.** Find the Taylor series for

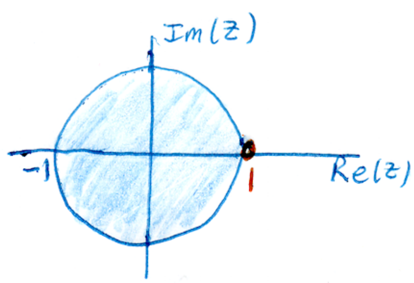
$$f(z) = \frac{e^z}{1-z}$$

around $z = 0$. Give the radius of convergence.

*Solution:* We start by writing the Taylor series for each of the factors and then multiply them out.

$$\begin{aligned} f(z) &= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right)\left(1 + z + z^2 + z^3 + \ldots\right) \\ &= 1 + (1+1)z + \left(1 + 1 + \frac{1}{2!}\right)z^2 + \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!}\right)z^3 + \ldots \end{aligned}$$

The biggest disk around $z = 0$ where $f$ is analytic is $|z| < 1$. Therefore, by Taylor's theorem, the radius of convergence is $R = 1$.




$f(z)$ is analytic on $|z| < 1$ and has a singularity at $z = 1$.

**Example 7.12.** Find the Taylor series for

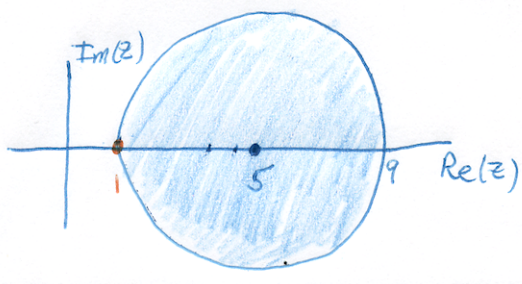
$$f(z) = \frac{1}{1-z}$$

around $z = 5$. Give the radius of convergence.

*Solution:* We have to manipulate this into standard geometric series form.

$$f(z) = \frac{1}{-4(1 + (z-5)/4)} = -\frac{1}{4}\left(1 - \left(\frac{z-5}{4}\right) + \left(\frac{z-5}{4}\right)^2 - \left(\frac{z-5}{4}\right)^3 + \ldots\right)$$

Since $f(z)$ has a singularity at $z = 1$ the radius of convergence is $R = 4$. We can also see this by considering the geometric series. The geometric series ratio is $(z-5)/4$. So the series converges when $|z-5|/4 < 1$, i.e. when $|z-5| < 4$, i.e. $R = 4$.




Disk of convergence stops at the singularity at $z = 1$.

**Example 7.13.** Find the Taylor series for

$$f(z) = \log(1 + z)$$

around $z = 0$. Give the radius of convergence.

*Solution:* We know that $f$ is analytic for $|z| < 1$ and not analytic at $z = -1$. So, the radius of convergence is $R = 1$. To find the series representation we take the derivative and use the geometric series.

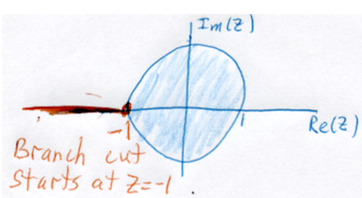
$$f'(z) = \frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - \ldots$$

Integrating term by term (allowed by Theorem 7.1) we have

$$f(z) = a_0 + z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots = a_0 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$$

Here $a_0$ is the constant of integration. We find it by evalating at $z = 0$.

$$f(0) = a_0 = \log(1) = 0.$$




Disk of convergence for $\log(1 + z)$ around $z = 0$.

**Example 7.14.** Can the series

$$\sum a_n (z-2)^n$$

converge at $z = 0$ and diverge at $z = 3$.

*Solution:* No! We have $z_0 = 2$. We know the series diverges everywhere outside its radius of convergence. So, if the series converges at $z = 0$, then the radius of convergence is at least 2. Since $|3 - z_0| < 2$ we would also have that $z = 3$ is inside the disk of convergence.

### 7.5.1 Proof of Taylor's theorem

For convenience we restate Taylor's Theorem 7.5.

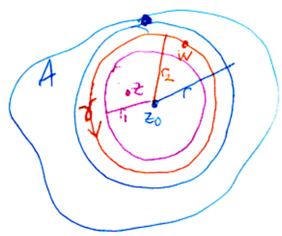
**Taylor's theorem.** (Taylor series) Suppose $f(z)$ is an analytic function in a region $A$. Let $z_0 \in A$. Then,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the series converges on any disk $|z - z_0| < r$ contained in $A$. Furthermore, we have formulas for the coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz \tag{7}$$

*Proof.* In order to handle convergence issues we fix $0 < r_1 < r_2 < r$. We let $\gamma$ be the circle $|w - z_0| = r_2$ (traversed counterclockise).




Disk of convergence extends to the boundary of $A$
$r_1 < r_2 < r$, but $r_1$ and $r_2$ can be arbitrarily close to $r$.

Take $z$ inside the disk $|z - z_0| < r_1$. We want to express $f(z)$ as a power series around $z_0$. To do this we start with the Cauchy integral formula and then use the geometric series.

As preparation we note that for $w$ on $\gamma$ and $|z - z_0| < r_1$ we have

$$|z - z_0| < r_1 < r_2 = |w - z_0|,$$

so

$$\frac{|z - z_0|}{|w - z_0|} < 1.$$

Therefore,

$$\frac{1}{w - z} = \frac{1}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}} = \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left( \frac{z - z_0}{w - z_0} \right)^n = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}$$

Using this and the Cauchy formula gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw \\ &= \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} \frac{f(w)}{(w-z_0)^{n+1}}\,(z-z_0)^n\,dw \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)(z-z_0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}\,(z-z_0)^n \end{aligned}$$

The last equality follows from Cauchy's formula for derivatives. Taken together the last two equalities give Taylor's formula. QED

## 7.6 Singularities

**Definition.** A function $f(z)$ is singular at a point $z_0$ if it is not analytic at $z_0$

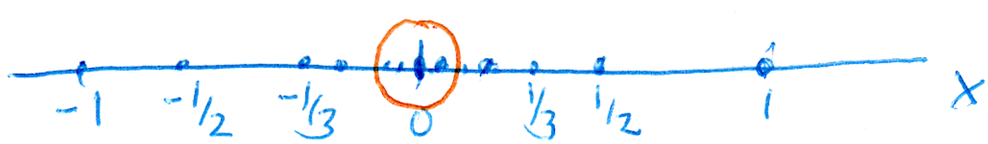
**Definition.** For a function $f(z)$, the singularity $z_0$ is an isolated singularity if $f$ is analytic on the deleted disk $0 < |z - z_0| < r$ for some $r > 0$.

**Example 7.15.** $f(z) = \dfrac{z+1}{z^3(z^2+1)}$ has isolated singularities at $z = 0, \pm i$.

**Example 7.16.** $f(z) = \mathrm{e}^{1/z}$ has an isolated singularity at $z = 0$.

**Example 7.17.** $f(z) = \log(z)$ has a singularity at $z = 0$, but it is not isolated because a branch cut, starting at $z = 0$, is needed to have a region where $f$ is analytic.

**Example 7.18.** $f(z) = \dfrac{1}{\sin(\pi/z)}$ has singularities at $z = 0$ and $z = 1/n$ for $n = \pm 1, \pm 2, \ldots$ The singularities at $\pm 1/n$ are isolated, but the one at $z = 0$ is not isolated.




Every neighborhood of 0 contains zeros at $1/n$ for large $n$.

## 7.7 Laurent series

**Theorem 7.19.** (Laurent series). Suppose that $f(z)$ is analytic on the annulus

$$A:\ r_1 < |z - z_0| < r_2.$$

Then $f(z)$ can be expressed as a series

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

The coefficients have the formulus

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw$$
$$b_n = \frac{1}{2\pi i}\int_\gamma f(w)(w-z_0)^{n-1}\,dw,$$

where $\gamma$ is any circle $|w-z_0| = r$ inside the annulus, i.e. $r_1 < r < r_2$.

Furthermore

- The series $\displaystyle\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges to an analytic function for $|z-z_0| < r_2$.
- The series $\displaystyle\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ converges to an analytic function for $|z-z_0| > r_1$.
- Together, the series both converge on the annulus $A$ where $f$ is analytic.

The proof is given below. First we define a few terms.

**Definition.** The entire series is called the Laurent series for $f$ around $z_0$. The series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

is called the analytic or regular part of the Laurent series. The series

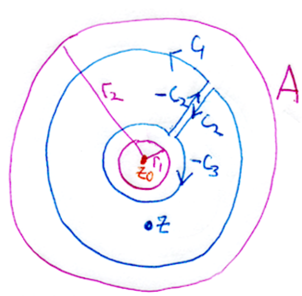
$$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

is called the singular or principal part of the Laurent series.

**Note.** Since $f(z)$ may not be analytic (or even defined) at $z_0$ we don't have any formulas for the coefficients using derivatives.

*Proof.* (Laurent series). Choose a point $z$ in $A$. Now set circles $C_1$ and $C_3$ close enough to the boundary that $z$ is inside $C_1 + C_2 - C_3 - C_2$ as shown. Since this curve and its interior are contained in $A$, Cauchy's integral formula says

$$f(z) = \frac{1}{2\pi i}\int_{C_1+C_2-C_3-C_2} \frac{f(w)}{w-z}\,dw$$




The contour used for proving the formulas for Laurent series.

The integrals over $C_2$ cancel, so we have

$$f(z) = \frac{1}{2\pi i} \int_{C_1 - C_3} \frac{f(w)}{w - z}\, dw.$$

Next, we divide this into two pieces and use our trick of converting to a geometric series. The calculuations are just like the proof of Taylor's theorem. On $C_1$ we have

$$\frac{|z - z_0|}{|w - z_0|} < 1,$$

so

$$\begin{aligned}
\frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w - z}\, dw &= \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w - z_0} \cdot \frac{1}{\left(1 - \frac{z - z_0}{w - z_0}\right)}\, dw \\
&= \frac{1}{2\pi i} \int_{C_1} \sum_{n=0}^{\infty} \frac{f(w)}{(w - z_0)^{n+1}} (z - z_0)^n\, dw \\
&= \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w - z_0)^{n+1}}\, dw \right) (z - z_0)^n \\
&= \sum_{n=0}^{\infty} a_n (z - z_0)^n.
\end{aligned}$$

Here $a_n$ is defined by the integral formula given in the statement of the theorem. Examining the above argument we see that the only requirement on $z$ is that $|z - z_0| < r_2$. So, this series converges for all such $z$.

Similarly on $C_3$ we have

$$\frac{|w - z_0|}{|z - z_0|} < 1,$$

so

$$\begin{aligned}
\frac{1}{2\pi i} \int_{C_3} \frac{f(w)}{w - z}\, dw &= \frac{1}{2\pi i} \int_{C_3} -\frac{f(w)}{z - z_0} \cdot \frac{1}{\left(1 - \frac{w - z_0}{z - z_0}\right)}\, dw \\
&= -\frac{1}{2\pi i} \int_{C_3} \sum_{n=0}^{\infty} f(w) \frac{(w - z_0)^n}{(z - z_0)^{n+1}}\, dw \\
&= -\frac{1}{2\pi i} \sum_{n=0}^{\infty} \left( \int_{C_1} f(w)(w - z_0)^n\, dw \right) (z - z_0)^{-n-1} \\
&= -\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}.
\end{aligned}$$

In the last equality we changed the indexing to match the indexing in the statement of the theorem. Here $b_n$ is defined by the integral formula given in the statement of the theorem. Examining the above argument we see that the only requirement on $z$ is that $|z - z_0| > r_1$. So, this series converges for all such $z$.

Combining these two formulas we have

$$f(z) = \frac{1}{2\pi i} \int_{C_1 - C_3} \frac{f(w)}{w - z}\, dw = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

The last thing to note is that the integrals defining $a_n$ and $b_n$ do not depend on the exact radius of the circle of integration. Any circle inside $A$ will produce the same values. We have proved all the statements in the theorem on Laurent series. QED

### 7.7.1 Examples of Laurent series

In general, the integral formulas are not a practical way of computing the Laurent coefficients. Instead we use various algebraic tricks. Even better, as we shall see, is the fact that often we don't really need all the coefficients and we will develop more techniques to compute those that we do need.

**Example 7.20.** Find the Laurent series for

$$f(z) = \frac{z+1}{z}$$

around $z_0 = 0$. Give the region where it is valid.

*Solution:* The answer is simply

$$f(z) = 1 + \frac{1}{z}.$$

This is a Laurent series, valid on the infinite region $0 < |z| < \infty$.

**Example 7.21.** Find the Laurent series for

$$f(z) = \frac{z}{z^2+1}$$

around $z_0 = i$. Give the region where your answer is valid. Identify the singular (principal) part.

*Solution:* Using partial fractions we have

$$f(z) = \frac{1}{2} \cdot \frac{1}{z-i} + \frac{1}{2} \cdot \frac{1}{z+i}.$$

Since $\dfrac{1}{z+i}$ is analytic at $z = i$ it has a Taylor series expansion. We find it using geometric series.

$$\frac{1}{z+i} = \frac{1}{2i} \cdot \frac{1}{1+(z-i)/(2i)} = \frac{1}{2i} \sum_{n=0}^{\infty} \left(-\frac{z-i}{2i}\right)^n$$

So the Laurent series is

$$f(z) = \frac{1}{2} \cdot \frac{1}{z-i} + \frac{1}{4i} \sum_{n=0}^{\infty} \left(-\frac{z-i}{2i}\right)^n$$

The singular (principal) part is given by the first term. The region of convergence is $0 < |z-i| < 2$.

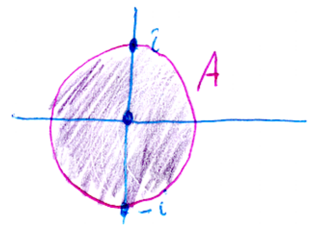
**Note.** We could have looked at $f(z)$ on the region $2 < |z-i| < \infty$. This would have produced a different Laurent series. We discuss this further in an upcoming example.

**Example 7.22.** Compute the Laurent series for

$$f(z) = \frac{z+1}{z^3(z^2+1)}$$

on the region $A: \ 0 < |z| < 1$ centered at $z = 0$.

*Solution:* This function has isolated singularities at $z = 0, \pm i$. Therefore it is analytic on the region $A$.




$f(z)$ has singularities at $z = 0, \pm i$.

At $z = 0$ we have

$$f(z) = \frac{1}{z^3}(1+z)(1 - z^2 + z^4 - z^6 + \ldots).$$

Multiplying this out we get

$$f(z) = \frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{z} - 1 + z + z^2 - z^3 - \ldots.$$

The following example shows that the Laurent series depends on the region under consideration.

**Example 7.23.** Find the Laurent series around $z = 0$ for $f(z) = \dfrac{1}{z(z-1)}$ in each of the following regions:

(i) the region $A_1: \ 0 < |z| < 1$

(ii) the region $A_2: \ 1 < |z| < \infty$.

*Solution:* For (i)

$$f(z) = -\frac{1}{z} \cdot \frac{1}{1-z} = -\frac{1}{z}(1 + z + z^2 + \ldots) = -\frac{1}{z} - 1 - z - z^2 - \ldots$$

For (ii): Since the usual geometric series for $1/(1-z)$ does not converge on $A_2$ we need a different form,

$$f(z) = \frac{1}{z} \cdot \frac{1}{z(1-1/z)} = \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right)$$

Since $|1/z| < 1$ on $A_2$ our use of the geometric series is justified.

One lesson from this example is that the Laurent series depends on the region as well as the formula for the function.

## 7.8 Digression to differential equations

Here is a standard use of series for solving differential equations.

**Example 7.24.** Find a power series solution to the equation

$$f'(x) = f(x) + 2, \qquad f(0) = 0.$$

*Solution:* We look for a solution of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Using the initial condition we find $f(0) = 0 = a_0$. Substituting the series into the differential equation we get

$$f'(x) = a_1 + 2a_2x + 3a_3x^3 + \ldots = f(x) + 2 = a_0 + 2 + a_1x + a_2x^2 + \ldots$$

Equating coefficients and using $a_0 = 0$ we have

$$\begin{aligned}
a_1 &= a_0 + 2 & \Rightarrow\ a1 &= 2 \\
2a_2 &= a_1 & \Rightarrow\ a_2 &= a_1/2 = 1 \\
3a_3 &= a_2 & \Rightarrow\ a_3 &= 1/3 \\
4a_4 &= a_3 & \Rightarrow\ a_4 &= 1/(3\cdot 4)
\end{aligned}$$

In general

$$(n+1)a_{n+1} = a_n \quad \Rightarrow \quad a_{n+1} = \frac{a_n}{(n+1)} = \frac{1}{3\cdot 4\cdot 5\cdots(n+1)}.$$

You can check using the ratio test that this function is entire.

### 7.9 Poles

Poles refer to isolated singularities. So, we suppose $f(z)$ is analytic on $0 < |z - z_0| < r$ and has Laurent series

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n.$$

**Definition of poles.** If only a finite number of the coefficients $b_n$ are nonzero we say $z_0$ is a finite pole of $f$. In this case, if $b_k \neq 0$ and $b_n = 0$ for all $n > k$ then we say $z_0$ is a pole of order $k$.

- If $z_0$ is a pole of order 1 we say it is a simple pole of $f$.
- If an infinite number of the $b_n$ are nonzero we say that $z_0$ is an essential singularity or a pole of infinite order of $f$.
- If all the $b_n$ are 0, then $z_0$ is called a removable singularity. That is, if we define $f(z_0) = a_0$ then $f$ is analytic on the disk $|z - z_0| < r$.

The terminology can be a bit confusing. So, imagine that I tell you that $f$ is defined and analytic on the punctured disk $0 < |z - z_0| < r$. Then, a priori we assume $f$ has a singularity at $z_0$. But, if after computing the Laurent series we see there is no singular part we can extend the definition of $f$ to the full disk, thereby 'removing the singularity'.

We can explain the term essential singularity as follows. If $f(z)$ has a pole of order $k$ at $z_0$ then $(z-z_0)^k f(z)$ is analytic (has a removable singularity) at $z_0$. So, $f(z)$ itself is not much harder to work with than an analytic function. On the other hand, if $z_0$ is an essential singularity then no algebraic trick will change $f(z)$ into an analytic function at $z_0$.

### 7.9.1 Examples of poles

We'll go back through many of the examples from the previous sections.

**Example 7.25.** The rational function

$$f(z) = \frac{1+2z^2}{z^3+z^5}$$

expanded to

$$f(z) = \left(\frac{1}{z^3} + \frac{1}{z}\right) - \sum_{n=0}^{\infty} (-1)^n z^{2n+1}.$$

Thus, $z = 0$ is a pole of order 3.

**Example 7.26.** Consider

$$f(z) = \frac{z+1}{z} = 1 + \frac{1}{z}.$$

Thus, $z = 0$ is a pole of order 1, i.e. a simple pole.

**Example 7.27.** Consider

$$f(z) = \frac{z}{z^2+1} = \frac{1}{2} \cdot \frac{1}{z-i} + g(z),$$

where $g(z)$ is analytic at $z = i$. So, $z = i$ is a simple pole.

**Example 7.28.** The function

$$f(z) = \frac{1}{z(z-1)}$$

has isolated singularities at $z = 0$ and $z = 1$. Show that both are simple poles.

*Solution:* In a neighborhood of $z = 0$ we can write

$$f(z) = \frac{g(z)}{z}, \quad \text{where} \quad g(z) = \frac{1}{z-1}.$$

Since $g(z)$ is analytic at 0, $z = 0$ is a finite pole. Since $g(0) \neq 0$, the pole has order 1, i.e. it is simple.

Likewise, in a neighborhood of $z = 1$,

$$f(z) = \frac{h(z)}{z-1}, \quad \text{where} \quad h(z) = \frac{1}{z}.$$

Since $h$ is analytic at $z = 1$, $f$ has a finite pole there. Since $h(1) \neq 0$ it is simple.

**Example 7.29.** Consider

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots$$

So, $z = 0$ is an essential singularity.

**Example 7.30.** $\log(z)$ has a singularity at $z = 0$. Since the singularity is not isolated, it can't be classified as either a pole or an essential singularity.

### 7.9.2 Residues

In preparation for discussing the residue theorem in the next topic we give the definition and an example here.

Note well, residues have to do with isolated singularites.

**Definition 7.31.** Consider the function $f(z)$ with an isolated singularity at $z_0$, i.e. defined on $0 < |z - z_0| < r$ and with Laurent series

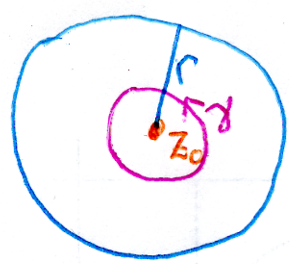
$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The residue of $f$ at $z_0$ is $b_1$. This is denoted

$$\text{Res}(f, z_0) \quad \text{or} \quad \operatorname*{Res}_{z=z_0} f = b_1.$$

What is the importance of the residue? If $\gamma$ is a small, simple closed curve that goes counterclockwise around $z_0$ then

$$\int_\gamma f(z) = 2\pi i b_1.$$




$\gamma$ is small enough to be inside $|z - z_0| < r$, and surround $z_0$.

This is easy to see by integrating the Laurent series term by term. The only nonzero integral comes from the term $b_1/z$.

**Example 7.32.** The function

$$f(z) = \mathrm{e}^{1/(2z)} = 1 + \frac{1}{2z} + \frac{1}{2(2z)^2} + \ldots$$

has an isolated singularity at 0. From the Laurent series we see that

$$\text{Res}(f, 0) = \frac{1}{2}.$$

## 7.10 Appendix: convergence

This section needs to be completed. It will give some of the careful technical definitions and arguments regarding convergence and manipulation of series. In particular it will define the notion of uniform convergence. The short description is that all of our manipulations of power series are justified on any closed bounded region. Almost, everything we did can be restricted to a closed disk or annulus, and so was valid.