Solutions for Problem Set 8: Models, compactifications, identifying spaces, characterizing subspaces

Read the notes on the course website related to adjunctions and compactifications (sections 1–5).

Problem 1 (8)**.** Have you done the reading?

Solution 1. Yes!

Problem 2 (B8)**.** Let X be a compact space and Y be a metric space. Prove that the metric topology on the set of continuous functions $\hom(X,Y)$ (defined by the metric $d(f,g) = \sup_{x\in X}\{d_Y(f(x),g(x))\}$) is the same as the compact open topology.

Solution 2. Let K be compact subset of X and U be an open subset of Y. Let $V = V(K,U)$ be the set of functions $f : X \to Y$ such that $f(K) \subset U$. Such subsets V form a basis for the compact open topology; hence showing that V is metric open shows that metric topology is finer than compact open topology.

Now, let $f \in V$. Then, $f(K) \subset U$ is a compact subset, Hence we can find $\epsilon > 0$ such that $B(x;\epsilon) \subset U$ for all $x \in f(K)$. Then consider the open ball in the mapping space, centered at f and with radius ϵ. Then, one easily check this ball lies in V. Hence, V is metric open.

Conversely, start with a metric open subset V' and $f \in V'$. Let r be such that $B = B(f,r) \subset V'$. The open balls with radius $r/2$ cover Y, hence, they cover $f(X)$. Thus, there exists $x_1,\ldots,x_n \in Y$ such that $f(X) \subset \bigcup_{i=1}^n B(x_i,r/2)$. Let K_i denote $f^{-1}(\overline{B(x_i,r/2)})$, which is a compact subset of X. Then, clearly $f \in \bigcap V(K_i, B(x_i;2r/3)$, as $f(K_i) \subset \overline{B(x_i,r/2)}$. On the other hand, it is easy to show that no function in $\bigcap V(K_i, B(x_i;2r/3)$ can have metric distance to f bigger than $2r/3$. Hence, $\bigcap V(K_i, B(x_i;2r/3)$ is in the metric open ball $B(f,r)$. This shows B and V' are open in the metric topology as well. Hence, the two topologies coincide.

Problem 3 (8)**.** Give a model for *the space of rays starting at the origin in* $\mathbb{R}^n$ and prove that it is homeomorphic to S^{n-1}.

Solution 3. Call this space $\mathcal{R}$. Each $x \in \mathbb{R}^n \setminus \{0\}$ gives a ray $\{tx : t \in \mathbb{R}_{\geq 0}\}$. Hence we have a surjection of sets

$$\mathbb{R}^n \setminus \{0\} \to \mathcal{R}$$

The fiber of a ray $[x] = \mathbb{R}_{\geq 0}x$ is $\mathbb{R}_{\geq 0}x$. It is reasonable to endow $\mathcal{R}$ with the quotient topology. If one wants to relate things that he/she already knows, one can either see it as a subset of the mapping space $Map(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ endowed with compact open topology, or see it as the set $\{(l, p) : l \in \mathbb{RP}^2, p \text{ is a direction in } l\}$ and endowed with the coarsest topology making $\mathcal{R} \to \mathbb{RP}^2$ and $(l, p) \mapsto (l, -p)$ continuous. Another way of topologizing this is giving a description of $Top(\ ,\mathcal{R})$, as the functor that sends a space B into rays parametrized by B, i.e. the set of closed subsets of $B \times \mathbb{R}^n \setminus \{0\}$ that specialize to a ray over each point of B. Note I am highly unsure of this last one, check it for yourself. Then, Yoneda tells us it is uniquely determined in Top and we can check that our model represents it.

Now, my model was $\mathbb{R}^n \setminus \{0\}/\sim$, where $\sim$ is identifying vectors that are positive real multiples of each other. But we have a map

$$p : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$$

such that $x \mapsto \frac{x}{||x||}$. This map is clearly surjective, with the fibers the same as those of the projection to $\mathcal{R}$. Hence, to show that S^n has the quotient topology, hence homeomorphic to $\mathcal{R}$, we only need to show p is a quotient map. But, this is easy, it is obviously continuous and open, indeed images of small enough open balls are open balls again. Hence, we are done.

Problem 4 (16)**.** Prove that $\mathbb{RP}^2$ is not homeomorphic to $\mathbb{D}^2$.

Solution 4. The easiest way of proving this would be showing that every loop in D^2 contracts, whereas there are loops in $\mathbb{RP}^2$ that does not contract to a constant loop. Namely, if you have a loop $\gamma : S^1 \to \mathbb{D}^2$, then there exists a path of loops(see Problem 5), or a map $\Gamma : [0, 1] \times S^1 \to \mathbb{D}^2$ such that $\Gamma(0, t) = \gamma(t)$ and $\Gamma(1, t)$ is constant in t (e.g. take $\Gamma(s, t) = (1 - s)\gamma(t)$). You can show this does not hold for $\mathbb{RP}^2$ in various ways. Take for instance the standard quotient map $\pi : S^2 \to \mathbb{RP}^2$. Take one of the big equitorial circles and parametrize half of it, which we call $\tilde{\theta} : [0, 1] \to S^2$. Then its endpoints are antipodal to each other and its image $\theta = \pi \circ \tilde{\theta}$ in $\mathbb{RP}^2$ defines a loop. If it contracts as before, i.e. if there exists a $\Theta : [0, 1] \times [0, 1] \to \mathbb{RP}^2$ such that $\Theta(0, t) = \theta(t)$, $\Theta(s, 0) = \Theta(s, 1)$ and $\Theta(1, t)$ is constant in t, then one can show this has a lift $\tilde{\Theta} : [0, 1] \times [0, 1] \to S^2$ such that $\tilde{\Theta}(0, t) = \tilde{\theta}(t)$ and $\pi \circ \tilde{\Theta} = \Theta$. Then the endpoints of $\tilde{\Theta}(s,\ )$ are antipodal always(either antipodal or the same as it projects to Θ, but actually antipodal everywhere as it is antipodal for $s = 0$) Hence, this has to be the case for $\tilde{\Theta}(1, t)$ as well, which projects to a constant path, hence which is itself constant. This contradiction shows that θ does not contract, hence the two spaces are not homeomorphic.

Another way would be proceeding locally, namely the boundary points cannot be identified with any point of $\mathbb{RP}^2$ under any homeomorphism/diffeomorphism. Showing this for diffeomorphisms is easier, it just uses basic calculus(there is a notion of outward tangent vector on the boundary points) A nice, purely topological way of showing this would be as follows: Every boundary point p of $\mathbb{D}^2$ has a neighborhood basis $\{U_i\}$ so that $U_i \setminus \{p\}$ is contractible or at least every loop in it is equivalent in the above sense to a constant loop. But for points

$q \in \mathbb{RP}^2$, small neighborhoods are homeomorphic to neighborhoods of 0 in $\mathbb{R}^2$. And one can show if $0 \in U \subset \mathbb{R}^2$ an open subset, then $U \setminus \{0\}$ has circles that does not contract, e.g. using integration over certain closed 1-forms as explained in the question below.

Problem 5 (B9)**.** Consider two points, α and β, in the mapping space $\hom(S^1, \mathbb{R}^2 \setminus \{0\})$.

a. Describe geometrically what it would mean for there to be a path from α to β in this space.

b. Give an example of two points that are in the same path component of this space.

c. Give an example of two points that are not in the same path component.

Solution 5. a. We have the adjunction,

$$Top(X, hom(Y, Z)) = Top(X \times Y, Z)$$

for nice enough topological spaces(a very geometric start:). Appllied in our situation, this just tells us that a map from $I = [0, 1]$ to $hom(S^1, \mathbb{R}^2 \setminus \{0\})$ is the same thing as a map $I \times S^1 \to \mathbb{R}^2 \setminus \{0\}$. Hence, a path from α to β is just a homotopy,i.e. a map

$$\phi : I \times S^1 \to \mathbb{R}^2 \setminus \{0\}$$

such that $\phi_{|\{0\} \times S^1} = \alpha$ and $\phi_{|\{1\} \times S^1} = \beta$.

b. Well, we can take $\alpha = \beta$ and the path/homotopy to be the constant path/homotopy, i.e. $(t, x) \mapsto \alpha(x) = \beta(x)$. A less cheap example is when there exists an $r > 0$ such that $\alpha(x) = r\beta(x)$ for all x. Then,

$$(t, x) \mapsto (t - tr + r)\beta$$

is such a path.

c. Let α be the standard circle $x \mapsto x$ and β be the constant map, $x \mapsto x_0$, for a fixed $x_0 \in \mathbb{R}^n \setminus \{0\}$. Now I don't know how much you have seen about fundamental groups or covering spaces so far, but an elementary proof of this can be given as follows:

Consider the 1-form $Im(\frac{dz}{z})$, where $z = x + iy$, i.e. the closed 1-form $\omega = \frac{xdy - ydx}{x^2 + y^2}$. If you parametrize α by $\theta \mapsto (cos\theta, \sin\theta)$, then you can calculate $\int_\alpha \omega = 2\pi$, whereas $\int_\beta \omega = 0$. But by Stokes theorem, if they were related by an homotopy as above, these integrals would be the same. Hence, they are in different components.

Problem 6 (8)**.** Let $f : S^n \to \mathbb{R}$ be a continuous function. For any point $x \in S^n$, denote by τx the antipodal point to x. Prove that there is at least one point $x \in S^n$ so that $f(x) = f(\tau x)$.

Solution 6. Consider the function $g(x) := f(x) - f(\tau x)$. Notice that, $g(x) = -g(\tau x)$. Hence, $g(S^n)$ is symmetric with respect to $0 \in \mathbb{R}$ and connected as S^n is connected. This implies $0 \in g(S^n)$. Hence, there exists an $x \in S^n$ such that $0 = g(x) = f(x) - f(\tau x)$. We are done.

Problem 7 (24)**.** This problem deals with a variant of $\mathbb{RP}^2$.

a. Give a model for the *space of two distinct lines passing through the origin in* $\mathbb{R}^2$—I will call this space L.

b. Prove that L is not compact, and describe a compactification Y for this space.

c. What does the space $Y \setminus L$ look like?

d. When does a continuous function $L \to \mathbb{R}$ extend to a continuous function $Y \to \mathbb{R}$?

Solution 7. a. You have seen that $\mathbb{RP}^1$ is a model for the space of lines through the origin. Hence, space of pairs of lines (l_1, l_2) is $\mathbb{RP}^1 \times \mathbb{RP}^1$. But, we want only distinct lines, hence take of the diagonal, i.e. consider $M = \mathbb{RP}^1 \times \mathbb{RP}^1 \setminus \{(l, l) : l \in \mathbb{RP}^1\}$. But in M, the set of distinct lines $\{l_1, l_2\}$ occurs twice: as (l_1, l_2) and as (l_2, l_1). Hence this suggests taling the quotient of M by the equivalence relation $(l_1, l_2) \sim (l_2, l_1)$, denote the quotient map by

$$\pi : M \to L$$

This is the model for the desired space. Other models can be given by modifying models of $\mathbb{RP}^1$ such as

$$\big(S^1 \times S^1 \setminus \{(x, x)\}\big)/(x_1, x_2) \sim (\pm x_2, \pm x_1)$$

or

$$\left((\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}^2 \setminus \{0\}) \setminus \{(x, x)\}\right) \Big/ (x_1, x_2) \sim (r_2 x_2, r_1 x_1), r_1, r_2 \in \mathbb{R} \setminus \{0\}$$

It is not hard to show they are equivalent. Also, note that by replacing 1 above by n, and 2 by $n+1$ we get the same space for $\mathbb{R}^{n+1}$.

b. The quotient map

$$\pi : M \to L$$

is clearly an open and closed map. Now, M is an open subset of $\mathbb{RP}^1 \times \mathbb{RP}^1$, which is itself a Hausdorff and connected space. Hence, M cannot be closed in there, so it cannot be compact. Now, note an interesting property of π: For every $p \in L$, there exists a neighborhood $U \subset L$ such that $\pi^{-1}(U) = U_1 \cup U_2$, where U_i are disjoint, open and $\pi_{|U_i} : U_i \to U$ is an homeomorphism. This can easily be proven by taking seperating neighborhoods V_1, V_2 of $(l_1, l_2), (l_2, l_1) \in \pi^{-1}(p)$ and letting $U_1 = V_1 \cap \tau(V_2)$ and $U_2 = \tau(U_1) = V_2 \cap \tau(V_1)$ and $U = \pi(U_i)$.

Hence, if L is compact, one can cover it with finitely many compact sets F_k, $k = 1, \dots m$ each of which lie in an open set like $U \subset L$ above. Hence, the preimage $\pi^{-1}(F_k)$ is homeomorphic to two disjoint copies of F_k, thus it is compact. This would imply $M = \bigcup_{k=1}^{m} \pi^{-1}(F_k)$ is compact, but we showed this is not the case. Hence, L cannot be compact.

Now, the cheapest way of compactifying L is extending the involution τ to act on $\mathbb{RP}^1 \times \mathbb{RP}^1$, i.e. by $(l_1, l_2) \mapsto (l_2, l_1)$ and taking the quotient of $\mathbb{RP}^1 \times \mathbb{RP}^1$. That this is compact is clear, Hausdorff property follows by normality of $\mathbb{RP}^1 \times \mathbb{RP}^1$ or can be checked by hand, and that $L \hookrightarrow Y$ open, is also clear. Note, making a compactification with 2-point fibers everywhere is trickier, here, we roughly added a point when two lines coincide, but you can also keep track of the direction the two lines approaching to each other, and add a limit point for that. We will not pursure this here.(also this adds one extra point only in dimension 2)

c. Y is obtained by dividing $\mathbb{RP}^1 \times \mathbb{RP}^1$ by the equivalence relation $(l_1, l_2) \sim (l_2, l_1)$ and L is obtained by dividing $Y \setminus \Delta$ by the same relation, where Δ is the diagonal as above. Now the involution acts trivially on it, in other words Δ is invariant under $\sim$ and π is a bijection on it. This shows that

$$\pi_{|\Delta} : \Delta \to Y \setminus L$$

is a bijection. It is clearly continuous and both spaces are compact Hausdorff, hence it is a homeomorphism. Thus the homeomorphism type of $Y \setminus L$ is the same as $\Delta \cong \mathbb{RP}^1$.

d. Well, I should have taken one point compactification instead:). Anyway, in this case, the answer is whenever the composition

$$\mathbb{RP}^1 \times \mathbb{RP}^1 \setminus \Delta \to L \to \mathbb{R}$$

extends to $\mathbb{RP}^1 \times \mathbb{RP}^1$. This is true since if you know there exists an extension $\mathbb{RP}^1 \times \mathbb{RP}^1 \to \mathbb{R}$, then you already know it is invariant under

$$(l_1, l_2) \mapsto (l_2, l_1)$$

($f(l_1, l_2) = f(l_2, l_1)$ is a closed property and holds over the dense subset $\mathbb{RP}^1 \times \mathbb{RP}^1 \setminus \Delta$) I don't know a good criteria for this, though it is more visible here, it holds if whenever l_1, l_2, l_1', l_2' converges to $l \in \mathbb{RP}^1$, then $f(l_1, l_2)$ and $f(l_1', l_2')$ converge to the same real number.