Solutions for Problem Set 7: Models, compactifications, identifying spaces, characterizing subspaces

Read the notes on the course website related to the subspace topology and quotient topology (sections 3 and 5, though you might enjoy 1 and 2 as well).

Problem 1 (8)**.** Have you done the reading?

Solution 1. Yes.

Problem 2 (8)**.** Prove that a map $f : A \to X$ is a homeomorphism on its image if and only if $f : A \to X$ satisfies the universal property of the subspace topology. (If you get stuck, go back and read the notes, which contains a proof of a very similar statement. Then try to prove this statement again. Repeat this procedure until you can understand and prove this result without consulting the notes.)

Solution 2. Let $B = f(A) \xrightarrow{i} X$ be the image with the subspace topology. Then $B \xrightarrow{i} X$ satisfies the mentioned universal property. If f satisfies it too, then as $Im(i) = Im(f)$, there are unique maps $\phi : A \to B$ and $\psi : B \to A$ such that $i \circ \phi = f$ and $f \circ \psi = i$. Then $\psi \circ \phi : A \to A$ is a map commuting with f. But id_A is also such a map and by universal property this map is unique; hence, $\psi \circ \phi = id_A$. Similarly, $\phi \circ \psi = id_B$. Thus, ϕ and ψ are homeomorphisms. But, ϕ is obtained from f by just restricting the target, hence f is a homeomorphism onto its image.

The converse is easier, if f is homeomorphism onto B then, let ϕ again denote the map with restricted target and i denote the inclusion of B. But it is generally true that composition with isomorphisms of a category preserve universal properties, hence, f satisfies it as $i \circ \phi = f$ and ϕ is an isomorphism(More concretely, you can use f to identify A with B).

Problem 3 (9)**.** Let $f : X \to Y$ be a continuous surjection of topological spaces.

a. Give an example to show that f may be an open map and at the same time not be a closed map.

b. Give an example to show that f may be a closed map and at the same time not be an open map.

c. Prove that if f is either open or closed, then the topology on Y is equal to the quotient topology coming from the relation: $r, s \in X$ are equivalent iff $f(r) = f(s)$.

Solution 3. a. An example for this can be given by taking X to be coproduct of open subsets of Y, for which at least one of the open subsets is not closed. For instance, let $Y = \mathbb{R}$ and $X = \mathbb{R}_{>-1} \times \{0\} \cup \mathbb{R}_{<0} \times \{1\} \subset \mathbb{R}^2$ and let f be the projection to the first component. Then, this is clearly open, but the closed subset $\mathbb{R}_{>-1} \times \{0\}$ maps onto $\mathbb{R}_{>-1}$, which is not closed.

b. An example of the above spirit can be given easily for instance replace $\mathbb{R}_{>-1}$ and $\mathbb{R}_{<0}$ by $\mathbb{R}_{\geq -1}$ and $\mathbb{R}_{\leq 0}$ respectively. Then, $\mathbb{R}_{\geq -1} \times \{0\}$ is open but maps to the set $\mathbb{R}_{\geq -1}$, which is not open.

c. Recall that a subset $A \subset Y$ is open(resp. closed) in the quotient topology if and only if $f^{-1}(A)$ is open(resp. closed) in X. We have to show this statement is true for subsets of Y. The only if part follows from continuity of f. But, if f is open then as $f(f^{-1}(A)) = A$ (by surjectivity), the if part also follows, i.e. if $f^{-1}(A)$ is open, then $f(f^{-1}(A)) = A$ is open. It is similar for closed maps.

Problem 4 (12)**.** This problem will define the one-point compactification and ask you to prove some theorems about it. Recall that a space X is called *locally compact* if for every point $p \in X$, there exists a compact set $K \subset X$ such that $p \in K$. Note that every compact space is locally compact (prove this to yourself).

a. Give an example of space which is locally compact and not compact.

Recall that the *one-point compactification* of a space X, if it exists, consists of

- a compact Hausdorff space Y
- an embedding $i : X \hookrightarrow Y$

such that

- $Y \setminus i(X)$ is a set containing exactly one element (we call that element "∞" for convenience).
- $\overline{i(X)} = Y$.

b. Prove that if X is a space with one-point compactifications $i : X \hookrightarrow Y$ and $j : X \hookrightarrow Z$ that there is a unique homeomorphism $h : Y \to Z$ such that $h \circ i = j$.

Observe that if X has a one-point compactification, then X must be Hausdorff (if this is not clear, prove it to yourself!). We might also imagine that if X fits inside a compact space, then it can't be that far off from being compact itself.

c. Prove that if X has a one-point compactification, then X must be locally compact.

So far in this problem you have proved that if X has a one-point compactification, X must be locally compact and Hausdorff. These seem to be the only "obvious" pieces of information we can extract about X if we know it has a one-point compactification. One question to ask, then, is the following:

d. if a space X is locally compact (but not compact) and Hausdorff, does it have a one-point compactification? (Hint: try using the construction we gave in class to build one.)

Solution 4. a. The simplest ones are $\mathbb{R}$ with Euclidean topology and any infinite set with discrete topology.

b. For simplicity identify X with its image under i and j. then $id_X : X \to X$ uniquely extends to a bijection $Y \xrightarrow{f} Z$, namely $\infty_Y \mapsto \infty_Z$. We check that the map is closed and applied to f^{-1} this proves continuity as well. Let $F \subset Y$ be a closed subset. Then it is compact, as Y is compact Hausdorff. If $F \subset X$, then it carries the same topology as a subset of Y and Z(as X carries the subspace topology). Hence, $f(F)$, it is compact and so is closed. If $F \not\subseteq X$, i.e. $\infty_Y \in F$, then $F \setminus \{\infty_Y\}$ is closed in X, hence $\overline{f(F)} \cap X = f(F) \setminus \{\infty_Z\}$ closed in X. Thus $f(F) = \overline{f(F)}$, hence it is closed.

Thus, f is a homeomorphism. Uniqueness is almost a set theoretic statement in this case.

c. A one sentence proof can be given by stating that compact Hausdorff spaces are locally compact and open subsets of locally compact spaces are locally compact(the proof for this may be easier or harder depending on the definition of local compactness you start with). For a longer proof, let $x \in X \subset Y$, where Y is a one point compactification as above. Then there exists open subsets $U, V \subset Y$ such that $x \in U$, $\infty \in V$ and $U \cap V = \emptyset$. Then, $Y \setminus V \subset X$ is a closed subset of Y, hence compact. $U \subset Y \setminus V$; hence, $Y \setminus V$ is compact neighborhood of x in X. Hence, X is locally compact.

d. Yes it is. Set theoretically we know what it should be so start with $X \cup \{\infty\}$. Declare a subset to be open if it is contained in X and open in there or it contains ∞ and its complement is a compact subset of X. Hausdorff property of X is enough to prove this is a topology. That it is compact also follows from that, if we have an open cover, then at least one of them contains ∞, hence has a compact complement which lives in X. Then, finitely many of the remaining open subsets can be chosen to cover this compact set, hence, these finitely many subset with the first chosen one cover $X \cup \{\infty\}$.

Locally compactness of X is necessary to prove $X \cup \{\infty\}$ is Hausdorff. To see this we just need to see ∞ can be separated from any point $x \in X$:

just choose $x \in U \subset K \subset X$, where U is open and K is compact. Then U and $(X \cup \{\infty\}) \setminus K$ separate x and ∞. Hence, $X \cup \{\infty\}$ is Hausdorff.

Problem 5 (12)**.** For the following problem, the notation $(0,1)$ will refer to the open interval in the real line.

a. Prove that $\mathbb{D}^n \cong \mathbb{R}^n$.

b. Prove that $(\mathrm{int}\mathbb{D}^n)^+ \cong S^n$.

c. Many of you argued that $\big(S^1 \times (0,1)\big)^+$ is homeomorphic to a "pinched torus." Give a model for the pinched torus and prove that it is homeomorphic to the one-point compactification of $S^1 \times (0,1)$.

Solution 5. a. I assume $\mathbb{D}^n$ refers to open unit disc $\{x \in \mathbb{R}^n : ||x|| < 1\}$. Then I can write down a homoemorphism which just rescales elements by a number that only depends on its norm. To be more precise let $\phi : [0,1) \to \mathbb{R}_{\geq 0}$ be an order preserving homeomorphism, such as tangent function $tan(\frac{2x}{\pi})$. Then the map $x \mapsto \phi(||x||)x$ is a homeomorphism, with an inverse of the same type.

b. Stereographic projection

$$(x_1, \ldots, x_{n+1}) \mapsto \Big(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}}\Big)$$

gives us a homeomorphism $S^n \setminus \{(0, \ldots, 0, 1)\}$ onto $\mathbb{R}^n$ with an obvious inverse

$$(y_1, \ldots, y_n) \mapsto \Big(\frac{2y_1}{||y||^2 + 1}, \ldots, \frac{2y_n}{||y||^2 + 1}, \frac{||y||^2 - 1}{||y||^2 + 1}\Big)$$

Hence, $\mathbb{R}^n$ embeds into S^n as a dense open subset with a complement consisting of a single point. As we know S^n is compact Hausdorff, this implies it is one point compactification of $\mathbb{R}^n$. By part (a), it is also one point compactification of $int(\mathbb{D}^n) = \mathbb{D}^n$.

c. Well, you can take this to be your model and the question becomes a tautology. But, this is not the simplest model. Let $T^2 = S^1 \times S^1$. choose a base point $e \in S^1$ and I would call $T^2/(S^1 \times \{e\})$ a pinched torus. Note for a closed subset $A \subset X$ of a space X, I denote X/A to be the space obtained by identifying the points of A and nothing else, endowed with quotient topology. In this simple situation it is easy to show it is Hausdorff. In general if X is strongly regular, i.e. you can separate closed subsets from points by functions, this still holds.

Now, take an open embedding $i : (0,1) \hookrightarrow S^1$ with image $S^1 \setminus \{e\}$. Then, it is easy to see that the composition

$$S^1 \times (0,1) \xrightarrow{id \times i} S^1 \times S^1 \to T^2/(S^1 \times \{e\})$$

is an open embedding with a dense image which has a complement consisting of a single point namely the pinched part $(S^1 \times \{e\})/(S^1 \times \{e\})$.