Solutions for Problem Set 6: Compactness and The Zariski topology

Problem 1 (8)**.** Did you do your reading assignment?

Solution 1. Yes.

Problem 2 (24)**.**

a. Let R be a commutative ring with unit ($1 \in R$). Prove that $\text{spec}(R)$ is compact.

b. When is $\text{spec}(R)$ Hausdorff?

Consider the map

$$i : \mathbb{R}^n \to \text{spec}(\mathbb{R}[x_1, \ldots, x_n])$$

that is defined by taking the n-tuple $(r_1, \ldots, r_n)$ to the ideal generated by the degree one polynomials

$$x_1 - r_1, \ldots, x_n - r_n.$$

c. Prove that the map i is injective. Identifying $\mathbb{R}^n$ with its image under the map, it acquires a topology as a subspace of $\text{spec}(\mathbb{R}[x_1, \ldots, x_n])$. Denote $\mathbb{R}^n$ with this topology by $\mathbb{A}^n$

d. What is the closure of $\left\{(n, \frac{1}{n}) : n \in \mathbb{N} \setminus \{0\}\right\} \subset \mathbb{A}^n$?

Solution 2. a. Let $\{U_i\}$ be an open cover of $spec(R)$. Then there exist ideals I_i such that $U_i = spec(R) \setminus V(I_i)$. As $\bigcup U_i = spec(R)$, $\bigcap V(I_i) = V(\sum I_i) = \emptyset$. If $\sum I_i \neq R$, then there would exist a maximal(hence prime) ideal containing it, thus $V(\sum I_i)$ would be non-empty. Thus, $1 \in R = \sum I_i$, hence there exist $i_1, \ldots, i_r$ and $x_j \in I_{i_j}$ such that $\sum x_j = 1$. Thus, $1 \in \sum_{j=1}^r I_{i_j}$, i.e. $\sum_{j=1}^r I_{i_j} = R$. Thus, $\bigcap_{j=1}^r V(I_{i_j}) = \emptyset$ and $\bigcup_{j=1}^r U_{i_j} = spec(R)$. Hence, there exist a finite subcover for any open cover, i.e. the space is compact.

b. If $spec(R)$ is Haussdorff, then all the points should be closed. But, it is easy to see that $\overline{\{p\}} = V(p)$. Hence, this holds if and only if $V(p) = \{p\}$, i.e. all the prime ideals are maximal. In other words, the ring has 0 Krull dimension.

The converse is also true, but it is harder to show. Even if we assume R is Noetherian, it requires some knowledge in commutative algebra. You may also try to give stronger criteria implying Hausdorff condition. For instance, the assumption that for all $p, q \in spec(R)$, there exist $f, g \in R$ such that $fg = 0$, $f \notin p$ and $g \notin q$ implies Hausdorff. In this case $spec(R) \setminus V(f)$ and $spec(R) \setminus V(g)$ seperate points p and q. Note this is also equivalent to Hausdorff condition.

c. For a given ideal I in $\mathbb{R}[x_1, \ldots, x_n]$, consider the set $Z(I) := \{(a_1, \ldots, a_n) \in \mathbb{R}^n | f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I\}$. But then it is easy to see that $Z(i(a_1, \ldots, a_n)) = \{(b_1, \ldots, b_n) | b_i - a_i = 0, i = 1, \ldots, n\} = \{(a_1, \ldots, a_n)\}$. Hence, $i(a_1, \ldots, a_n)$ determines $(a_1, \ldots, a_n)$ and i is injective.

d. $i(n, \frac{1}{n}) = (x_1 - n, x_2 - \frac{1}{n})$ and let $V(I)$ be the closure of $\{i(n, \frac{1}{n})\}$ in $spec(\mathbb{R}[x_1, \ldots x_n])$. Then, it easy to see that I can be taken to be $\bigcap i(n, \frac{1}{n}) = \bigcap (x_1 - n, x_2 - \frac{1}{n})$. If $f \in \bigcap (x_1 - n, x_2 - \frac{1}{n})$, then $f(n, \frac{1}{n}) = 0$ for all $n \in \mathbb{N} \setminus \{0\}$. That implies the rational function $f(x, \frac{1}{x})$ has infinitely many zeroes, hence it is equal to 0. The converse is also true, thus $I = \{f : f(x, \frac{1}{x}) = 0\}$. Thus, the closure in $\mathbb{R}^n$ is $i^{-1}(V(I)) = Z(I) = \{(a_1, a_2) : f(a_1, a_2) = 0 \text{ for all } f \in I\}$. In particular, letting $f = x_1 x_2 - 1$ we see that each point in this set should be of the form $(a, \frac{1}{a})$. Converse is tautological, hence the closure is $\{(a, \frac{1}{a}) : a \in \mathbb{R} \setminus \{0\}$.